Implicit and explicit iterations for quasi-nonexpansive mappings From Mann to Midpoint method

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A problem in Fixed Point Theory

Let X be a real Banach space, C a nonempty, closed and convex subset of X and $T : C \to C$ a nonlinear operator. Provided that T is not a strict contraction, i.e.

$$\|Tx - Ty\| \leq k \|x - y\|$$

for all $x, y \in C$ and some $k \in (0, 1)$.
Under which conditions on T, C and X, the sequence

$$\begin{cases} x_0 \in C \\ x_n = Tx_{n-1} = T^n x_0, \quad n = 1, 2, \dots \end{cases}$$

converges to a fixed point of T?

A new contractive condition

Many results in literature are concerning the convergence of iterative methods for the class of nonexpansive mappings. Appeared to be of interest to investigate the same problem, dealing with mappings $T : C \to C$ such that $Fix(T) = \{z \in C : Tz = z\} \neq \emptyset$ and

$$\|Tx - p\| \leq \|x - p\|, \quad \forall x \in C, \forall p \in Fix(T)$$

Such operators are known as Quasi-nonexpansive mappings. In general, a quasi-nonexpansive mapping is discontinuous.

A result for real-valued functions

Theorem (F. Tricomi, Lecture Series, Inst. Fluid Dynamics and Appl. Math., Univ. of Maryland, 1966)

Let A be a real-valued function on the (finite or infinite) interval $a < x < b$, whose values lie in the same interval; and such that A^k is continuous, for a certain positive integer k. Suppose

- *there exists a number p, with $a < p < b$, such that $A(p) = p$;*
- *$|A(x) - p| < |x - p|$ for $a < x < b$, $x \neq p$.*

  Then, for every $a < x < b$,

$$\lim_{m \to +\infty} A^m(x) = p,$$

where A^m denotes the m-th iterate of the function A.

Diaz and Metcalf, 1967

Let S be a nonempty metric space and $T : S \to S$ a continuous mapping, $Fix(T)$ the set of fixed points of T. The authors focused on the structure of the set

$$\mathcal{L}(x) = \{y \in S : y = \lim_{i \to +\infty} T^{n_i}(x)\}, \quad \forall x \in S$$

with $(T^{n_i}(x))_{i \in \mathbb{N}} \subset (T^n(x))_{n \in \mathbb{N}}$.
They proved that $\mathcal{L}(x)$ is a closed and connected subset of $Fix(T)$, under suitable conditions, including

$$d(Tx, Fix(T)) < d(x, Fix(T)),$$

$\forall x \in S \setminus Fix(T)$.

Theorem (*J. B. Diaz and F. T. Metcalf, Bull. Amer. Math. Soc., 1967*)

Let S a metric space and $T : S \to S$ be continuous. Suppose

1. $Fix(T) \neq \emptyset$,
2. *For each $x \in X \setminus Fix(T)$ and $p \in Fix(T)$, one has $d(Tx, p) < d(x, p)$.*

Let $x \in S$. Then, either $\mathcal{L}(x)$ is empty, or $\mathcal{L}(x)$ consists of exactly one point, in this case $\lim_{n\to\infty} T^n x$ exists and belongs to $Fix(T)$.

Remark

Assumed $\mathcal{L}(x)$ non empty, if condition (2) is strenghtned by

$$d(Tx, Ty) < d(x, y)$$

for all $x, y \in S$ with $x \neq y$, the last result reduces to a previous by Edelstein (J. London Math., 1962).

Further developments

Let X be a real Banach space, C a nonempty, closed and convex subset of X and $x_0 \in C$.

- **Mann** iterative process:

$$x_n = (1 - t_n)x_n + t_n Tx_n, \quad n = 1, 2, 3 \ldots$$

where $(t_n)_{n \in \mathbb{N}} \in (0, 1)$
Introduced for nonexpansive mappings by W. R. Mann (1953) and then generalized for quasi-nonexpansive mappings by W. G. Dotson (*Trans. Amer. Math. Soc., 1970*).

- **Krasnoselskij-Mann**:

$$x_n = {S_\lambda}^n x_0, \quad n = 1, 2, 3 \ldots$$

where $S_\lambda = \lambda I + (1-\lambda)T$, for $\lambda \in (0,1)$, and I denotes the identity map of C.
Studied by Schaefer (1957), Opial (1967) for nonexpansive operators and the extended to special classes of quasi-nonexpansive mappings by Williamson and Petryshyn (JMAA, 1973), even in more general Banach spaces.

