Variable fractional Fourier processor: a simple implementation

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A new set of optical implementations of the fractional Fourier transform (FRT) is developed by use of Wigner matrix algebra. The reinterpretation of some elementary operations that synthesize a rotation in the phase-space domain allows us to propose a lensless setup for obtaining the FRT. This compact configuration is also very flexible, because the fractional degree of the transformation can be varied continuously by shifting the input and the output planes along the optical axis by proper amounts. The above results permit one to build an optical FRT processor formed by two FRT systems in cascade, with a spatial filter between them. We present the design of such a variable FRT processor, which contains only one lens. © 1997 Optical Society of America [S0740-3232(97)00304-9]

1. INTRODUCTION

The introduction of the fractional Fourier transform (FRT) in the field of optics in 1993[1] has engaged the interest of many researchers. Many of the potential applications, among which filtering and correlation stand out because of the space variance of the FRT, are now emerging.[2–4]

The bulk-optical implementation of the FRT of an input function was first developed in Ref. 5. The proposed procedure was based on the Wigner distribution function associated with the input. Since then other optical implementations have been proposed.[6–8] In Ref. 5 Lohmann also gives an expression for the optical FRT of a one-dimensional function $u_0(x_0)$, namely,

$$\begin{aligned} u_P(x) &= F^P[u_0(x_0)] \\ &= C\int u_0(x_0)\exp\{i\pi/[\lambda f_1 \tan(P\pi/2)](x^2+x_0{}^2)\} \\ &\quad\times \exp\{-2\pi i/[\lambda f_1 \sin(P\pi/2)]xx_0\}\mathrm{d}x_0, \end{aligned} \tag{1}$$

where P is the fractional degree of the FRT, λ is the wavelength of the light, f_1 is a constant focal distance, and C satisfies the equation

$$1/C^2 = \lambda f_1|\sin(P\pi/2)|. \tag{2}$$

The optical configurations proposed in Ref. 5 provide the FRT with a fixed fractional degree that cannot be varied unless ordinary lenses are replaced by "fake zoom lenses."[9]

Our aim in this paper is twofold. First, we present new bulk-optics implementations of the FRT. The design strategy relates the FRT operation to a rotation in the phase space. The new synthesis procedures are obtained with use of Wigner matrix algebra with the three matrices F (free space), L (lens), and M (magnification) as elements.

Second, on the basis of the matrix representation we develop a lensless setup to obtain optically a FRT of variable degree. By lensless we mean no lens anywhere between input and output, but possibly one or two lenses between the laser and the input. This setup allows us to propose, in a second step, two FRT processors for tunable filtering in the fractional domain. The variation of the fractional degree at the filtering and the output planes is performed simply by shifting the different elements of the processor along the optical axis.

In Section 2 we present the new complete set of triplets of matrices F, L, and M. We summarize the values of the different matrix parameters for obtaining the FRT of any arbitrary degree. The reinterpretation of certain configurations is the key for developing, in Section 3, a lensless FRT transformer. Finally, in Section 4 two variable FRT processors are proposed. The first one is a lensless device, which is affected by a scale error at the output plane. In the second one, this error is compensated by introducing a single positive lens.

2. NEW COMPLETE TRIPLETS

As was shown in Ref. 5, the FRT has a simple interpretation in the framework of Wigner space. A rotation of the Wigner distribution function associated with the input function by an angle ϕ is related to the fractional degree P of the input by the relationship $\phi = P\pi/2$. This means that the FRT is periodic in P, with period 4. Besides, since $u_{P+2}(x) = u_P(-x)$, complete information about the FRT of any order can be obtained by considering values of P ranging in the interval [0, 2]. A rotation in the Wigner domain can be synthesized by use of three shearing steps.

