JOURNAL OF MODERN OPTICS, 1998, VOL. 45, NO. 8, 1671–1679

Analytical formula for calculating the focal shift in apodized systems

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(*Received 10 June 1997; revision received 12 November 1997*)

Abstract. We report a quite simple analytical formula for the evaluation of the focal shift in apodized systems, with or without rotational invariance. Specifically it is shown that the magnitude of the focal shift is determined by the product of the Fresnel number of the focusing geometry and the standard deviation of a mapped version of the azimuthal average of the pupil transmittance. To illustrate our approach, several examples are examined.

1. Introduction

It is well known that, when a monochromatic uniform converging spherical wave is diffracted by a circular aperture, the point of maximum irradiance in the focal region is not at the geometrical focus but is displaced towards the aperture, resulting in the so-called focal-shift effect [1–3]. This effect has been found to appear also in other geometries, for example in uniformly illuminated systems with obscured pupils [4, 5], in focused Gaussian beams [6, 7], or, in general, in any type of diffracting screen [8–10].

Although in general it has been recognized that the magnitude of the focal shift is closely related to the Fresnel number of the focusing geometry, only for the particular case of the circular aperture under uniform [1] or Gaussian [7] spherical illumination has an approximate analytical formula been reported for its evaluation. In connection with this last focusing geometry it is of relevance that the formula, which was reported by Li [11] and is based on the focal-shift formulae for the cases of the circular aperture and the non-truncated Gaussian beam, gives the focal shift with quite good accuracy.

The aim of this paper is to present an analytical formulation for the evaluation of the focal shift for the general case of spherically illuminated, rotationally asymmetric diffracting screens, that is, for diffracting screens whose amplitude–transmittance function $t(r, \theta)$ does not exhibit, in general, rotational invariance. The approach, which is based on the moment expansion of a mapped version of the azimuthal average on the screen amplitude transmittance, permits us not only to calculate in quite a simple way the focal shift but also to define a new merit

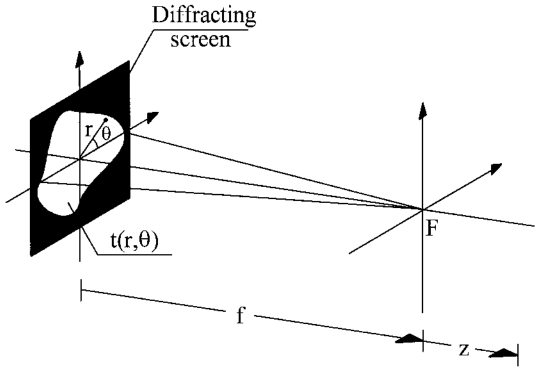


Figure 1. Geometry of the focusing set-up.

function for estimating the tendency of a diffracting screen to suffer the focal-shift effect.

In section 2 we formulate the basic theory for evaluating the axial-irradiance distribution produced by an arbitrary, radially asymmetric diffracting screen, at any Fresnel number. In section 3 we obtain an analytical formula that explicitly depends on the product of the Fresnel number of the focusing geometry and the standard deviation of the pupil, for evaluating in quite a simple way the relative focal shift. Finally, in section 4 we illustrate our approach by calculating the focal shift in some focusing geometries.

2. Basic theory

Let us start by considering a purely absorbing, rotationally asymmetric diffracting screen, with amplitude transmittance $t(r, \theta)$, which is illuminated by a uniform monochromatic spherical wave of focal length f, as shown in figure 1. Then, the amplitude distribution along the optical axis in the vicinity of the focal point, within the paraxial approximation, is [9]

$$U(z) = \frac{\exp(\mathrm{i}kz)}{\mathrm{i}\lambda f(f+z)} \int_0^{2\pi} \int_0^{r_0} t(r, \theta) \exp\left(-\mathrm{i}2\pi \frac{z}{2\lambda f(f+z)} r^2 \right) r \, \mathrm{d}r \, \mathrm{d}\theta, \qquad (1)$$

where r_0 is the maximum radial extent of the diffracting screen and z is the axial coordinate as measured from the paraxial focal point F.

