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Zone plates with cells apodized by Legendre profiles

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By apodizing the cells of a zone plate and changing the opening ratio, it is possible to shape the relative power spectrum of its foci. We describe a novel procedure that leads to an analytical formula for shaping the focus power spectrum by using apodizers expressible as the Legendre series; these act on cells of arbitrary opening ratio. Our general result is used to design zone plates that have missing foci and to discuss a synthesis procedure using apodizers with various opening ratios. Our applications can also be used for shaping the power spectrum of 1-D gratings.

I. Introduction

Multiple images of an input picture are used in microelectronics for mask generation, producing new patterns in the textile industry and automatic recognition by pyramidal image processing.[1] Multiple in-plane impulse responses are generated by gratings[2,3]; while multiple on-axis impulse responses are created by a Fabry-Perot interferometer[4] or using zone plates.[5,6]

For some applications, it is convenient that the multiple impulse responses have prespecified characteristics. For example, in microelectronics it is useful for the multiple responses to have high focal depth. The impulse response is usually tailored in instrumental spectroscopy and in imaging systems by apodization.[7] Recently, some efforts have been addressed to increasing the focal depth or reducing the influence of spherical aberration by apodization.[8–12] However, except for a few examples,[13,14] it seems that apodization has not been applied to shape multiple impulse responses along the optical axis.

The aim of this paper is to report a novel procedure that gives an analytical formula for evaluating the relative peak energy of the multiple responses of zone plates with an arbitrary opening ratio. The elementary cells are expressible as a series of Legendre polynomials. This approach is illustrated by designing zone plates that do not produce certain foci.

In Sec. II, we use Bauer's formula[15] for discussing the relative irradiances of an in-plane multiple impulse response. In Sec. III, we extend the discussion to on-axis multiple impulse responses, using the quasiperiodic approach of Lohmann and Paris.[16] In Sec. IV, we define novel profiles for achieving zone plates with missing foci. In Sec. V, we describe a profile synthesis procedure that uses apodizers with different opening ratios.

II. In-Plane Multiple Impulse Responses

We start by considering a grating with period d. In each period, the grating transmittance is assumed to be zero for all the points inside a band whose width is $(1 - s)d$. The parameter s, where $0 < s \leq 1$, is called here the opening ratio of the grating. The complex amplitude transmittance of a grating with opening ratio s can be written in terms of a Fourier series as

$$F(x,s) = \sum_{m=-\infty}^{\infty} C_m(s) \exp(i2\pi xm/d), \tag{1}$$

where

$$C_m(s) = (1/d) \int_{-sd/2}^{sd/2} F(x,s) \exp(-i2\pi xm/d) dx. \tag{2}$$

Next we recognize that the kernel in the integral transform in Eq. (2) can be written using Bauer's formula. Since

$$\exp(-ixy) = \sum_{n=0}^{\infty} (-i)^n (2n+1) P_n(x) j_n(y), \tag{3}$$

we find that

$$\exp(-i2\pi xm/d) = \sum_{n=0}^{\infty} (-i)^n (2n+1) P_n(2x/sd) j_n(m\pi s). \tag{4}$$

In Eqs. (3) and (4), P_n denotes the n-order Legendre polynomial, and j_n represents the spherical Bessel function of the n-order, also known as the Bessel function of the fractional order.

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Now, by substituting Eq. (4) in Eq. (2), we obtain

$$C_m(s) = \sum_{n=0}^{\infty} (-i)^n(2n+1)\left[(1/d)\int_{-sd/2}^{sd/2} F(x,s)P_n(2x/sd)dx\right] j_n(m\pi s), \tag{5}$$

which, using the change of variable, can be rewritten

$$t = 2x/sd, \qquad G(t) = F(x,s), \tag{6}$$

as

$$C_m(s) = (s/2)\sum_{n=0}^{\infty} (-i)^n(2n+1)j_n(m\pi s)\left[\int_{-1}^{1} G(t)P_n(t)dt\right]. \tag{7}$$