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Optically, each one corresponds either to a free propagation over a distance Rf_1 or to a passage through a lens with focal power Q/f_1. For a one-dimensional input signal, the above operations can be represented by a matrix F or L, namely,

$$F \equiv \begin{bmatrix} 1 & -R \\ 0 & 1 \end{bmatrix}, \qquad L \equiv \begin{bmatrix} 1 & 0 \\ Q & 1 \end{bmatrix}. \tag{3}$$

These matrices describe how the location vector (x, ν) in the Wigner space will change as a consequence of free-space propagation or transition through a lens, respectively. The ν stands for spatial frequency. Two complete triplets that can synthesize a rotation in a Wigner domain are FLF and LFL, which correspond to Lohmann's systems I and II, respectively, proposed in Ref. 5.

Here we recognize that other optical setups that use a telescopic system, which in Wigner space is represented by the magnification matrix

$$M \equiv \begin{bmatrix} 1/M & 0 \\ 0 & M \end{bmatrix}, \tag{4}$$

can also be used to obtain the FRT. In other words, all the complete triplets involving L, F, and M (i.e., MFL, MLF, LFM, FLM, LMF, and FML) can also perform a rotation of the WDF by an arbitrary angle. In Fig. 1, the single action of each component of the above triplets in the Wigner domain is sketched. Together with the above-mentioned LFL and FLF triplets, they form a set of three-matrix products that are the basis for the bulk-optical implementation of the FRT.

We now derive explicitly the effect of one of the above new triplets to a given input function. In particular, we pay attention to the MFL triplet. The action of a lens of focal distance Q/f_1 followed by a free propagation over a distance Rf_1 is represented by the matrix

$$\begin{bmatrix} 1 & -R \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Q & 1 \end{bmatrix} = \begin{bmatrix} 1 - RQ & -R \\ Q & 1 \end{bmatrix}, \tag{5}$$

which obviously cannot be identified with a rotation matrix, i.e.,

$$ROT \equiv \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}. \tag{6}$$

In fact the identification can only be accomplished for the trivial case $\phi = 0$. The additional impact of the magnification matrix provides the final result

$$\begin{bmatrix} 1 - RQ/M & -R/M \\ QM & M \end{bmatrix}, \tag{7}$$

which can be transformed into a rotation matrix provided that

$$\begin{aligned} M &= \cos\phi, \\ Q &= \tan\phi, \\ R &= \sin\phi\cos\phi. \end{aligned} \tag{8}$$

Consequently, the MFL optical setup is composed by a lens of focal distance

$$f' = f_1/\tan\phi, \tag{9}$$

followed by a free propagation over the distance

$$d = Rf_1 = f_1 \sin\phi\cos\phi, \tag{10}$$

and finally a telescopic system with magnification $M = \cos\phi$, which rescales the output and at the same time

