Hybrid Waveforms for Precessing Binary Black holes for LIGO Data Analysis

Jam Sadiq
11th Iberian Gravitational Waves Meeting
June 9-11, 2021

June 10th, 2021
Waveform divided into three parts

- **Inspiral**: BHs far apart, described by post-Newtonian (PN) theory.
- **Merger**: Highly relativistic, needs Numerical Relativity (NR).
- **Ringdown**: Single BH, described by perturbation theory or NR.

**Idea**: Match NR simulation to PN, before PN becomes inaccurate.
With the improvements in low frequency detector sensitivity, longer waveforms will be required.

Numerical relativity waveforms are accurate but mostly available for late inspiral merger and ring down phase.

Analytical approximate model waveforms provide good accuracy waveforms in the early inspiral regime.

Numerical relativity waveforms have been hybridized to analytical model waveforms.
Non-precessing binaries have been hybridized and used for parameter estimation as done in Lange et. al (2017) [3]

The precessing-binary parameter space has been sampled by only a relatively small number of numerical simulations.

We want to **hybridize precessing binaries** for LIGO data analysis for such events.

Hybridizing precessing waveforms is a complicated process, as shown below.
In precessing binaries the orbital precession strongly affects the gravitational waveforms by modulating both amplitude and phase.

$\ell = 2, \ m = \pm 2$ are not necessarily the dominant mode for the waveforms.
Complications of precession can be reduced by transforming the waveform into co-precessing frame.

Time dependent 3D rotation of the waveform to align the orbital angular momentum along z-direction (non-inertial).

In this co-precessing frame the waveform behaves like a non-precessing waveform.

$SXS:BBH:0058$
Co-precessing Frame

- Developed in O’Shaughnessy et al. (2011), Schmidt et al. (2011), Boyle et al. (2011) [4, 5, 6]
- The idea is to continually rotate (3D) the waveform. Two of the Euler angles obtained from principal eigenvector of the orientation-averaged tensor

$$\langle L_{(ab)} \rangle = \frac{1}{\sum_{lm} |h_{lm}|^2} \begin{bmatrix} I_0 + \text{Re}(I_2) & \text{Im}I_2 & \text{Re}I_1 \\ \text{Im}I_2 & I_0 - \text{Re}(I_2) & \text{Im}I_1 \\ \text{Re}I_1 & \text{Im}I_1 & I_{zz} \end{bmatrix}$$

where $I_0, I_1, I_2, I_{zz}$ are related to quantum mechanical angular momentum notations.
Co-precessing Frame Rotations

- Two of the Euler angles are related to the principal eigenvector \( \hat{V} \) of the orientation-averaged tensor \( \langle L_{(ab)} \rangle \), O'Shaughnessy et al. (2011)[4]

\[
\alpha = \cos^{-1}[\hat{v}_z]
\]

\[
\beta = \text{Arg}[\hat{v}_x + i\hat{v}_y] - \frac{\pi}{2}
\]

- The third Euler angle chosen to account for the phase evolution of the waveform. Boyle et al. (2011)[6]

\[
\gamma = -\int \dot{\alpha} \cos \beta \ dt
\]
Hybridization Procedure: Instantaneous Rotation

- We choose a time and the corresponding instantaneous rotations \((\alpha, \beta, \gamma)\) of two waveforms that place the waveforms in the co-precessing frame at that time. This minimize the non-quadrupole modes for a short period

\[
H_{lm}^{\text{rot}}(t) = \sum_{m'=-l}^{l} e^{im'\gamma + im\alpha} d^l_{mm'}(\beta) h_{lm}(t)
\]
Hybridization Procedure: Instantaneous Rotation

- We rotate the two waveforms at these fixed angles

\[ H^\text{rot}_{lm}(t) = \sum_{m' = -l}^{l} e^{im'\gamma + im\alpha} d^l_{mm'}(\beta) h_{lm}(t) \]
Hybridization Procedure: Time and Phase shifts

- We rotate the two waveforms at these fixed angles

$$H_{lm}^{\text{rot}}(t) = \sum_{m'=-l}^{l} e^{im'\gamma+im\alpha} d_{mm'}(\beta) h_{lm}(t)$$