The integation of equation (1) over θ gives

$$U(z) = \exp(\mathrm{i}kz) \frac{2\pi}{\mathrm{i}\lambda f(f+z)} \int_0^{r_0} t_0(r) \exp\left(-\mathrm{i}2\pi \frac{z}{2\lambda f(f+z)} r^2 \right) r \, \mathrm{d}r, \qquad (2)$$

where

$$t_0(r) = \frac{1}{2\pi} \int_0^{2\pi} t(r, \theta) \, \mathrm{d}\theta, \qquad (3)$$

is a radially symmetric function that represents the azimuthal average of the screen amplitude transmittance $t(r, \theta)$ for each value of r. From equation (2), one infers that the axial behaviour of a radially asymmetric focusing set-up is governed by the one-dimensional (1D) Fourier transform of a radially symmetric version of the amplitude transmittance of the diffracting screen. In order to make the Fourier-mapping relation clear, it is convenient to employ the next geometrical mapping

$$\xi = \left(\frac{r}{r_0}\right)^2 - 0.5, \quad q_0(\xi) = t_0(r), \tag{4}$$

which translates the radial interval $[0, r_0]$ into the 1D interval $[-0.5, 0.5]$. Now, by substituting equation (4) into equation (2) we find that, aside from an irrelevant pre-multiplying phase factor, the axial amplitude distribution is

$$U(z) = Q(W_{20}) = \frac{\pi(N - 2W_{20})}{f} \int_{-0.5}^{0.5} q_0(\xi) \exp(-i2\pi W_{20}\xi)\, d\xi, \tag{5}$$

where

$$N = \frac{r_0^2}{\lambda f} \tag{6}$$

represents the Fresnel number of the focusing set-up, that is, the number of Fresnel zones that are covered by the diffracting screen as viewed from the geometrical focus, whereas the axial coordinate is expressed in terms of

$$W_{20} = \frac{Nz}{2(f + z)}, \tag{7}$$

which is the well-known defocus coefficient measured in units of wavelength.

As the aim of this work is the calculation of the position of the maximum of the axial-irradiance distribution, next we express the axial behaviour of the focusing set-up in terms of the normalized axial irradiance $I_N(W_{20}) = I(W_{20})/I(0)$, namely

$$\begin{aligned} I_N(W_{20}) &= \left(1 - \frac{2}{N} W_{20}\right)^2 \left|\int_{-0.5}^{0.5} q_0(\xi) \exp(-i2\pi W_{20}\xi)\, d\xi\right|^2 \bigg/ \left|\int_{-0.5}^{0.5} q_0(\xi)\, d\xi\right|^2 \\ &= \left(1 - \frac{2}{N} W_{20}\right)^2 \bar{I}_N(W_{20}). \end{aligned} \tag{8}$$

From this equation it follows that the axial-irradiance pattern is determined from the product of two terms. The first term $\bar{I}_N(W_{20})$ involves the 1D Fourier transform of the mapped transmittance $q_0(\xi)$ of the screen. Since $t(r, \theta)$ and consequently $q_0(\xi)$ are real and positive functions, this term is symmetrical about the geometrical focus, $W_{20} = 0$, where its maximum value is achieved. The second term $(1 - 2W_{20}/N)^2$, which has W_{20} as the functional variable, is responsible for the loss of symmetry in the axial-irradiance distribution and, therefore, for the displacement of the irradiance maximum towards the screen plane, as we discuss below.

From equation (8), it is clear that for a given profile of the diffracting screen, that is, for a fixed $q_0(\xi)$, the influence of the parabolic term $(1 - 2W_{20}/N)^2$ and, therefore, the magnitude of the focal shift depends exclusively on the Fresnel number of the focusing geometry. Consequently, for $N \gg \bar{W}_{20}$, where $\bar{W}_{20}$ represents the maximum axial coordinate where the term $\bar{I}_N(W_{20})$ takes significant values, the parabolic factor is nearly constant over the region of interest, and then the effect of the focal shift is absent. However, when the value of N is low, the parabolic term rapidly increases for negative values of W_{20} within the focal region and then produces a relevant displacement of the axial-irradiance maximum towards the screen plane.

From the above reasoning, it follows that the magnitude of the focal shift effect is governed by the value of the Fresnel number of the focusing set-up, but it also depends on the dimensions of the axial region where the Fourier transform of $q_0(\xi)$ takes significant values. Therefore, it is apparent that the smoother the slope of the Fourier transform of $q_0(\xi)$, the greater is the influence of the factor $(1 - 2W_{20}/N)^2$ in the axial-irradiance distribution, and thus the greater is the magnitude of the resulting focal shift. In this context, we hypothesize that a function of merit for evaluating the focal shift in focusing set-ups should depend not only on the Fresnel number of the geometry but also on a parameter that evaluates the capacity of the screen to produce a slowly varying axial-irradiance pattern, as we discuss in mathematical terms in the next section.