Remark

Mann scheme only yelds weak convergence in general.
In this regard recall the example provided by A. Genel and J. Lindenstrauss for nonexpansive mappings (Israel J. Math., 1975).

- **Moudafi**'s method:

$$\begin{cases} x_0 \in C \\ x_{n+1} = \frac{\epsilon_n}{1+\epsilon_n} f(x_n) + \frac{1}{1+\epsilon_n} Tx_n \quad n = 0, 1, 2, \dots \end{cases}$$

where f is a θ-contraction for some $\theta \in [0, 1)$.
The generated sequence $(x_n)_{n \in \mathbb{N}}$ strongly converges to the unique solution of the variational inequality

$$\langle p - f(p), p - x \rangle \leq 0, \quad \forall x \in Fix(T).$$

It has been introduced by Moudafi (2000) and successively extended for quasi nonexpansive mappings by Maingé (Computers & Mathematics with Applications, 2010).

Numerical integration

Consider the initial value problem

$$\begin{cases} x'(t) = f(x(t)) \\ x(t_0) = x_0 \end{cases}$$

Midpoint rules are methods employed for numerically solving such problems.

Given the time interval $[t_0, T]$, the integration procedure consists in providing for each positive integer N:

- The stepsize $h = \frac{T - t_0}{N}$,
- The mesh of nodes $\{t_n = t_0 + nh\}_{n=0}^{N}$,
- Approximate values y_n of the solution $x(t)$ at t_n, for $n = 0, 1, \ldots, N$,
- The polygonal $Y_N(t)$, connecting each pair of consecutive points (t_n, y_n), (t_{n+1}, y_{n+1}), for $n = 0, 1, \ldots, N$.

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The set of approximed values $\{y_n\}_{n=0}^{N}$ is generated by one of the following finite difference schemes:

- **Implicit Midpoint Rule (IMR)**:

$$\begin{cases} y_0 = x_0 \\ y_{n+1} = y_n + hf\left(\frac{y_n + y_{n+1}}{2}\right), \quad n = 0, \cdots, N-1. \end{cases}$$

- **Explicit Midpoint Rule (EMR)**:

$$\begin{cases} y_0 = x_0 \\ \bar{y}_{n+1} = y_n + hf(y_n) \\ y_{n+1} = y_n + hf\left(\frac{y_n + \bar{y}_{n+1}}{2}\right), \quad n = 0, \cdots, N-1 \end{cases}$$

Under suitable assumptions, the functions sequence $(Y_N(t))_{N \in \mathbb{N}}$ converges to the exact solution of the initial value problem, as $N \to \infty$.

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Midpoint methods for nonexpansive mappings

The framework:

- H real Hilbert space,
- $T : H \to H$ nonexpansive

In 2014, Xu et al., based on the previous numerical IMR, proposed an iterative process, presenting formal analogy with IMR finite difference scheme:

$$\begin{cases} x_0 \in H \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T(\frac{x_n + x_{n+1}}{2}), \quad n \geq 0 \end{cases}$$

for which they obtained a weak convergence result.
It is known as *Implicit Midpoint Rule for nonexpansive mappings* (M. A. Alghamdi, H.K., Xu, N. Shahzad, Fixed Point Theory Appl., 2014).

Let H be defined as before and C a nonempty convex and closed subset of H; strong convergence results have been obtained for:

- Viscosity implicit midpoint rule (H.K., Xu, M. A. Alghamdi, N. Shahzad, Fixed Point Theory Appl., 2015):

$$\begin{cases} x_0 \in C \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T\left(\frac{x_n + x_{n+1}}{2}\right), \quad n \geq 0 \end{cases}$$

- Generalized viscosity implicit midpoint rule (Y. Ke, C. Ma, Fixed Point Theory Appl., 2015):

$$\begin{cases} x_0 \in C \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T(s_n x_n + (1-s_n) x_{n+1}), \ n \geq 0 \end{cases}$$

A midpoint method for quasi-nonexpansive mappings

Based on numerical **EMR**

$$\begin{cases} y_0 = x_0 \\ \bar{y}_{n+1} = y_n + hf(y_n) \\ y_{n+1} = y_n + hf\left(\frac{y_n+\bar{y}_{n+1}}{2}\right), \quad n = 0, \cdots, N-1 \end{cases}$$

we focus on the following purposes:

- Construction of an explicit iterative method through formal analogy to the above numerical procedure;
- Weaken the assumption on the mapping, considering the class of quasi-nonexpansive mappings;
- Determination of suitable assumptions in order to get strong convergence of the proposed algorithm to a fixed point of the mapping.