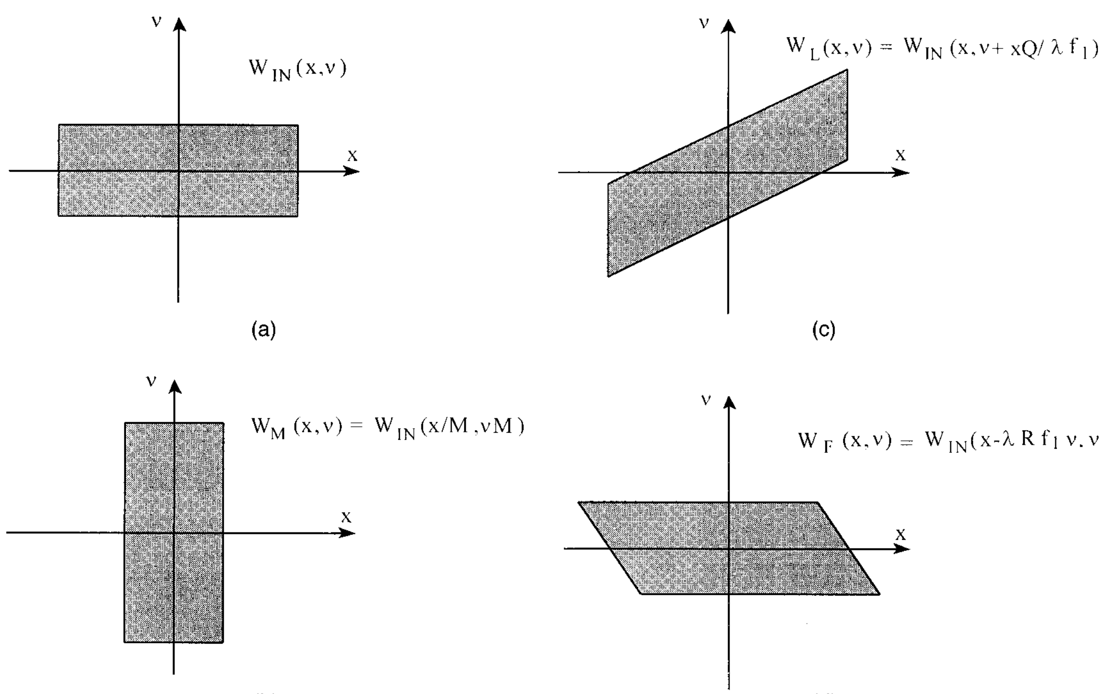

Fig. 1. (a) Schematic Wigner representation of the input signal, (b) after a magnification M, (c) after the action of a lens of focal power Q/f_1, and (d) after free propagation over a distance Rf_1.

Table 1. All Complete Triplets Involving the Magnification Matrix M

	MFL	*MLF*[a]	*FLM*	*LFM*	*FML*	*LMF*
M	$\cos\phi$	$\cos^{-1}\phi$	$\cos\phi$	$\cos^{-1}\phi$	$\cos\phi$	$\cos^{-1}\phi$
R	$\sin\phi\cos\phi$	$\tan\phi$	$\tan\phi$	$\sin\phi\cos\phi$	$\tan\phi$	$\tan\phi$
Q	$\tan\phi$	$\sin\phi\cos\phi$	$\sin\phi\cos\phi$	$\tan\phi$	$\tan\phi$	$\tan\phi$

[a] Although the authors did not interpret their result in this way, this particular solution was first obtained in Ref. 10.

preserves the correct quadratic phase factor at the output plane according to the definition of the FRT.

The effect of the other members of the set of triplet matrices involving L, F, and M can be interpreted in a similar way. The corresponding values for M, R, and Q for each new triplet are summarized in Table 1.

3. LENSLESS QUASI–FRACTIONAL FOURIER TRANSFORM

The action of the doublet FL gives the key to implementing a lensless optical system that provides the FRT of an input function apart from a quadratic phase factor. Let us consider the LFL system represented in Fig. 2(a), which corresponds to Lohmann's system II.[5] According to this configuration, the FRT of order P of the input is obtained just behind the second lens provided that the separation d between the two lenses and the focal length f' of both lenses are, respectively,

$$d = f_1 \sin\phi, \tag{11a}$$

$$f' = \frac{f_1}{\tan(\phi/2)}. \tag{11b}$$

The lens L_2 in front of the output plane can be removed if we are interested only in the square modulus of the FRT, since it simply introduces a quadratic phase factor. In this sense, we called the remaining FL system a quasi-FRT setup. In addition, the first lens L_1 can be replaced by a spherical wave front converging to its back focal point, here denoted S. The resulting lensless quasi-FRT setup is shown in Fig. 2(b). In Wigner matrix algebra the above discussion is reduced to transform the equality $ROT = L\ F\ L$ into $L^{-1}\ ROT = F\ L$.

Hence the FRT of order P is obtained—aside from a quadratic phase factor but with the proper magnification—by placing the input transparency $t(x)$ at a distance z_P from S such that

$$z_P = -f' = \frac{-f_1}{\tan(P\pi/4)}, \tag{12}$$

and the observation plane must be placed at the distance

$$R(P;\, z_P) = d = f_1 \sin(P\pi/2). \tag{13}$$

In this way, we are able to obtain the quasi-FRT of any order P of the input, by simply selecting the position of the input and the output planes following the prescriptions of Eqs. (12) and (13).

