Workshop on Fixed Point Theory and its Applications

On the occasion of Enrique Llorens' 70th birthday

ON TWO MIDPOINT RULE FOR QUASI-NONEXPANSIVE MAPPINGS

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- H Hilbert space,
- C closed, convex subset of H
- $T: C \to C$ a nonexpansive mapping

The orbit

$$T^n x$$

does not converge, in general (also if T has fixed points)

However,

- **Krasnoselskii (1955)**

If instead of T we consider the auxiliary mapping

$$T_{1/2} = \frac{1}{2}(I + T)$$

and T(C) is compact,

then the orbit

$$x_{n+1} = T_{1/2} x_n = \frac{1}{2}(x_n + T x_n)$$

is strongly convergent.

- **Opial (1967)**

Without the assumption of compactness of T(C) one has weak convergence for the orbit of the average map

$$x_{n+1} = T_{\alpha}^{n} x_o = (1-\alpha) x_n + \alpha T x_n$$

- **Mann (1953)**

$$x_{n+1} = (1 - t_n)x_n + t_n T x_n$$

- **Halpern (1967)**

$$x_{n+1} = (1 - t_n)u + t_n T x_n$$

- **Ishikawa (1974)**

$$x_{n+1} = (1 - t_n)x_n + t_n T[(1 - s_n)x_n + s_n T x_n]$$

- **Moudafi (2000)**

$$x_{n+1} = (1-t_n)f(x_n) + t_n T x_n$$

- **M. – Xu (2006)**

$$x_{n+1} = \alpha_n \gamma f(x_n) + (Id - \alpha_n A) T x_n$$

Strong convergence to the argument of the minimization problem

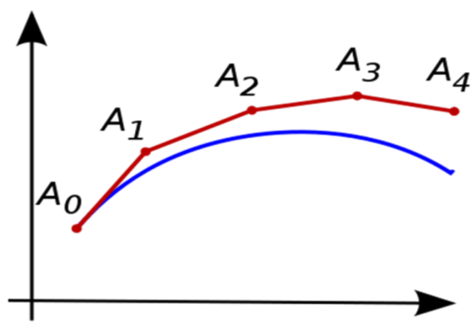
$$\min_{x \in Fix(T)} \frac{1}{2} < \mathrm{Ax}, \mathrm{x} > \ + \mathrm{h}(\mathrm{x})$$

where h is a potential function for γf.

- In the last two years he was born a new idea,

starting from the Euler method for the construction of a polygonal approximating the solution of an ODE

$$y_{n+1} = y_n + hf(t_n, y_n)$$



Drawback: numerical unstability

- Example: $y' = -2.3y, \quad y(0) = 1$

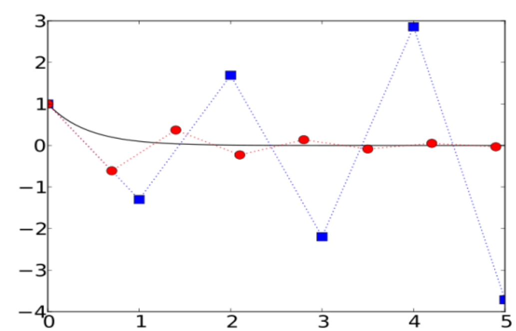
Solution $y(t) = e^{-2.3t}$, which decays to zero as t → ∞

However, if the Euler method is applied to this equation with step size h = 1, then the numerical solution is qualitatively wrong: it oscillates and grows (blue squares). Taking

h=0.7 one obtain the red circle

Polygonal.

The black curve is the solution.



- A simple modification of the Euler method which eliminates the stability problems is the

backward Euler method or implicit method

$$y_{n+1} = y_n + hf(t_{n+1}, y_{n+1})$$

The midpoint method further improves the situation

- We see some details.

- The starting point is the initial value problem for an autonomous ODE

$$\begin{cases} x'(t) = f(x(t)) \\ x(t_o) = x_o \end{cases}$$

- Given a time interval $[t_o, T]$;
- Fixed N natural number;
- Defined the step size $h_N = \frac{T - t_o}{N}$;
- Defined the mesh $t_n = t_o + nh$;

- For h sufficiently small, (i.e. for N sufficiently big)

$$f(x(t)) = x'(t) \approx \frac{x(t+h) - x(t)}{h}$$

So,

we obtain the approximed values of the solution at each time step $y_n \approx x(t_n)$ via the Euler method

$$y_{n+1} = y_n + hf(y_n).$$

Moreover, the Eulero polygonal $Y_N(t)$ obtained connecting two consecutive nodes (t_n, y_n) and (t_{n+1}, y_{n+1}),under suitable assumption on the function f, converges to the solution when N$\rightarrow \infty$.