Note that the change of variable in Eq. (6) makes the integral in Eq. (7) independent of the opening ratio s. Furthermore, Eq. (6) indicates that from a given apodizing function $G(t)$, it is possible to generate a whole family of apodized gratings, $F(x,s)$, which have the same transmittance profile with a different opening ratio, as is shown in Fig. 1. We point out the interesting fact that any pair of functions belonging to the same family is related by a scale transformation, namely,

$$F(x,s_1) = F(s_2x/s_1,s_2). \tag{8}$$

Since we are interested in apodization profiles that are expressible as a series of Legendre polynomials,

$$G(t) = \sum_{q=0}^{\infty} a_qP_q(t). \tag{9}$$

By substituting Eq. (9) into Eq. (7) and taking into account the orthogonal property of the Legendre polynomials, we obtain

$$C_m(s) = s\sum_{q=0}^{\infty} (-i)^q a_q j_q(\pi ms). \tag{10}$$

Note that as a particular case Eq. (10) contains the square groove grating, which is characterized by $a_0 = 1$ and $a_q = 0$ for $q \neq 0$. Then Eq. (10) becomes

$$C_m(s) = sj_0(\pi ms) = s\sin(\pi ms)/(\pi ms). \tag{11}$$

From Eq. (10) it is now possible to evaluate the relative power spectrum of in-plane multiple impulse responses, $|C_m(s)|^2$, for variable opening ratio s and for any apodizing function which can be expressed as a series of Legendre polynomials. This treatment is extended next to zone plates.

III. On-Axial Multiple Impulse Responses

The complex amplitude transmittance of any zone plate, with opening ratio s, can be written as

$$H(r^2,s) = \sum_{m=-\infty}^{\infty} h_m(s)\exp[i2\pi m(r/R)^2], \tag{12}$$

where the radial coordinate is r, the period of the zone plate in r^2 is R^2, and

$$h_m(s) = (1/R^2)\int_0^{sR^2} H(r^2,s)\exp[-i2\pi m(r/R)^2]d(r^2). \tag{13}$$

Next, it is convenient to make the following change of variables:

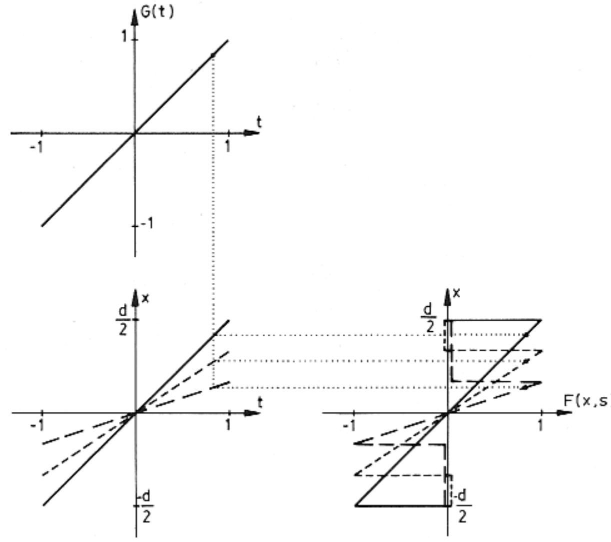

Fig. 1. Generation of apodizing functions with the same transmittance profile but different opening ratio.

$$x = r^2 - sR^2/2, \quad J(x,s) = H(r^2,s), \quad d = R^2. \tag{14}$$

By substituting Eq. (14) into Eq. (13) we obtain

$$h_m(s) = (1/d)\int_{-sd/2}^{sd/2} J(x,s)\exp(-i2\pi mx/d)dx, \tag{15}$$

which is recognized as Eq. (2). Consequently, by using the procedure discussed in Sec. II, we find that

$$h_m(s) = s\sum_{q=0}^{\infty} (-i)^q a_q j_q(\pi ms). \tag{16}$$

When using this formulation, it is important to remember that, from Eqs. (6), (9), and (14),

$$H(r^2) = J(x = r^2 - sR^2/2) = \sum_{q=0}^{\infty} a_qP_q(t = 2r^2/sR^2 - 1). \tag{17}$$

The above formulation is illustrated next with some examples.

IV. Zone Plates with Missing Foci

We consider first the trivial case of rectangular cells. Next, we discuss apodization by the first-order Legendre polynomial, and later we propose a compound Legendre apodizer.

A. Zero-Order Legendre Ruling

As we indicate in Eq. (11), for this example we find that

$$a_q = \delta_{0q}, \quad G(t) = P_0(t) = 1, \tag{18}$$

and consequently

$$h_m(s) = sj_0(\pi ms) = \sin(\pi ms)/\pi m. \tag{19}$$

In Eq. (18) δ_{0q} denotes Kronecker's delta. The result in Eq. (19) is the well-known formula for rectangular profiles. This formula predicts that for an opening ratio of one half, $s = 0.5$, the even orders vanish. This type of zone plate (or grating) is known in the optics literature as a Fresnel-Soret plate (or Ronchi ruling). The above results are shown graphically in Fig. 2.