In mathematical terms, the amplitude distribution at the output plane is represented by

$$U[x;\, z_P,\, R(P;\, z_P)] = \exp[-ik/(2z_P)x^2]F^P[t(x)]. \tag{14}$$

The quadratic phase term in Eq. (14) represents a wave front coming from a point source located at a distance $-z_P$ from the output plane. This situation is similar to what happens when the Fourier transform of an input is obtained by optical means with use of a single converging lens. In general, the optical Fourier transform appears multiplied by a quadratic phase factor. In many textbooks the optical Fourier transform is called exact only when this factor is not present, i.e., when the object is placed in the front focal plane of the lens.

Of course, Eq. (14) can be rewritten as

$$F^P[t(x)] = \exp[ik/(2z_P)x^2]U[x;\, z_P,\, R(P;\, z_P)]. \tag{15}$$

From this point of view, Eq. (15) states that, except for a quadratic phase term that may be irrelevant in many situations, the FRT of order P is a certain Fresnel diffraction pattern of the input signal when it is illuminated by a spherical wave front with the proper curvature.

If the input is placed at a distance $z \neq z_P$ from S, the quasi-FRT of order P can also be obtained by free propagation, but with a scale error. Its new location, $R(P;\, z)$, and magnification, $M(P;\, z)$, can be derived from the Fresnel diffraction theory. They are connected to the above results through the relations

$$\frac{1}{z} + \frac{1}{R(P,\, z)} = \frac{1}{z_P} + \frac{1}{R(P;\, z_P)}, \tag{16}$$

$$M(P;\, z) = \frac{R(P;\, z)}{R(P;\, z_P)}. \tag{17}$$

In the context of image formation, in each member of Eq. (16) one can recognize the defocus coefficient W_{20} as used by Stokseth.[11] Using Eqs. (12) and (13), we finally obtain

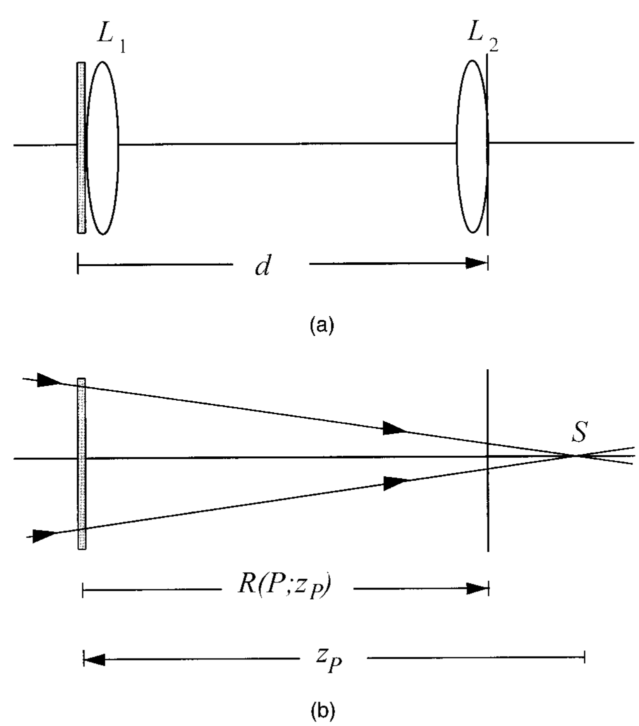

Fig. 2. Fractional Fourier transform systems: (a) Lohmann's setup of type II, (b) lensless configuration.

$$R(P;\, z) = \frac{f_1 \tan(P\pi/2)}{1 - f_1/z \tan(P\pi/2)}, \tag{18}$$

$$M(P;\, z) = \frac{1 + \tan(P\pi/2)\tan(P\pi/4)}{1 - f_1/z \tan(P\pi/2)}. \tag{19}$$