- On the other hand, one can replace
  - $x'(t) \approx \frac{x(t+h)-x(t)}{h}$

with

$$f\left(x\left(t+\frac{h}{2}\right)\right) = x'(t+h/2) \approx (x(t+h)-x(t))/h$$

getting so

$$x(t+h) \approx x(t) + hf\left(x\left(t+\frac{h}{2}\right)\right)$$

- Reached this stage, there are two different procedures for approximating the (unknown) value $x\left(t+\frac{h}{2}\right)$:

(1) One can apply Euler's method, obtaining

$$x\left(t+\frac{h}{2}\right) \approx x(t) + \frac{h}{2} f\big(x(t)\big)$$

that, substituted in

$$x(t+h) \approx x(t) + hf\left(x\left(t+\frac{h}{2}\right)\right)$$

leads to

$$x(t+h) \approx x(t) + hf\left(x(t) + \frac{h}{2} f(x(t))\right)$$

Taking t_n as t, we obtain the recursive scheme

$$\text{EMR}\begin{cases} y_o = x_o \\ \bar{y}_{n+1} = y_n + hf(y_n) \\ y_{n+1} = y_n + hf\left(\frac{y_n + \bar{y}_{n+1}}{2}\right) \end{cases}$$

Explicit midpoint rule

(2) The value $x(t+\frac{h}{2})$ is approximated with the midpoint of the line segment from $x(t)$ to $x(t+h)$, obtaining

$$x(t+h) \approx x(t) + hf\left(\frac{x(t)+x(t+h)}{2}\right)$$

Taking t_n as t, we obtain the recursive scheme

$$\text{IMR}\begin{cases} y_o = x_o \\ y_{n+1} = y_n + hf\left(\frac{y_n+y_{n+1}}{2}\right) \end{cases}$$

<u>Implicit midpoint rule</u>

- If we write the function f in the form

$$f(y) = y - g(y),$$

then our Cauchy problem becomes

$$\begin{cases} y' = y - g(y) \\ y(x_o) = y_o \end{cases}$$

And the IMR is rewritten as

$$y_{n+1} = y_n + h\left[\frac{y_n + y_{n+1}}{2} - g\left(\frac{y_n + y_{n+1}}{2}\right)\right]$$

- The equilibrium problem associated with the differential equation is the fixed point problem $y = g(y)$.
- This motivates us to trasplant IMR to the solving the fixed point equation $x = Tx$

where T is, in general, a nonlinear operator in a Hilbert space.

This is the new idea present in the paper «The implicit midpoint rule for nonexpansive mappings» by Hong Kun Xu , M. Alghamdi, M. A. Alghamdi and N. Shahazad, published on FPTA, 2014.

- According to the IMR (that, we underline, is a method for construct a single polygonal function approximating the solution of the differential equation) rewritten as

$$y_{n+1} = y_n + h\left[\frac{y_n+y_{n+1}}{2} - g\left(\frac{y_n+y_{n+1}}{2}\right)\right],$$

Xu and al., propose the following algorithm for the approximation of fixed points of a nonexpansive mapping T

$$x_{n+1} = x_n - t_n\left[\frac{x_n + x_{n+1}}{2} - T\left(\frac{x_n + x_{n+1}}{2}\right)\right]$$

- They show also that the previous is equivalent to the algorithm

$$x_{n+1} = (1 - t_n)x_n + t_n T\left(\frac{x_n + x_{n+1}}{2}\right)$$

Moreover, they prove <u>weak</u> convergence results under suitable assumptions on the sequence of parameters t_n.

- After their paper, this algorithm is known as IMR for nonexpansive mappings

- In 2015, Xu and al., again on FPTA, apply the viscosity technique introduced by Moudafi in 2000 and obtain the following Viscosity Implicit Midpoint Rule (VIMR)

$$x_{n+1} = (1 - t_n) f(x_n) + t_n T\left(\frac{x_n + x_{n+1}}{2}\right).$$

- After this, Ke and Ma, once again on FPTA, improve the VIMR, replacing the midpoint by any point of interval $[x_n, x_{n+1}]$. They obtain the following generalized VIMR

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T(s_n x_n + (1 - s_n) x_{n+1})$$

- Inspired by these works, we follow the EMR derived by Eulero’s method

$$\text{EMR}\begin{cases} y_o = x_o \\ \bar{y}_{n+1} = y_n + hf(y_n) \\ y_{n+1} = y_n + hf\left(\frac{y_n+\bar{y}_{n+1}}{2}\right) \end{cases}$$

with the introduction of a viscosity term, and propose the iterative algorithm

$$\begin{cases} x_o \in C \\ \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n) T x_n \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T(s_n x_n + (1-s_n)\bar{x}_{n+1}) \end{cases}$$

where f is a viscosity term, T is a nonexpansive mapping and all the coefficients are in (0,1).

We can call this

generalized VEMR

- In the attempt to prove the strong convergence of the generalized VEMR, we used the beautiful Maingé’s Lemma present in the paper

«*Strong convergence of projected subgradient methods for nonsmooth and non-strictly convex minimization*»,

Set Valued Analysis 16 (2008) 899-912

- Maingé Lemma: *Let* (γ_n) *be a sequence of real numbers such that there exists a subsequence* $\left(\gamma_{n_j}\right)$ *of* (γ_n) *such that* $\gamma_{n_j} < y_{n_j+1}$ *for all* j.