- we compute phase and time shifts to align waveforms in a hybridizing interval

$$\Delta = \min_{t_0, \phi_0, \Psi} \int_{t_1}^{t_2} \sum_{l,m} \left| H_{lm}^{\text{NR}}(t) - H_{lm}^{\text{PN}}(t - t_0) e^{i(m\phi_0 + 2\Psi)} \right| dt$$
Hybridization Procedure: Hybridization

- We rotate the two waveforms at these fixed angles

\[ H_{lm}^{\text{rot}}(t) = \sum_{m'=-l}^{l} e^{im'\gamma + im\alpha} d_{mm'}^{l} (\beta) h_{lm}(t) \]

- we compute phase and time shifts to align waveforms in a hybridizing interval

\[ \Delta = \min_{t_0,\phi_0,\Psi} \int_{t_1}^{t_2} \sum_{l,m} |H_{lm}^{\text{NR}}(t) - H_{lm}^{\text{PN}}(t-t_0)e^{i(m\phi_0 + 2\Psi)}| \, dt \]

- Construct the Hybrid

\[ h_{lm}^{\text{hyb}} = \tau(t) H_{lm}^{\text{NR}}(t) + [1 - \tau(t)] H_{lm}^{\text{PN}}(t - t'_0)e^{i(m\phi'_0 + 2\Psi')} \]

where \( \tau(t) \) is function that smoothly goes from 0 to 1 in hybrid interval
Complete Hybridization

- $\chi_1 = (0.239, 0.318, 0.244)$, $\chi_2 = (-0.361, 0.039, 0.289)$,
- $M_{tot} = 70M_\odot$ [7, 8, 9]

(SXS:BBH:1410) $q = 4$, Precessing

$h_{lm}(t) \times 10^{-21}$

$\tau[sec]$
Hybridization of Precessing Binary Waveform

- Precessing case **SXS:BBH:1410**, \( q = 4 \), \( \chi_1 = (0.239, 0.318, 0.244) \), \( \chi_2 = (-0.361, 0.039, 0.289) \), \( M_{tot} = 70 M_\odot \)
Quality of Hybrid Waveforms

- Compute mode by mode mismatch using advanced LIGO power spectral density

\[
\langle h_1 | h_2 \rangle = 2 \int_{-\infty}^{\infty} \frac{h_1^*(f) h_2(f)}{S_n(f)} \, df
\]

\[
\mathcal{O} = \frac{\langle h_1 | h_2 \rangle}{\sqrt{\langle h_1 | h_1 \rangle \langle h_2 | h_2 \rangle}}
\]

\[
\mathcal{M} = 1 - \mathcal{O}
\]

- Biased characterization at SNR 10: \( \mathcal{M} > 0.5\% \)
• Mode weighted Mismatch

$$W[\mathcal{M}] = \frac{\sum_{lm} \rho_{lm}^2 \mathcal{M}_{lm}}{\sum_{lm} \rho_{lm}^2}$$

where $\mathcal{M}_{lm}$ are the mode-by-mode, time-and-phase-maximized mismatches and $\rho_{lm}^2 = h_{lm} h_{lm}$. 
Precessing case SXS:BBH:1410 [7, 8]

PN-NR hybrid versus Full-NR

Longer NR waveform in hybrid is better
Hybrid Quality: Short-NR versus PN-NR hybrid

- Precessing case **SXS:BBH:1410**
- PN-NR hybrid versus Full-NR and short-NR versus Full-NR
- PN-NR hybrid are better than shorter NR waveforms
Hybrid Quality: The Length of NR Waveform

- Precessing case SXS:BBH:1410
- Longer NR waveforms are better than shorter ones

(SXS:BBH:1410) NRvsTruncatedNR (40 cycles)

(SXS:BBH:1410) NRvsTruncatedNR (20 cycles)

- $M_{tot}$ vs. $N_{cycle}$ for different modes ($l$, $m$) and with and without waveforms (WM).
Hybrid Quality: Hybrid versus Low Resolution NR Waveform