The proposed algorithm

Let T a quasi-nonexpansive self-mapping on C and $f : C \to C$ a θ-contraction for a certain $\theta \in [0, 1)$.
We call Generalized Viscosity Explicit Midpoint Rule the iterative method generating a sequence $(x_n)_{n\in\mathbb{N}}$ as follows

$$\begin{cases} x_0 \in C \\ \bar{x}_{n+1} = \beta_n x_n + (1-\beta_n) T x_n \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T(s_n x_n + (1-s_n)\bar{x}_{n+1}), \quad n \geq 0. \end{cases}$$

where $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}, (s_n)_{n\in\mathbb{N}}$ are sequences in $(0, 1)$.

Convergence result

Theorem

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. Let $T : C \to C$ be a quasi-nonexpansive mapping such that $Fix(T) \neq \emptyset$, with $I - T$ demiclosed at 0, and $f : C \to C$ be a contraction with coefficient $\theta \in [0, 1)$. Let $(\alpha_n)_{n\in\mathbb{N}}$, $(s_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$ be sequences in $(0, 1)$, satisfying the conditions

1. $\lim_{n\to\infty} \alpha_n = 0$,
2. $\sum_{n=0}^{\infty} \alpha_n = \infty$,
3. $\limsup_{n\to\infty} \beta_n(1 - \beta_n)(1 - s_n) > 0$.

Then, the sequence $(x_n)_{n\in\mathbb{N}}$, defined in (15), strongly converges to $p \in Fix(T)$, which is the unique solution in $Fix(T)$ of the variational inequality (VI)

$$\langle p - f(p), p - x\rangle \leq 0, \quad \forall x \in Fix(T).$$

Some preliminaries

Definition

A mapping T, with domain $D(T)$, is said to be demiclosed at y if whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in $D(T)$ such that $x_n \rightharpoonup x^* \in D(T)$ and Tx_n strongly converges to p, then $Tx* = p$.

Recall the following characterization for the metric projection P_C from a Hilbert space H onto a subset C of H:

Lemma (K. Goebel, W. A. Kirk, Cambridge Studies in Advanced Mathematics, 1990)

Let C be a closed, convex subset of a Hilbert space H. Given $x \in H$ and $y \in C$, then $y = P_C x$ if and only if there holds the inequality

$$\langle x - y, y - z\rangle \geq 0 \quad \forall z \in C$$

Lemma (H. K. Xu, J. London Math, 2002)

Let $(a_n)_{n\in\mathbb{N}}$ *be a sequence of non-negative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n \quad n \geq 0,$$

with

- $(\alpha_n)_{n\in\mathbb{N}} \in [0,1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- $\limsup_{n\to\infty} \sigma_n \leq 0$;
- $\gamma_n \geq 0$, $\sum_{n=1}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma (Maingé, Set-Valued Anal, 2008)

Let $(\gamma_n)_{n\in\mathbb{N}}$ be a sequence of real numbers such that there exists a subsequence $(\gamma_{n_i})_{i\in\mathbb{N}}$ of $(\gamma_n)_{n\in\mathbb{N}}$ such that

$$\gamma_{n_i} < \gamma_{n_{i+1}}, \quad \textit{for all } i \in \mathbb{N}.$$

Also consider the sequence of integers $(\tau(n))_{n\geq n_0}$ defined by

$$\tau(n) := \max\{k \leq n : \quad \gamma_k < \gamma_{k+1}\}.$$

Then $(\tau(n))_{n\in\mathbb{N}}$ is a nondecreasing sequence verifying

$$\lim_{n\to\infty} \tau(n) = \infty,$$

and, for all $n \geq n_0$, the following two estimates hold

$$\gamma_{\tau(n)} < \gamma_{\tau(n)+1}, \; \forall\, n \geq n_0;$$
$$\gamma_n < \gamma_{\tau(n)+1}, \; \forall\, n \geq n_0.$$

Proof lines

The real sequence $(\|x_n - p\|)_{n\in\mathbb{N}}$ is bounded for all $p \in Fix(T)$. Thus $(x_n)_{n\in\mathbb{N}}$ is bounded.
Two cases are distinguished

Case 1 The sequence $(\|x_n - p\|)_{n\in\mathbb{N}}$ in definitively nonincreasing.