3. The focal shift function of merit

Since, as stated in the previous section, the axial region of interest is located in the vicinities of the geometrical focus of the focusing set-up, it is then allowed to follow a reasoning equivalent to that of [12, 13], by expanding the normalized axial-irradiance distribution into a Taylor series, up to second-order approximation, that is

$$I_{\mathrm{N}}(W_{20}) \approx I_{\mathrm{N}}(0) + I'_{\mathrm{N}}(0)\,W_{20} + \frac{I''_{\mathrm{N}}(0)}{2}\,W_{20}^2. \tag{9}$$

Now, after a straightforward calculation, we obtain by using the moment theorem the following parabolic expression for the normalized axial-irradiance distribution:

$$I_{\mathrm{N}}(W_{20}) = 1 - \frac{4}{N}\,W_{20} + \frac{4}{N^2}(1 - \pi^2\sigma^2N^2)\,W_{20}^2, \tag{10}$$

where

$$\sigma = \left[\frac{m_2}{m_0} - \left(\frac{m_1}{m_0}\right)^2\right]^{1/2} \tag{11}$$

is the standard deviation of the mapped transmittance $q_0(\xi)$, m_n being the nth moment of $q_0(\xi)$.

Now, by derivation of equation (10), we find that the position and height of the maximum of the quadratic axial-irradiance distribution are given by

$$W_{20}^{\max} = \frac{N/2}{1 - \pi^2(N\sigma)^2} \tag{12}$$

and

$$I_{\mathrm{N}}(W_{20}^{\max}) = \frac{\pi^2(N\sigma)^2}{\pi^2(N\sigma)^2 - 1}, \tag{13}$$

respectively. Finally, by combining equations (12) and (7) we find that the relative axial position of the irradiance maximum, that is the relative focal shift, is given by

$$\frac{z_{\max}}{f} = -\frac{1}{\pi^2(N\sigma)^2}. \tag{14}$$

This relevant formula, which is a generalization of the analytical result obtained for the circular aperture [1], indicates that the axial-irradiance distribution provided by any diffracting screen, with or without radial symmetry, exhibits a focal shift whose magnitude depends exclusively on the function $N\sigma$. So, it is apparent that, independently of both the profile and the scale of the function $t(r, \theta)$, any pair of focusing set-ups having the same value for the product $N\sigma$ exhibits the same relative focal shift.

From the above result, it is then clear that the magnitude of the relative focal shift is determined not only by geometrical parameters of the focusing set-up, that is, by its Fresnel number, but also by the standard deviation of $q_0(\xi)$, which can be interpreted as a measure of the effective width of the diffracting screen. In other words, the function of merit $N\sigma$ evaluates not only the number of Fresnel zones of the focusing geometry but also their effective contribution to the axial-irradiance profile, and therefore to the magnitude of the relative focal shift.

Finally, we would like to remark that the function of merit $N\sigma$ is also useful for estimating when a focusing set-up tends to suffer the focusing effect known as focal switch [10].

4. Examples

To illustrate our approach we shall analyse the focal shift for three different kinds of diffracting screen, illuminated by a uniform monochromatic spherical wave: the circular aperture, a diffracting screen which produces axial apodization, and an axially superresolving screen.

For the case of the circular aperture the mapped transmittance is $q_0(\xi) = \mathrm{rect}(\xi)$, whose standard deviation is $\sigma_c = 1/12^{1/2}$. Then, according to equation (14), the value of the relative focal shift, expressed in terms of the Fresnel number, is

$$\frac{z_{\max}}{f} = -\frac{12}{\pi^2 N^2}, \tag{15}$$

which, within quite a good approximation, is equivalent to the result reported by Li and Wolf [1].

As an axially apodizing diffracting screen, we select the annular filter of the mapped transmittance $q_0(\xi) = \cos(\pi\xi)\,\mathrm{rect}(\pi\xi)$ (figure 2), which provides an optimum focal depth in the sense that it generates an axial-irradiance distribution with minimum second moment for pre-specified light throughput [14]. In this case, the value of the standard deviation is $\sigma_a = (\frac{1}{4} - 2/\pi^2)^{1/2} \approx 0.2176$. Then the relative focal shift formula for this screen is given by

$$\frac{z_{\max}}{f} = -\frac{21.12}{\pi^2 N^2}. \tag{16}$$

Note that, owing to the inverse square relation between the relative focal shift and the standard deviation of the mapped transmittance, the tendency of the axially apodizing diffracting screen to the focal shift effect is greater than that of the circular aperture.

