# Coupled fixed point problems and applications

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Dedicated to Enrique on the occasion of his 70th anniversary

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# The concept of $b$-metric space

**Definition-Bakhtin, Czerwik, ...**

Let $X$ be a nonempty set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to \mathbb{R}_+$ is said to be a $b$-metric (quasi-metric, almost metric) on $X$ if the following axioms are satisfied:
1) if $x, y \in X$, then $d(x, y) = 0$ if and only if $x = y$;
2) $d(x, y) = d(y, x)$, for all $x, y \in X$;
3) $d(x, z) \leq s[d(x, y) + d(y, z)]$, for all $x, y, z \in X$.

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A pair $(X, d)$ with the first two properties is called a semi-metric space, while a pair $(X, d)$ with above three properties is called a $b$-metric space.

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For the concept of $b$-metric space see also N. Bourbaki, D. Kurepa, L.M. Blumenthal, J. Heinonen.

# Examples.

**Example**

The set $l_p(\mathbb{R})$ with $0 < p < 1$, where

$l_p(\mathbb{R}) := \{(x_n) \subset \mathbb{R} | \sum_{n=1}^{\infty} |x_n|^p < \infty\}$, together with the function

$d : l_p(\mathbb{R}) \times l_p(\mathbb{R}) \to \mathbb{R}$,

$$d(x, y) := (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{1/p},$$

is a $b$-metric space with constant $s = 2^{1/p} > 1$.

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For $0 < p < 1$, the space $L_p[a,b]$ of all real functions $x(t)$, $t \in [a,b]$ such that $\int_a^b |x(t)|^p dt < \infty$, together with the function

$$d(x,y) := \left(\int_a^b |x(t) - y(t)|^p dt\right)^{1/p}, \text{ for each } x, y \in L_p[a,b],$$

is a $b$-metric space. Notice that in this case $s = 2^{1/p} > 1$.

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If $X$ is a nonempty set, then a mapping $d : X \times X \to E$ is called a cone metric on $X$ if the usual axioms of the metric take place with respect to "$\leq$".
The cone $P$ is called normal if there is a number $K \geq 1$ such that, for all $x, y \in E$, the following implication holds:

$$0 \leq x \leq y \Longrightarrow \|x\| \leq K\|y\|.$$

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If the cone $P$ is normal with the coefficient of normality $K \geq 1$, then the functional

$$\hat{d} : X \times X \to \mathbb{R}_+, \ \hat{d}(x, y) := \|d(x, y)\|$$

is a b-metric on $X$ with constant $s := K$.

# Czerwik's fixed point theorem for single-valued nonlinear contractions

**Theorem. (Czerwik (1993), Kirk-Shahzad)**

Let $(X, d)$ be a complete $b$-metric space with constant $s \geq 1$ and $f : X \to X$ be an operator, for which there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ (i.e., $\varphi$ is increasing and $\lim_{n\to\infty} \varphi^n(t) = 0$, for every $t > 0$) such that

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \ \forall x, y \in X.$$

Then, $f$ has a unique fixed point $x^* \in X$ and, for all, $x \in X$ $\lim_{n\to\infty} d(f^n(x), x^*) = 0$, i.e., $f$ is a Picard operator.

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# Czerwik's fixed point theorem for multi-valued nonlinear contractions

If $(X, d)$ is a metric space, then we denote

$$H_d(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(a, b)\}.$$

**Theorem 1. (Czerwik-1998)**

Let $(X, d)$ be a complete $b$-metric space with constant $s \geq 1$ and $F : X \to P_{cp}(X)$ be a multivalued operator. Suppose that $d$ is continuous and there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$H_d(F(x), F(y)) \leq \varphi(d(x, y)), \ \forall x, y \in X.$$

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**Theorem 2. (Czerwik-1998)**

Let $(X, d)$ be a complete $b$-metric space with constant $s \geq 1$ and $F : X \to P_{cl}(X)$ be such that there exists $k \in (0, \frac{1}{s})$ such that

$$H_d(F(x), F(y)) \leq kd(x, y), \ \forall x, y \in X.$$

Then, $F$ is a multivalued weakly Picard operator, i.e., there exists $x^* \in X$ such that $x^* \in F(x^*)$ and, for every $(x, y) \in Graph(F)$, there is a sequence of successive approximations for $F$ starting from $(x, y)$ which converges to $x^*(x, y) \in Fix(F)$.