- Precessing case **SXS:BBH:1410**
- PN-NR hybrid and Low-High resolution NR hybrid versus Full-NR
- Low resolution NR in hybridization perform better than PN

(SXS:BBH:1410) Hybridize 40 cycles before merger

(SXS:BBH:1410) Mismatch Res3 vs Res2

\[ M_{tot} \]

\[ M \]

\[ l = 2, m = 1 \]
\[ l = 2, m = 2 \]
\[ l = 3, m = 1 \]
\[ l = 3, m = 2 \]
\[ l = 3, m = 3 \]
\[ l = 4, m = 1 \]
\[ l = 4, m = 2 \]
\[ l = 4, m = 3 \]
\[ l = 4, m = 4 \]

WM
In general higher order PN approximant is better. We found phase $\phi$ order with 3 PN order terms works better than available higher orders in phase with the maximum available amplitude $\alpha$ order.
Mismatch as Function of Angles

\[ h(t) = \sum_{lm} h_{lm}^{-2} Y_{lm}(\theta, \phi), \quad M_{\text{tot}} = 40 M_\odot \]

Higher order modes are important
We introduce an automated method to hybridize precessing binary waveforms.

We tested the accuracy of our hybridization procedure using mismatch.

We tested effects of the length of the numerical waveforms, hybrid versus different numerical resolutions as well as post-Newtonian errors on the hybrid.

All presented work is published in [10]

**Future:**
- Testing length of hybrid interval and how it affects the mismatch.
- Analyzing other hybridization errors (truncation and numerical) for many different waveform models.
Thank You
M. Coleman Miller and Nicolás Yunes.  
The new frontier of gravitational waves.  

Frank Ohme.  

J. Lange et al.  
Parameter estimation method that directly compares gravitational wave observations to numerical relativity.  

Efficient asymptotic frame selection for binary black hole spacetimes using asymptotic radiation.  
Patricia Schmidt, Mark Hannam, and Sascha Husa.  
Towards models of gravitational waveforms from generic binaries: A simple approximate mapping between precessing and non-precessing inspiral signals.  

Michael Boyle, Robert Owen, and Harald P. Pfeiffer.  
A geometric approach to the precession of compact binaries.  

SXS Collaboration.  

P. Ajith.  
Addressing the spin question in gravitational-wave searches: Waveform templates for inspiralling compact binaries with nonprecessing spins.  

Luc Blanchet.  
Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries.  
Co-precessing Frame

\[ I_2 \equiv \frac{1}{2} (h|L_+L_+|h) \]
\[ = \frac{1}{2} \sum_{lm} c_{lm} c_{l,m+1} h_{l,m+2}^* h_{lm} \]

\[ I_1 \equiv (h|L_+(L_z + 1/2)|h) \]
\[ = \sum_{lm} c_{lm} (m + 1/2) h_{l,m+1}^* h_{lm} \]

\[ I_0 \equiv \frac{1}{2} (h|L^2 - L_z^2|h) \]
\[ = \frac{1}{2} \sum_{lm} [l(l+1) - m^2]|h_{lm}|^2 \]

\[ I_{zz} \equiv (h|L_z L_z|h) = \sum_{lm} m^2 |h_{lm}|^2 \]

where \( c_{lm} = \sqrt{l(l+1) - m(m+1)}. \)
Wigner Matrix

\[ d_{m'm}^l(\beta) \] given by

\[
d_{m'm}^l(\beta) = \sqrt{(l + m)! (l - m)! (l + m')! (l - m')!} \times \sum_k (-1)^{k+m'-m} \frac{k! (l + m - k)! (l - m' - k)! (m' - m + k)!}{(2k+m'-m)(2l-2k-m'+m)}.
\]

(1)
Hybrid Quality: IMRPhenomXHM-NR hybrid versus PN-NR hybrid

- Precessing case **SXS:BBH:1410**
- PN-NR hybrid versus Full-NR and IMR-NR hybrid versus Full-NR
- PN-NR hybrid are better than IMR-NR hybrid