Case 2 There exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that

$$\|x_{n_k} - p\| < \|x_{n_k+1} - p\| \text{ for all } k \in \mathbb{N}$$

In both assumptions, the following main results can be listed:

- $(x_n)_{n\in\mathbb{N}}$ is an approximate fixed point sequence for T, i. e. $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$,
- $\omega_w(x_n) \subset Fix(T)$,
- $\lim_{n\to\infty} \|x_n - p\| = 0$.

Some remarks

Rem.1 Conditions

- T is quasi-nonexpansive,
- $I - T$ is demiclosed at 0,

are not related.

(i) Let $H = \mathbb{R}$, $C = [0, +\infty)$ and T a mapping defined by

$$Tx = \begin{cases} \frac{2x}{x^2+1} & x \in (1, +\infty) \\ 0 & x \in [0, 1] \end{cases}$$

- $Fix(T) = \{0\}$,
- T is discontinuous and quasi-nonexpansive,
- $x_n = 1 + \frac{1}{n} \to 1 \notin Fix(T)$ and $|x_n - Tx_n| \to 0$, thus $I - T$ is not demiclosed at 0.

(ii) Let $H = \mathbb{R}$, $C = [0, 1]$ and $T : C \to H$ defined by $Tx = 1 - x^{\frac{3}{2}}$.

- $Fix(T) = \{p\}$,
- T is a continuous pseudo-contraction, therefore $I - T$ is demiclosed at 0,
- T is not quasi-nonexpansive since:
  - if $x = 0$, then $|Tx - p| \leq |x - p|$ implies $1 - p \leq p$, and hence $p \geq \frac{1}{2}$
  - if $x = 1$, then $|Tx - p| \leq |x - p|$ implies $p \leq 1 - p$, and hence $p \leq \frac{1}{2}$

Thus it must be $p = \frac{1}{2}$, that is a contradiction.

Rem.2 There exist mappings $T : C \to C$, with $Fix(T)$ nonempty, which are quasi-nonexpansive and such that $I - T$ is demiclosed at 0:

- Nonexpansive mappings,
- Nonspreading mappings,
- L-Hybrid mappings.

The convergence result holds for these classes of mappings.

Rem.3 No additional assumption has been formulated about the limit of the sequence $(s_n)_{n\in\mathbb{N}}$, hence it could be that $\lim_{n\to\infty} s_n = 0$

An example

Consider
$H = \mathbb{R}$,
$\alpha_n = \frac{1}{n}$, $\beta_n = \frac{n-1}{2n}$, $s_n = \frac{1}{n}$,
$f(x) = \frac{x}{2}$,

$$Sx = \begin{cases} 0 \quad if \quad x \in [-2,2] \\ P_{[-1,1]}(x) \quad if \quad x \in [-3,3] \setminus [-2,2], \end{cases}$$

where $P_{[-1,1]}$ is the metric projection of $\mathbb{R}$ onto $[-1,1]$.

- $Fix(S) = \{0\}$,
- S is discontinuous and quasi-nonexpansive,
- $I - S$ is demiclosed at 0.

Fixed $x_1 = 3$, the sequence $\{x_n\}$ is given by

$$\bar{x}_{n+1} = \frac{n-1}{2n} x_n + \frac{n+1}{2n} S x_n$$

$$x_{n+1} = \frac{x_n}{2n} + \left(\frac{n-1}{n}\right) S \left(\frac{x_n}{n} + \frac{n-1}{n} \bar{x}_{n+1}\right).$$

Explicitely:

$x_1 = 3$,
$x_2 = \frac{3}{2}$,
$x_3 = \frac{3}{8}$,
$\ldots$,
$x_{n+1} = \frac{3}{n! 2^n}$.

Thus

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{3}{n! 2^n} = 0.$$

Gracias a todos por su attención!