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If additionally, $f$ is continuous and increasing (or decreasing) and there exists an element $x_0 \in X$ such that $x_0 \leq f(x_0)$, then $f$ has at least one fixed point.

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If additionally, for every $x, y \in X$ there exists $z \in X$ which is comparable to $x$ and $y$ **or** every pair of elements of $X$ has a lower bound or an upper bound, then $f$ is a Picard operator.

# Ran-Reurings theorem in b-metric space (I)

**Theorem.**

Let $X$ be a nonempty set endowed with a partial order "$\preceq$" and $d : X \times X \to X$ be a complete $b$-metric with constant $s \geq 1$. Let $f : X \to X$ be an operator which has closed graph with respect to $d$ and increasing with respect to "$\preceq$".

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Then $Fix(f) \neq \emptyset$ and the sequence $(f^n(x))_{n \in \mathbb{N}}$ converges to a fixed point $x^*(x)$ of $f$, for each $x \in X$ which is comparable to $x_0$.

Moreover, if $d$ is continuous, then we also have

$$d(f^n(x), x^*) \leq \frac{sk^n}{1 - sk} d(x, f(x)), \ \forall n \in \mathbb{N}^*.$$

# Ran-Reurings theorem in b-metric space (II)

**Theorem.**

Let $X$ be a nonempty set endowed with a partial order "$\preceq$" and $d : X \times X \to X$ be a complete $b$-metric with constant $s \geq 1$.

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(ii) $x_0 \preceq f(x_0)$;

(iii) for every $x, y \in X$ there exists $z \in X$ which is comparable to $x$ and $y$.

Then $f$ is a Picard operator.

# Coupled fixed point problems

## The coupled fixed point problem

If $(X, d)$ is a metric space and $T : X \times X \to X$ is an operator, then, by definition, a coupled fixed point for $T$ is a pair $(x^*, y^*) \in X \times X$ satisfying

$$\begin{cases} x^* = T(x^*, y^*) \\ y^* = T(y^*, x^*). \end{cases} \quad (1)$$

We will denote by $CFix(T)$ the coupled fixed point set for $T$.

# Coupled fixed point theorems

**Theorem.**

Let $(X, \leq)$ be a partially ordered set and let $d : X \times X \to \mathbb{R}_+$ be a complete $b$-metric on $X$ with constant $s \geq 1$. Let $T : X \times X \to X$ be an operator with closed graph which has the mixed monotone property on $X \times X$. Assume that the following conditions are satisfied:

(i) there exists $k \in (0, \frac{1}{s})$ such that, $\forall x \leq u, y \geq v$, we have:

$$d(T(x,y), T(u,v)) + d(T(y,x), T(v,u)) \leq k[d(x,u) + d(y,v)];$$

(ii) there exist $x_0, y_0 \in X$ such that $x_0 \leq T(x_0, y_0)$ and $y_0 \geq T(y_0, x_0)$.

Then, the following conclusions hold:

(a) there exists $(x^*, y^*) \in X \times X$ a solution of the coupled fixed point problem (7) and the sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ in $X$ defined by

$$\begin{cases} x_{n+1} = T(x_n, y_n) := T^n(x_0, y_0) \\ y_{n+1} = T(y_n, x_n) := T^n(y_0, x_0), \end{cases} \tag{2}$$

have the property that $(x_n)_{n\in\mathbb{N}} \to x^*$, $(y_n)_{n\in\mathbb{N}} \to y^*$ as $n \to \infty$. Moreover, for every pair $(x, y) \in X \times X$ with $x \leq x_0$ and $y \geq y_0$ (or reversely), we have that $(T^n(x, y))_{n\in\mathbb{N}}$ converges to $x^*$ and $(T^n(y, x))_{n\in\mathbb{N}}$ converges to $y^*$.