B. First-Order Legendre Ruling

In this case the only coefficient different from zero is a_1, that is, $a_q = \delta_{1q}$ or equivalently $G(t) = P_1(t) = t$; hence

$$h_m(s) = (-is)j_1(\pi ms) = (-is)[\sin(\pi ms)/(\pi ms)^2 - \cos(\pi ms)/(\pi ms)]. \quad (20)$$

As can be seen in Fig. 3, there are some values of s for which the coefficients h_m, in Eq. (20), are eliminated. For example, the value of s can be chosen to satisfy the following roots of j_1:

$$\pi ms = \pi(1.43) \quad \text{or} \quad \pi ms = \pi(2.47). \quad (21)$$

Note from Fig. 3 that the first on-axis diffraction order cannot be canceled by using an apodizer proportional to the first-order Legendre polynomial. However, by using the first root in Eq. (21), the second focus can be canceled by setting $s = 0.71$. The third focus vanishes for two different values of s. The fourth focus can disappear for $s = 0.36$ and so on. Any interested reader can use the above procedure for eliminating certain foci with suitable values of s.

C. Compound Legendre Ruling

The same procedure holds for other Legendre rulings. In Fig. 4 we show the amplitude transmittance profile obtained combining the zero-order Legendre polynomial and the second-order Legendre polynomial:

$$G(t) = (2/3)P_0(t) - (2/3)P_2(t) = 1 - t^2. \quad (22)$$

The focal power spectrum vs the opening ratio of this kind of apodizer is

$$|h_m(s)|^2 = 4s^2[\sin(\pi ms)/(\pi ms)^3 - \cos(\pi ms)/(\pi ms)^2]^2, \quad (23)$$

as shown in Fig. 5.

Instead of discussing other Legendre apodizers, we next outline a synthesis procedure which considers the possibility of adding functions with the same profile but with a different opening ratio.

V. Synthesis Procedure: Various Opening Ratios

We show now that it is possible to synthesize coefficients $h_m(s)$ by the weighted sum, and/or difference, of coefficients $h_m(s_k s)$, where $k = 1,2,3,\ldots$ These later coefficients are obtained from individual functions $J(x,s_k)$, which are members of the set of apodizers generated with the same apodizing function $G(t)$.

In other words, we can start with a certain generating apodizer $G(t)$. Then one can obtain any member of the family of apodizers $J(x,s)$ having the same profile but different opening ratio, as expressed in Eq. (6), namely,

$$x = (sd/2)t, \quad J(x,s) = G(t). \quad (24)$$

It is valid to consider an apodizing profile that is the

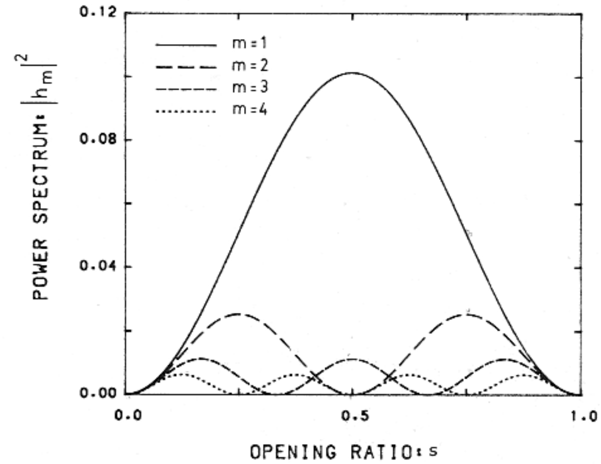

Fig. 2. Traditional method of focus elimination by changing the opening ratio of rectangular cells.

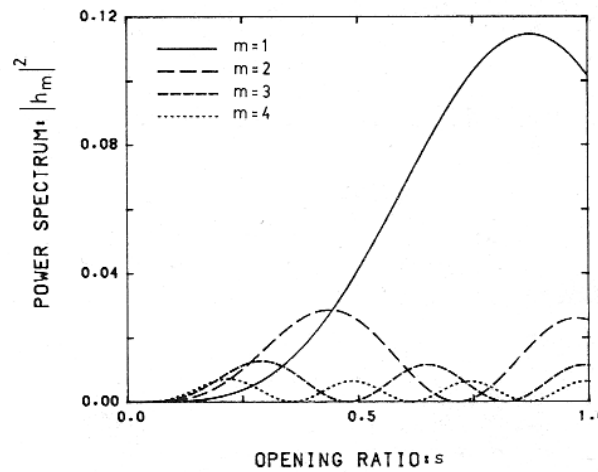

Fig. 3. Focus elimination by changing the opening ratio of cells apodized with the first-order Legendre polynomial, as in Fig. 1.

