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The Fixed Point Property in c_0 by E.Llorens-Fuster and B. Sims: The beginning of the adventure

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Nonexpansive mappings

- Let K be a nonempty subset of a Banach space X. A mapping $T : K \to X$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in K$.
- It is said that K has the fixed point property (FPP in short) for nonexpansive mappings provided that every nonexpansive self-mapping defined on K has a fixed point.

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The earliest fixed point theorems for nonexpansive mappings

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The failure of the FPP in c_0

- Let B be the closed unit ball of c_0. Define $T : B \to B$ by $T(x_1, x_2, x_3, ...) = (1, x_1, x_2, ...)$.
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Maurey's result

- B. Maurey (1980) proved, by using a ultraproduct technique, that every convex weakly compact subset of c_0 satisfies the FPP for nonexpansive mappings.
- How sharp Maurey's result is? Is weak compactness necessary or is it possible to replace weak compactness by a weaker assumption?

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- How sharp Maurey's result is? Is weak compactness necessary or is it possible to replace weak compactness by a weaker assumption?

A sharp example

E. Llorens-Fuster, B. Sims 1998

Considerer the topology $\sigma(c_0, E)$ where E is a subspace of ℓ_1 of codimension 1. There exists a convex set K which is $\sigma(c_0, E)$-compact and fails the FPP.

$K = \overline{\mathrm{co}}\{u_n : n \geq 0\}$ where $u_0 = 0$ and (u_n) is the summing basis of c_0, $u_n = (1, 1, 1, ..., 1_{\frown n}, 0, 0)$

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$$T\left(\sum_{n=0}^{\infty} t_n u_n\right) = \sum_{n=0}^{\infty} t_n u_{n+1}$$

$$T(1-t_0, 1-t_0-t_1, \ldots) = (1, 1-t_0, 1-t_0-t_1, \ldots)$$

T is a fixed point free affine isometry.

Note that $Tx = x$ implies $t_n = 0$ for every $n \geq 0$ but $T(0) = (1, 0, 0, \ldots)$.

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Uniformly Lipschitz mappings

Theorem

(**TDB, M. Japón, S. Prus, 2002**) *For every convex bounded closed subset C of c_0 which is not weakly compact and for every positive ϵ, there exists a convex closed subset K of C and a mapping $T : K \to K$ which is $(1+\epsilon)$-uniformly Lipschitz (i.e. all iterates are $(1+\epsilon)$-Lipschitz) and fixed point free.*

A mapping is nonexpansive if and only if it is 1-uniformly Lipschitz !!!

The summing basis of c_0

Assume that $\{u_n\}$ is the summing basis of c_0. For any sequence $(t_n) \in \ell_1$ we have that

$$\sum_{n=1}^{\infty} t_n u_n = \left(\sum_{k=1}^{\infty} t_k, \sum_{k=2}^{\infty} t_k, \sum_{k=3}^{\infty} t_k, \dots \right)$$

which implies that

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Isometric copies of the summing basis of c_0

If we consider a basic sequence as:
$v_1 = (B_1, 0, 0, 0, ...)$
$v_2 = (B_1, B_2, 0, 0, ...)$
$v_3 = (B_1, B_2, B_3, 0, 0, ...)$
....
where B_i is a finite-dimensional vector with $\|B_i\|_\infty = 1$ we still obtain

$$\left\| \sum_{n=1}^{\infty} t_n v_n \right\| = \max_n \left| \sum_{k=n}^{\infty} t_k \right|$$

The basis (v_n) is said to be an isometric copy of the summing basis.

Almost isometric copies of the summing basis of c_0

If C is a closed convex bounded subset of c_0 which is not weakly compact, we can find a basic sequence $\{v_n\}$ in C which is very "similar" to the summing basis of c_0.

y

How to find this almost isometric copy

Assume that C is bcc and it is not weakly compact. By Eberlein-Smulian and Alaoglu Theorems there exists a sequence $\{x_n\}$ in C which converges to a point $x = (\xi_n) \in \ell_\infty \setminus c_0$ in the weak-star topology of ℓ_∞.

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Since $x \notin c_0$ we know that $\limsup_n \xi_n \neq 0$. By multiplication we can assume $\limsup_n \xi_n = 1$ and for an arbitrary $\epsilon > 0$ we can write $x = (A_1, A_2, A_3, \ldots)$ where $\|A_1\| \geq 1 - \epsilon$ and $|\|A_i\| - 1| < \epsilon$ for $i \geq 2$.

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Since x_n converges to x coordinatewise we can choose a first vector in the sequence x_n, denoted again x_1 such that $x_1 = (A_1', U_1, Z_1)$ where A_1' is close to A_1, Z_1 is close to 0 and U_1 is an "uncontrolled" finite-dimensional vector.

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In the same way we find a second vector, denoted again x_2 in the form
$x_2 = (A'1, A_2', U_2, Z_2)$
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The right shift $T : \overline{\mathrm{co}}\,\{y_n : n \in \mathbb{N}\} \to \overline{\mathrm{co}}\,\{y_n : n \in \mathbb{N}\}$ is not, in general nonexpansive, but it is still uniformly Lipschitz with Lipschitz constant close to 1.

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The solution in 2004

COMING SOON !!!

Theorem

Let C be a nonempty, closed, bounded, convex subset of c_0. Then C is weakly compact if and only if C has the fixed point property ; i.e., every nonexpansive mapping $T : C \to C$ has a fixed point.