Then there exists a nondecreasing sequence of natural numbers $\tau(n)$ *such that*

- $\lim_n \tau(n) = \infty$
- $\gamma_{\tau(n)} < \gamma_{\tau(n)+1}$
- $\gamma_n < \gamma_{\tau(n)+1}$

The use of Maingé Lemma permitted us to obtain strong convergence results for both generalized VEMR and VIMR not only for nonexpansive mappings, but for the much bigger class of quasi-nonexpansive mappings (i.e. mappings T for which $Fix(T) \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\| \; \forall p \in Fix(T).$$

Moreover permitted also to weaken the assumption on the coefficients with respect to previous results.

- Convergence result:

The generalized VEMR

$$\begin{cases} x_o \in C \\ \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T(s_n x_n + (1-s_n)\bar{x}_{n+1}) \end{cases}$$

converges to

$$p \in Fix(T) \ that\ is\ the\ unique\ solution\ in\ Fix(T)$$

$$of\ the\ variational\ inequality$$

$$< p - f(p), p - x > \leq 0 \qquad \forall x \in Fix(T)$$

under the assumptions

- Assumptions on space and on mappings:
- H real Hilbert space, C nonempty closed convex subset of H, T:$C \to C$ quasi-nonexpansive, I-T demiclosed in 0, $f: C \to C$ contraction.
- Assumptions on the coefficients (all in $[0,1]$):
- $\lim_n \alpha_n = 0,$
- $\sum \alpha_n = \infty$
- $limsup_n \beta_n (1 - \beta_n)(1 - s_n) > 0$

- Example 1. A nonspreading mapping
- H=$\mathbb{R}$, S:$[-3,3] \to [-3,3]$

$$Sx = \begin{cases} 0 & if\ x \in [-2,2] \\ -1 & if\ x \in [-3,-2] \\ 1 & if\ x \in [2,3] \end{cases}$$

S is nonspreading (i.e.

$$2\|Sx - Sy\|^2 \le \|Sx - y\|^2 + \|x - Sy\|^2)$$

and so quasi-nonexpansive, S is not continuous and I-S is demiclosed in 0

- If we take
- $\alpha_n = \frac{1}{n}, \beta_n = \frac{n-1}{2n}, s_n = \frac{1}{n}, f(x) = \frac{x}{2}, x_1 = 3,$

then the VEMR is given by

$$\begin{cases} \bar{x}_{n+1} = \dfrac{n-1}{2n} x_n + \dfrac{n+1}{2n} S x_n \\ x_{n+1} = \dfrac{x_n}{2n} + \left(\dfrac{n-1}{n}\right) S\left(\dfrac{x_n}{n} + \dfrac{n-1}{n}\bar{x}_{n+1}\right) \end{cases}$$

so that

- $x_1 = 3, \quad x_2 = \frac{3}{2}, \quad x_3 = \frac{3}{8}, \quad \ldots, x_{n+1} = \frac{3}{n!2^n},$

That quikly converges to 0.

- Example 2. Fredholm integral equation

$$x(t) = g(t) + \int_0^1 F(t, s, x(s)ds, \qquad t \in [0,1]$$

where g is a given continuous function on [0,1] and $F: [0,1] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

If F satisfies the Lipshitz continuity condition

$$|F(t, s, x) - F(t, s, y)| \leq |x - y|$$

then the Fredholm equation has solutions in $L^2[0,1]$ (classical)

- The fixed points of the nonexpansive mapping $T: L^2[0,1] \to L^2[0,1]$ defined by

$$Tx(t) = g(t) + \int_0^1 F(t, s, x(s)ds, \qquad t \in [0,1]$$

re the solutions of the Fredholm equation.

So, sequence of functions in $L^2[0,1]$ defined by

$$x_{n+1} = (1 - t_n)x_n + t_n T\left(\frac{x_n + x_{n+1}}{2}\right)$$

converges weakly in $L^2[0,1]$ to a solution of Fredholm equation.

- Example 3. Nonlinear Evolution problem

Browder in 1965 proved the existence of a periodic solution of the time-dependent nonlinear evolution equation in a real Hilbert space H

$$\frac{du}{dt} + A(t)u = f(t,u), t > 0$$

where $A(t)$ is a family of closed linear operator in H and $f: \mathbb{R} \times H \to H$ satisfy suitable conditions that ensure periodic solutions of period τ.

Define a nonexpansive mapping $T: H \to H$ as

$$T(v) := u(\tau),$$

where u is the solution of the nonlinear evolution equation satisfying the initial value $u(0) = v$.

Then the periodic solutions are the fixed points of T.

To approximate such solutions we can use each of our algorithms.