Finally, as axially superresolving screens we chose the set of annular binary filters reported by Martínez-Corral *et al.* [13], which have the attribute of producing tunable axial superresolution. The mapped transmittance for the family

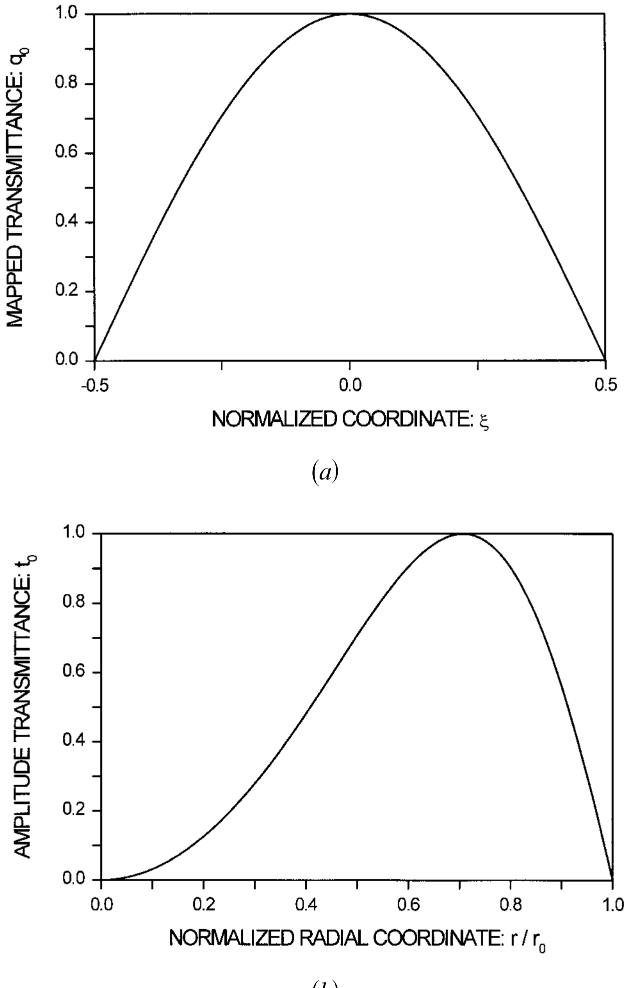


Figure 2. Amplitude transmittance of the axially apodizing diffracting screen, $q_0(\xi) = \cos(\pi\xi)\,\mathrm{rect}(\xi)$: (*a*) ξ-space representation: (*b*) radial coordinate representation.

of filters is

$$q_0(\xi) = \mathrm{rect}(\xi) - \mathrm{rect}\left(\frac{\xi}{\mu}\right), \quad \text{with } 0 < \mu < 1. \tag{17}$$

The profile of these screens in the ξ coordinate and in its actual two-dimensional (2D) representation are shown, for two values of the obscuration ratio parameter μ, in figure 3.

The value of the standard deviation for these filters is $\sigma_s = \left[(\mu^2 + \mu + 1)/12\right]^{1/2}$. Thus the corresponding relative focal shift is

$$\frac{z_{\max}}{f} = -\frac{12}{\pi^2 N^2(\mu^2 + \mu + 1)}, \tag{18}$$

whose magnitude, depending on the selected value of μ, is clearly lower than that corresponding to the circular aperture.