The unbounded case

The previous result completely solves the problem when the closed convex set C is bounded. But, what happens in the unbounded case?

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For instance, assume that C is a line in c_0. Then, C is convex closed and non-weakly compact but it is clear that we cannot find a sequence in C convergent to a vector $x \in \ell_\infty \setminus c_0$ in the weak-star topology of ℓ_∞.

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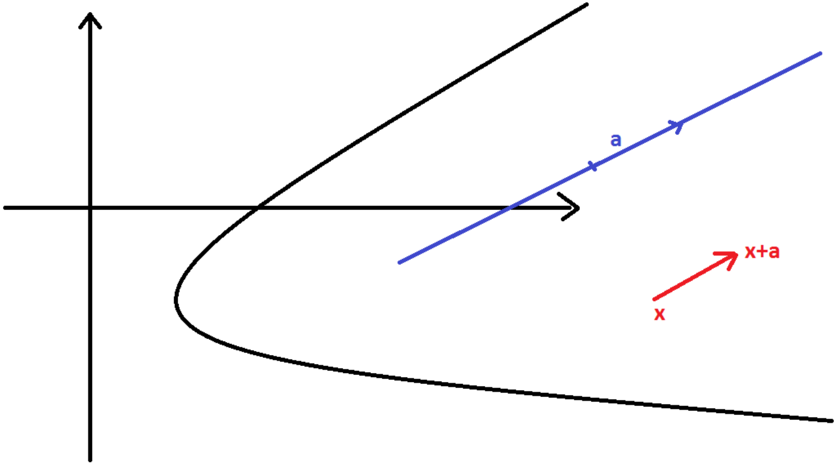
The Approximate Fixed Point Property

Definition

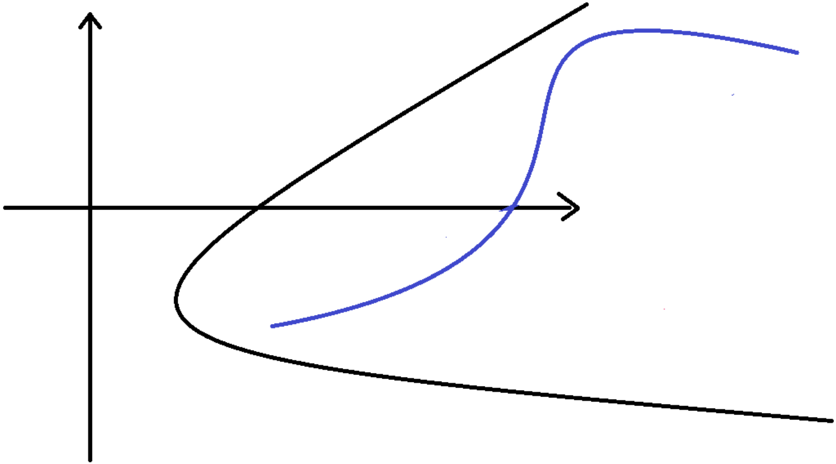
A subset C of a Banach space X is said to satisfy the AFPP if for every nonexpansive mapping $T : C \to C$ the minimal displacement is null. i. e. $\inf\{\|x - Tx\| : x \in C\} = 0$.

EVERY BOUNDED CONVEX SUBSET HAS THE AFPP !!!

Sets which are not linearly unbounded



Sets which are not directionally unbounded



Directionally bounded sets

Definition

(SHAFRIR 1990) Let X be a Banach space. A curve $\gamma : [0, \infty) \to X$ is said to be directional (with constant b) if there is $b \geq 0$ such that

$$t - s - b \leq \|\gamma(t) - \gamma(s)\| \leq t - s$$

for all $t \geq s \geq 0$. A convex subset is called directionally bounded if it contains no directional curve.

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(SHAFRIR 1990) Let C be a closed convex subset of a Banach space X. Then, C satisfies the AFPP if and only if C is directionally bounded.

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Split the proof in two cases: If C is not directionally bounded then C fails the AFPP.

Theorem

(TDB 2012) Assume that C is a closed convex unbounded subset of c_0 which is directionally bounded. Then, there exists a sequence in C which converges in the topology $\sigma(\ell_\infty, \ell_1)$ to a vector in $\ell_\infty \setminus c_0$.

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The standard basis of c_0 is 1-unconditional and shrinking

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The proof of this equivalence is strongly based on the existence of asymptotically isometric copies of the summing basis of c_0 in every convex closed bounded non-weakly compact set of c_0. However, there exist some equivalent norms in c_0 failing to have asymptotically isometric copies of c_0 for which the standard Schauder basis is 1-unconditional.

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Cascading nonexpansive mappings

Definition

Let X be a Banach space and C be a closed convex subset of X. Let $T : C \to C$ be a mapping and define $C_0 := C$, $C_n := \overline{\text{co}}\, T(C_{n-1})$ for every $n \in \mathbb{N}$. The mapping T is said to be cascading nonexpansive if there exists a sequence $(\lambda_n)_{n\geq 0} \subset [1, \infty)$ with $\lim_n \lambda_n = 1$ and such that $\|Tx - Ty\| \leq \lambda_n \|x - y\|$ for all $x, y \in C_n$ and for all $n \geq 0$.

Theorem

Let X be a Banach space with a shrinking 1-unconditional Schauder basis (e_n). Let C be a closed convex bounded subset of X. The following are equivalent: a) C is weakly compact. b) C has the FPP for cascading nonexpansive mappings.

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