Equations (18) and (19) allow us to recognize that with a spherical illuminating wave front, i.e., with a fixed distance z, we are able to obtain simultaneously the quasi-FRT of all the fractional orders with a different scale error and in a different axial position. Only the FRT of order P such that $z = z_P$, where z_P is given by Eq. (12), is achieved without scale error. Since the above equations hold for any value of P, one may regard the order of the FRT as a parameter that identifies the Fresnel diffraction patterns of the input. Thus the optical setup we propose can also be used in a simple experimental verification of the role played by the FRT in optical propagation problems.[12]

4. FRACTIONAL FOURIER TRANSFORM PROCESSOR: FILTERING SETUP IN FRACTIONAL DOMAIN

There are different solutions for implementing a FRT spatial filtering system. Here we present two simple optical configurations, which are based on the quasi-FRT concept discussed in Section 3. The first system we propose is a lensless setup, and the second one is a single-lens device. In both cases we assume a general filtering configuration in which, at the output, a composition of two FRT's is detected. The system performs a first transformation of order P, which is followed by a second transformation of order Q. We assume that the filtering process is carried out in the fractional domain of order P and that the output is detected at the FRT of order $P + Q$. Note that the overall filtering process does not prevent the signal from having a quadratic phase factor while passing through the spatial filter.

A. Lensless Setup

Working with spherical wave-front illumination, we have recognized that free-space propagation, i.e., the diffraction phenomenon, is able to provide simultaneously the quasi-FRT's of orders P and $P + Q$. However, depending on the curvature of the incoming spherical wave front, it is possible to achieve only one of the two with the proper scale. Therefore two possibilities arise.

In the first case, the distance z is selected equal to z_{P+Q}, where z_{P+Q} is obtained by replacing P with $P + Q$ in Eq. (12). In this way, the quasi-FRT of order $P + Q$ at the output is properly scaled, but the quasi-FRT of order P at the filter plane does not have the proper size. Referring to Fig. 3(a), we see that the output plane is located at the distance $R(P + Q;\, z_{P+Q})$ given by Eq. (13), with P replaced by $P + Q$, while the separation of the filter plane from the input, $R(P;\, z_{P+Q})$, and the scale of the filter, $M(P;\, z_{P+Q})$, are given by Eqs. (18) and (19), with z replaced by z_{P+Q}. This approach has already been used by Granieri *et al.*[10] to describe certain spatial filtering devices based on the self-imaging phenomenon. However, the result is an inflexible FRT spatial filtering system,

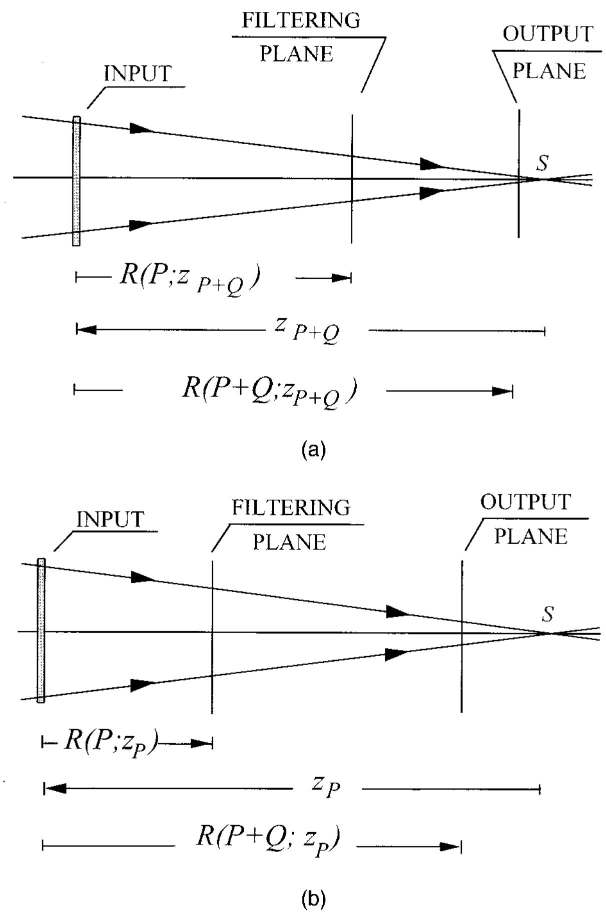

Fig. 3. Two configurations for a lensless FRT processor: (a) proper scale at the output plane, wrong scale at the filtering plane; (b) proper scale at the filtering plane, wrong scale at the output plane.