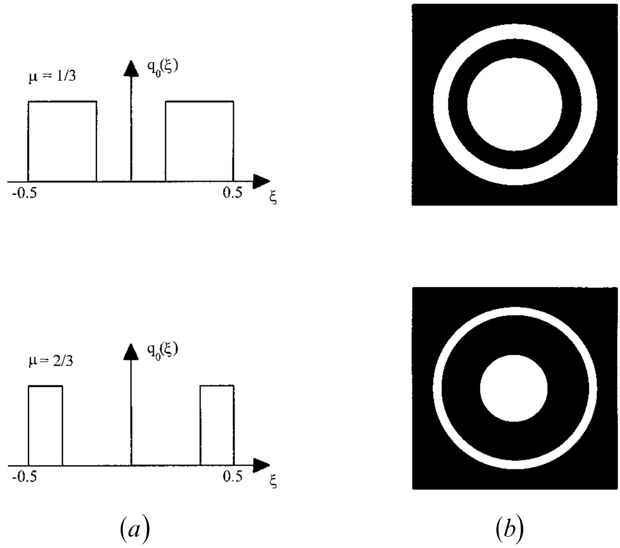


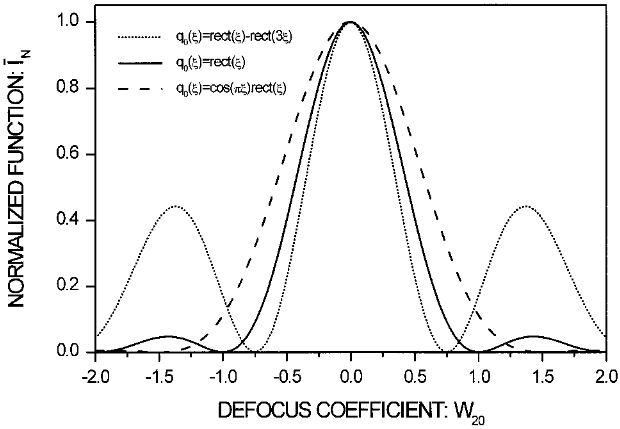
Figure 3. Two members of the family of axially superresolving diffracting screens of equation (17): (*a*) mapped function $q_0(\xi)$ for $\mu = \frac{1}{3}$ and $\mu = \frac{2}{3}$; (*b*) actual 2D representation.

To illustrate the influence of the parameter σ on the capacity for producing a slowly varying axial-irradiance pattern, and therefore on the magnitude of the relative focal shift, in figures 4(*a*) and (*b*) we have plotted the function $I_N(W_{20})$ and the relative focal shift respectively against N, for the above-described diffracting screen. As a representative of the set of axially superresolving filters, we have chosen that corresponding to $\mu = \frac{1}{3}$, which provides a significant narrowing of the central lobe of the axial-irradiance distribution but no drastic increase in the height of the secondary side lobes. From these figures it is clear that the tendency of a diffraction screen to suffer focal shift is closely related to is capacity to produce an axially apodizing effect. In other words, to obtain the same amount of focal shift the Fresnel number in the axially superresolving case should be lower than that corresponding to the case of the circular aperture and even lower than that of the axially apodizing case.

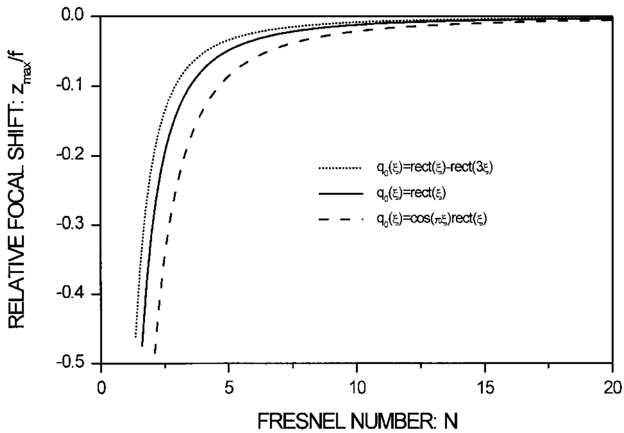
Finally, we would like to point out that comparison between the results provided by our formula and rigorous calculations already reported in the literature can only be done for the case of the spherically illuminated circular aperture [1] and the truncated Gaussian beam [11]. For the case of the circular aperture, experimental measurements of the magnitude of the focal shift are also available [15].

5. Conclusions

We have presented an analytical approach for evaluating in quite a simple way the relative focal shift in apodized systems. Specifically it has been stated that the magnitude of the relative focal shift exclusively depends on the product of the Fresnel number of the focusing geometry and the standard deviation of the mapped transmittance of the diffracting screen. Moreover, we have shown that



(*a*)



(*b*)

Figure 4. (*a*) Normalized function $\bar{I}_N(W_{20})$ corresponding to the proposed diffracting screens; (*b*) relative focal shift against the Fresnel number of the focusing geometry for the same screens as in (*a*).

this product constitutes a function of merit for evaluation of the tendency of a diffracting screen to suffer the focal-shift effect. Finally, we have illustrated our approach by examining the focal shift produced by several diffracting screens.

Acknowledgments

This work was supported by the Dirección General de Investigación Científica y Enseñanza Superior (grant PB93-0354-C02-01), Ministerio de Educación y Ciencia, Spain. C. J. Zapata-Rodríguez gratefully acknowledges financial support from this institution. M. Kowalczyk gratefully acknowledges financial support by Komitet Badan Naukowych, Poland (project 8T11F 020 12).

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