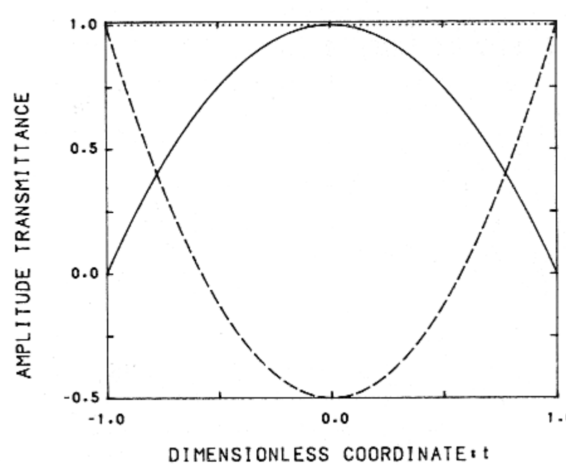

Fig. 4. Amplitude transmittance: (a) dotted line, the zero-order Legendre polynomial; (b) dashed line, the second-order Legendre polynomial; (c) solid line, the combination of (a) and (b) as in Eq. (22).

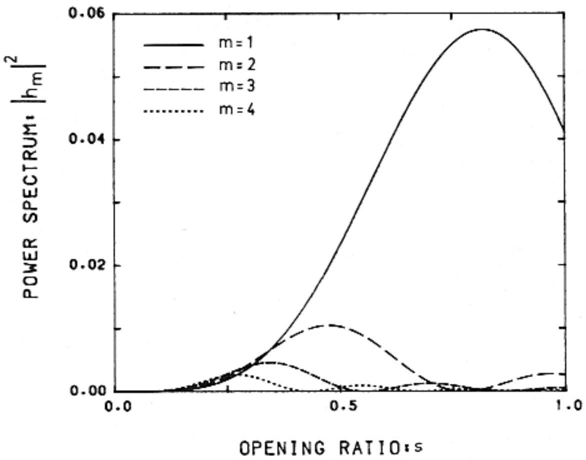

Fig. 5. Focal power spectrum of the apodized zone plate in Fig. 4.

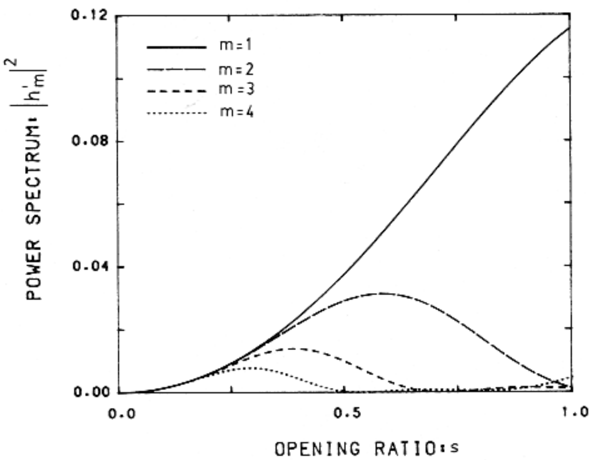

Fig. 7. Focal power spectrum of pyramidlike apodized zone plate in Fig. 6.

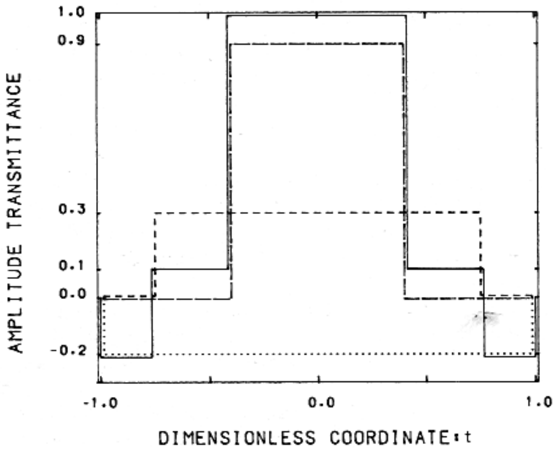

Fig. 6. Pyramidlike apodizer (solid line) generated by adding with various weights three rectangular functions (discontinuous lines) of different opening ratio.

weighted sum of several apodizers of equal profile but different opening ratio, that is,

$$f(x) = \sum_{k=1}^{K} e_k J(x,s_k). \tag{25}$$

We consider now that the resultant profile, $f(x)$ in Eq. (25), can be thought of as a new generating apodizer $G'(t)$, namely,

$$G'(t) = f[(d/2)t] = \sum_{k=1}^{K} e_k J[(d/2)t,s_k]. \tag{26}$$

As in Eq. (24), we can generate a new family of apodizers, $J'(x,s)$, that have the same profile as the generating function $G'(t)$ but with a variable opening ratio. In this case the formula equivalent to that in Eq. (24) is

$$x = (sd/2)t, \quad J'(x,s) = \sum_{k=1}^{K} e_k J(x/s,s_k)$$
$$= \sum_{k=1}^{K} e_k J(x,s_k s), \tag{27}$$

where, taking into account Eq. (8), $J(x,s_k s) = J(s_k x/s_k s,s_k) = J(x/s,s_k)$.