since the scale of the filter must be recalculated singly if we are interested in changing the order of the FRT at the intermediate filtering plane.

On the other hand, proper scale at the filter plane but arbitrary scale at the output is obviously a more convenient solution since the scale at the output can be easily modified with the image-detection system. As is sketched in Fig. 3(b), we select the distance z as z_P in Eq. (12). In this way, the separation between the input and the filter, $R(P;\, z_P)$, is given at once by Eq. (13). So the location of the output plane, $R(P + Q;\, z_P)$, and the scale at the output, $M(P + Q;\, z_P)$, are given by Eqs. (18) and (19) by replacing z with z_P and P with $P + Q$. Mathematically,

$$R(P + Q;\, z_P) = \frac{f_1 \tan[(P + Q)\pi/2]}{1 + \tan[(P + Q)\pi/2]\tan(P\pi/4)}, \tag{20}$$

$$M(P + Q;\, z_P) = \frac{1 + \tan[(P + Q)\pi/2]\tan[(P + Q)\pi/4]}{1 + \tan[(P + Q)\pi/2]\tan(P\pi/4)}. \tag{21}$$

It is a straightforward matter to show that $R(P + Q;\, z_P)$ and $M(P + Q;\, z_P)$ can be rewritten as

$$R(P + Q; z_P) = R(P; z_P) + R(Q; -z_P), \tag{22}$$

$$M(P + Q; z_P) = M(Q; -z_P). \tag{23}$$

Equation (22) is a consequence of the commutative additivity of the FRT, whereas Eq. (23) proves in mathematical terms that the error in the scale is produced only in the fractional Fourier transformation of order Q.

The versatility of this approach becomes evident when one realizes that the filtering order P can be tuned by simply changing the axial position of the input to the corresponding value z_P—the filtering plane will be located at $R(P; z_P)$. The order of the filtered output $P + Q$ can also be tuned easily by moving the output plane along the optical axis.

B. Single-Lens Setup

A FRT processor with the correct scale both at the output and at the filter plane arises as a direct consequence of the latter lensless setup. The magnification error $M(P + Q; z_P)$ at the output plane can be compensated by the use of a single lens, of back focal distance f, which forms an image of the filtered FRT of order $P + Q$ (produced by free propagation) with the proper scale, as is shown in Fig. 4. Therefore the magnification provided by the lens L must be

$$M_L = \frac{\pm 1}{M(P + Q; z_P)}. \tag{24}$$

Using the Gaussian lens equation and taking into account that M_L can be written as

$$M_L = \frac{-f}{a - f}, \tag{25}$$

we obtain

$$a = [1 \mp M(P + Q; z_P)]f, \tag{26a}$$

$$a' = \frac{[M(P + Q; z_P) \mp 1]}{M(P + Q; z_P)} f, \tag{26b}$$

where a and a' are sketched in Fig. 4. We would like to emphasize that the same results can be obtained with use of the Fresnel diffraction formula for propagation over the distances a and a' and taking into account the quadratic phase factor corresponding to the amplitude transmittance of the lens L.