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(b) In particular, if the $b$-metric $d$ is continuous, then:

$$d(T^n(x_0, y_0), x^*) + d(T^n(y_0, x_0), y^*) \le$$

$$\frac{sk^n}{1 - sk} \cdot [d(x_0, T(x_0, y_0)) + d(y_0, T(y_0, x_0))], \text{ for all } n \in \mathbb{N}^*.$$

# Proof.

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We also introduce the functional $\tilde{d} : Z \times Z \to \mathbb{R}_+$ defined by

$$\tilde{d}((x, y), (u, v)) := d(x, u) + d(y, v).$$

It is easy to see that $\tilde{d}$ is a $b$-metric on $Z$ with the same constant $s \geq 1$ and if the space $(X, d)$ is complete, then $(Z, \tilde{d})$ is complete too.

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$$\tilde{d}(F(z), F(w)) \leq k\tilde{d}(z, w), \text{ for all } z, w \in Z \text{ with } z \leq_P w.$$

Hence we can apply Ran-Reurings Theorem for b-metric spaces.

We consider now the operator $F : Z \to Z$ given by

$$F(x,y) := (T(x,y), T(y,x)).$$

As a consequence of our hypotheses and the construction of $F$, we have the following properties for $F$:

1) $F : Z \to Z$ has closed graph on $Z$;
2) $F : Z \to Z$ is increasing on $Z$ with respect to $\leq_P$;
3) there exists $z_0 := (x_0, y_0) \in Z$ such that $z_0 \leq_P F(z_0)$;
4) there exists $k \in (0, \frac{1}{s})$ such that

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Hence we can apply Ran-Reurings Theorem for b-metric spaces.

# Uniqueness of the fixed point

- under some additional assumptions

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# A more general case

If $(X, d), (Y, \rho)$ are two $b$-metric spaces and $T_1 : X \times Y \to X$, $T_2 : X \times Y \to Y$ are two single-valued operators, find $(x^*, y^*) \in X \times Y$ satisfying

$$\begin{cases} x^* = T_1(x^*, y^*) \\ y^* = T_2(x^*, y^*). \end{cases} \tag{3}$$

# The approach

## Definition

Let $(X, \leq_1)$ and $(Y, \leq_2)$ be a two partially ordered sets and $T_1 : X \times Y \to X$ and $T_2 : X \times Y \to Y$ be two mappings. We say that the operators $T_1$ and $T_2$ have the inverse mixed-monotone property if the following conditions hold:

(i) if $x_1, x_2 \in X$ with $x_1 \leq_1 x_2$ then

$$\begin{array}{l} T_1(x_1, y) \leq_1 T_1(x_2, y) \\ T_2(x_1, y) \ {}_2\geq T_2(x_2, y) \end{array}, \quad \forall\ y \in Y$$

(ii) if $y_1, y_2 \in Y$ with $y_1 \ {}_2\geq y_2$ then

$$\begin{array}{l} T_1(x, y_1) \leq_1 T_1(x, y_2) \\ T_2(x, y_1) \ {}_2\geq T_2(x, y_2) \end{array}, \quad \forall\ x \in X$$

# An existence result

**Theorem**

Let $(X, \leq_1)$ and $(Y, \leq_2)$ be a two partially ordered sets and suppose that we have two complete b-metrics $d : X \times X \to \mathbb{R}_+$ and $\rho : Y \times Y \to \mathbb{R}_+$ with the same constant $s \geq 1$.

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Let $T_1 : X \times Y \to X$ and $T_2 : X \times Y \to Y$ be two operators with closed graphic which have the inverse mixed-monotone property.