As the formation of multiple impulse responses is a linear process in complex amplitude, coefficients $h'_m(s)$ for the new apodizers $J'(x,s)$ are

$$h'_m(s) = \sum_{k=1}^{K} e_k h_m(s_k s). \tag{28}$$

The result in Eq. (28) is remarkable, since it allows one to calculate the coefficients h'_m for the variable opening ratio of a synthesized apodizer composed of a series of apodizers with the same profile and variable opening ratio. Next we consider some examples.

In Fig. 6 we show the synthesis of a discontinuous function, obtained by properly adding and subtracting rectangular functions (zero-order Legendre polynomials) with a different opening ratio. From the generating apodizer, we have apodizers with a different opening ratio and the same profile; consequently, one is able to shape the coefficients $h'_m(s)$ as shown in Fig. 7.

The same procedure applies for synthesizing continuous profiles, as shown in Fig. 8, where we display a piecewise continuous apodizer that results from adding and subtracting the apodizer of Fig. 4 with a different opening ratio. The coefficients $|h'_m|^2$, for $m = 1, 2, 3$, and 4, vs the opening ratio are displayed in Fig. 9.

The two examples indicate how to shape the focus power spectrum by using novel apodizing profiles obtained from the same Legendre ruling with a different opening ratio.

VI. Conclusions

We describe a novel approach for evaluating analytically the relative power spectrum of the multiple impulse responses, which are generated by gratings or zone plates, if the cells of these diffraction elements are apodized, by functions expressible as a Legendre series. Our formula considers explicitly the opening ratio of the cell, and it allows us to synthesize the apodizing function by using Legendre polynomials of

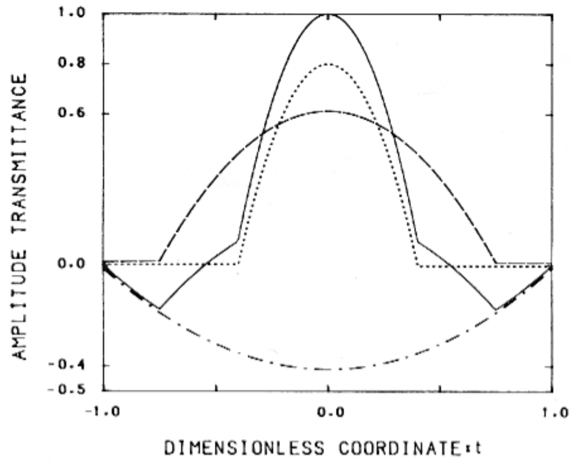

Fig. 8. Continuous apodizing function (solid line) generated by adding with different weights three functions (discontinuous lines) like that of Fig. 4 but with a different opening ratio.

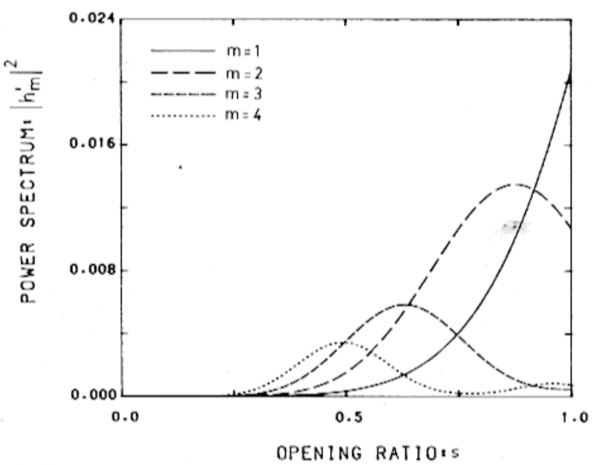

Fig. 9. Focal power spectrum of the continuous apodizer in Fig. 8.

any degree and any opening ratio. We illustrate our formula by designing apodizers, called Legendre rulings, that eliminate certain on-axis diffraction orders, and in this way we obtain zone plates that exhibit missing foci. Finally, we propose a synthesis procedure for designing apodizers by adding the same transmittance profile with a different opening ratio.

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