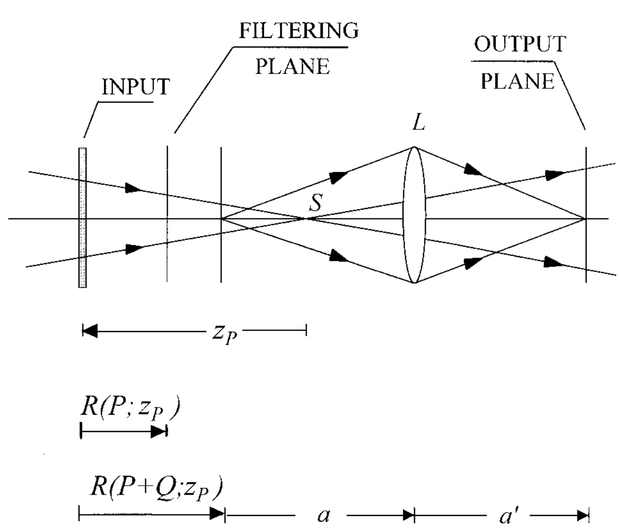

Fig. 4. Single-lens FRT processor.

Arguing practical considerations, next we impose that the filtering plane and the output plane should both be real planes. By inspection of Fig. 4, in which a lens of positive focal distance ($f > 0$) was assumed, we can write the above conditions as

$$a + R(P + Q; z_P) - R(P; z_P) > 0, \tag{27a}$$

$$a' > 0. \tag{27b}$$

If one takes into account Eqs. (22), (23), and (26), the above inequalities can be rewritten in terms of $M(Q; -z_p)$ and $R(Q; -z_P)$ as

$$[1 \mp M(Q; -z_P)]f + R(Q; -z_P) > 0, \tag{28a}$$

$$\frac{M(Q; -z_P) \mp 1}{M(Q; -z_P)} f > 0. \tag{28b}$$

We recognize that the focal length f of the lens L must fulfill Eqs. (28a) and (28b) for any value of P and Q to build a single-lens variable FRT processor.

The feasibility of our tunable FRT processor is discussed with the aid of two particular examples. First, we consider the case in which the output plane coalesces with the Fraunhofer plane, i.e., $P + Q = 1$. In this case, the values for $R(Q; -z_P)$ and $M(Q; -z_P)$ are

$$R(1-P; -z_P) = \frac{f_1 \cos(P\pi/2)}{\tan(P\pi/4)}, \tag{29a}$$

$$M(1-P; -z_P) = \frac{1}{\tan(P\pi/4)}. \tag{29b}$$

The last equalities are derived by replacing P with $1 - P$ and $-z$ with z_P, which is given by Eq. (12), in Eqs. (18) and (19), respectively.

For values of P ranging in the interval $0 \leqslant P \leqslant 1$, the above quantities are always positive. By selecting the negative sign in Eq. (24), and placing Eqs. (29a) and (29b) in Eqs. (28a) and (28b), we infer that these inequalities are always fulfilled. In other words, it is always possible to obtain a filtered version of any fractional order of the Fraunhofer diffraction pattern of the input with correct scales on both the output and the filter planes by use of a single positive lens regardless of the value of its optical power.

The last particular situation that we explore is $P + Q = 2$. In this case the output plane coincides with the image of the object. Now,

$$R(2 - P; -z_P) = \frac{-2f_1}{\mathrm{cotan}(P\pi/2) + \tan(P\pi/2)}, \tag{30}$$

and obviously

$$M(2 - P; -z_P) = -1. \tag{31}$$

Here, by taking the positive sign in Eq. (24), we obtain that the inequalities given by Eqs. (28a) and (28b) are fulfilled if

$$f \geqslant f_1 \frac{\sin(P\pi/2)}{2}. \tag{32}$$

So $f \geqslant f_1$ ensures a real filtering plane for any value of P, and consequently under this requirement the setup provides a filtered version of the image with arbitrary degree P.

5. CONCLUSIONS

We have presented some simple designs of a FRT transformer and a FRT processor. Both systems are variable in their fractional degree P. Our concern was to achieve those goals with a minimum number of lenses in a tunable fashion.

To this end, first we derived several new optical configurations that produce a rotation in the Wigner space using the magnification matrix. The Wigner matrix algebra is quite simple and was the key to the proposal of a set of new and flexible FRT optical designs.

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