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Let $T_1 : X \times Y \to X$ and $T_2 : X \times Y \to Y$ be two operators with closed graphic which have the inverse mixed-monotone property.
Suppose that there exists a constant $k \in \left(0, \frac{1}{s}\right)$ such that for each $(x, y), (u, v) \in X \times Y$ with $x \leq_1 u$, $y \;_2\geq v$ we have:

$$d(T_1(x, y), T_1(u, v)) + \rho(T_2(x, y), T_2(u, v)) \leq k[d(x, u) + \rho(y, v)].$$

If there exists $(x_0, y_0) \in X \times Y$ such that

$$x_0 \leq_1 T_1(x_0, y_0), \; y_0 \; {}_2\!\geq T_2(x_0, y_0)$$

or

$$x_0 \; {}_1\!\geq T_1(x_0, y_0), \; y_0 \leq_2 T_2(x_0, y_0)$$

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then:
(a) there exists $(x^*, y^*) \in X \times Y$ such that the sequence $(x_n)_{n\in\mathbb{N}}$ in $X$ and $(y_n)_{n\in\mathbb{N}}$ in $Y$, defined by

$$x_{n+1} = T_1(x_n, y_n) \text{ and } y_{n+1} = T_2(x_n, y_n) \text{ for all } n \in \mathbb{N},$$

have the property $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$ and

$$\begin{cases} x^* = T_1(x^*, y^*) \\ y^* = T_2(x^*, y^*) \end{cases}$$

Moreover, for any $(x, y) \in X \times Y$ such that

$$(x \leq_1 x_0,\ \ y\ {}_2\!\geq y_0) \text{ or } (x\ {}_1\!\geq x_0,\ \ y \leq_2 y_0)$$

the sequences $u_{n+1} = T_1(T^n(x, y))$ and $v_{n+1} = T_2(T^n(x, y))$ where

$$T(x, y) = (T_1(x, y), T_2(x, y))$$

converges to $x^*$ and respectively to $y^*$.
(b) If, in addition, $d$ and $\rho$ are two continuous b-metrics, then we have:

$$d(x_n, x^*) + \rho(y_n, y^*) \leq \frac{sk^n}{1 - sk}[d(x_0, T_1(x_0, y_0)) + \rho(y_0, T_2(x_0, y_0))].$$

Moreover, for any $(x, y) \in X \times Y$ such that

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4) there is $k \in \left(0, \frac{1}{s}\right)$ such that
$\tilde{d}(T(z), T(w)) \leq k\tilde{d}(z,w)$ for all $z, w \in Z$ with $z \leq_p w$.

Then, by Ran-Reurins Theorem in $b$-metric space, we get that $T$ has at least one fixed point $z^* = (x^*, y^*) \in Z$ and, for any $z \in Z$ which is comparable with $z_0$, the sequence of successive approximation for $T$ starting from $z$ converges to $z^* = (x^*, y^*)$.

**Remark.** If, in addition to the hypotheses of above Theorem, we suppose that every pair of elements $X \times Y$ has a lower bound or an upper bound with respect to $\leq_p$, Then, by Ran-Reurins Theorem in $b$-metric space, we get that $T$ has at least one fixed point $z^* = (x^*, y^*) \in Z$ and, for any $z \in Z$ which is comparable with $z_0$, the sequence of successive approximation for $T$ starting from $z$ converges to $z^* = (x^*, y^*)$.

**Remark.** If, in addition to the hypotheses of above Theorem, we suppose that every pair of elements $X \times Y$ has a lower bound or an upper bound with respect to $\leq_p$,then the system of equations

$$x = T_1(x, y), \quad y = T_2(x, y) \text{ has a unique solution.}$$

## A global version of the previous theorem

Let $(X, d)$ and $(Y, \rho)$ be two complete b-metric spaces with constant $s \geq 1$. Let $T_1 : X \times Y \to X$ and $T_2 : X \times Y \to Y$ be two operators with closed graph. Suppose there is $k \in (0, 1)$ such that

$$d(T_1(x, y), T_1(u, v)) + \rho(T_2(x, y), T_2(u, v))$$

$$\leq k[d(x, u) + \rho(y, v)], \ \forall \ (x, y), (u, v) \in X \times Y.$$

Then, there exists a unique $(x^*, y^*) \in X \times Y$ with

$$x^* = T_1(x^*, y^*), \quad y^* = T_2(x^*, y^*)$$

such that the sequences $(x_n)_{n \in \mathbb{N}}$ in $X$ and $(y_n)_{n \in \mathbb{N}}$ in $Y$ defined by

$$x_{n+1} = T_1(x_n, y_n) \text{ and } y_{n+1} = T_2(x_n, y_n) \text{ for all } n \in \mathbb{N},$$

have the property $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$.

## An application

We will consider, for a given continuous function $f : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$, the following system:

$$\begin{cases} -x''(t) = f(t, x(t), y'(t)) \\ -y''(t) = f(s, y(t), x'(t)), \\ x(a) = x(b) = y(a) = y(b) = 0. \end{cases} \tag{4}$$

This problem is equivalent to

$$\begin{cases} x(t) = \int_a^b G(t,s) f(s, x(s), y'(s)) ds \\ y(t) = \int_a^b G(t,s) f(s, y(s), x'(s)) ds, \end{cases}$$

where $G(t,s) := \begin{cases} \frac{(s-a)(b-t)}{b-a}, & \text{if } s \leq t \\ \frac{(t-a)(b-s)}{b-a}, & \text{if } s \geq t. \end{cases}$

# Application (II)

We denote $X := \{x \in C^1[a,b] : x(a) = x(b) = 0\}$ and we consider on $X$ the following norms

$$\|x\|_C := \max_{t\in[a,b]} |x(t)| \text{ and } \|x\|_S := \max_{t\in[a,b]} |x'(t)|.$$

Then, both of them are Banach spaces.
Denote

$$T : X \times X \to X, \text{ by } T(x,y)(t) := \int_a^b G(t,s)f(s,x(s),y'(s))ds.$$

Then $T$ is well defined and our problem can be re-written as follows

$$\begin{cases} x = T(x,y) \\ y = T(y,x) \end{cases}$$

# Application (III)

Let us suppose that there exist $\alpha, \beta > 0$ such that, for each $u_1, v_1, u_2, v_2 \in \mathbb{R}$ and for all $s \in [a, b]$, we have

$$|f(s, u_1, v_1) - f(s, u_2, v_2)| \leq \alpha |u_1 - u_2| + \beta |v_1 - v_2| . \quad (5)$$

Moreover assume that

$$\max\{\alpha \left( \frac{(b-a)^2}{8} + \frac{b-a}{2} \right), \beta \left( \frac{(b-a)^2}{8} + \frac{b-a}{2} \right)\} < 1. \quad (6)$$

Then, we obtain:

$$\|T(x_1, y_1) - T(x_2, y_2)\|_C + \|T(x_1, y_1) - T(x_2, y_2)\|_S \leq$$
$$\max\{\alpha \left( \frac{(b-a)^2}{8} + \frac{b-a}{2} \right), \beta \left( \frac{(b-a)^2}{8} + \frac{b-a}{2} \right)\} \cdot$$
$$(\|x_1 - x_2\|_C + \|y_1 - y_2\|_S) .$$

## An existence, uniqueness and approximation theorem

Let us consider the problem (4), where $f : [a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous mapping. Assume that that the above conditions (5) and (6) hold. Then, problem (4) has a unique solution $(x^*, y^*) \in C^2[a, b] \times C^2[a, b]$ and the sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ given, for $n \in \mathbb{N}^*$, by

$$x_{n+1}(t) := \int_a^b G(t, s)f(s, x_n(s), y_n'(s))ds,$$

and

$$y_{n+1}(t) := \int_a^b G(t, s)f(s, y_n(s), x_n'(s))ds,$$

converges to $x^*$ and respectively to $y^*$ as $n \to \infty$, for any arbitrary elements $x_0, y_0 \in C^1[a, b]$.

# The coupled fixed point problem in the multivalued case

Let $(X, d)$ be a metric space and $P(X)$ be the family of all nonempty subsets of $X$.
If $G : X \times X \to P(X)$ is a multi-valued operator, then, by definition, a coupled fixed point for $G$ is a pair $(x^*, y^*) \in X \times X$ satisfying

$$\begin{cases} x^* \in G(x^*, y^*) \\ y^* \in G(y^*, x^*). \end{cases} \tag{7}$$

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