Effective Fresnel-number concept for evaluating the relative focal shift in focused beams

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We report on an analytical formulation, based on the concept of effective Fresnel number, to evaluate in a simple way the relative focal shift of rotationally nonsymmetric scalar fields that have geometrical focus and moderate Fresnel number. To illustrate our approach, certain previously known results and also some new focusing setups are analytically examined. © 1998 Optical Society of America [S0740-3232(98)00902-8] OCIS codes: 050.1940, 220.2560.

1. INTRODUCTION

The evaluation of the electromagnetic field diffracted by a coherently illuminated circular aperture has constituted the aim of several research efforts over the last few decades. Classical studies,^{1,2} which are based on the Debye integral representation of focused fields, reveal that when a monochromatic, uniform, converging spherical wave is diffracted by a circular aperture, the irradiance distribution in the focal region is symmetric about the geometrical focus, and the point of maximum irradiance is located at the focal point. However, since the description in the focal region by this representation is valid only when the Fresnel number of the focusing geometry is much higher than unity,³ these studies could not explain the shift toward the aperture of the irradiance maximum, which, under certain circumstances, was found to appear by some researchers during the 1970's.⁴⁻⁶

It was in the early 1980's when, on the basis of the Fresnel diffraction integral, it was analytically established,⁷⁻⁹ and experimentally verified,¹⁰ that the magnitude of the shift suffered by the axial irradiance peak, i.e., the magnitude of the so-called focal-shift effect, is governed by the Fresnel number of the focusing geometry. More recently, it has been recognized that the focal-shift effect is also present in systems with an obscured pupil,^{11,12} in axially superresolving setups,¹³ and, in general, in any type of diffracting screen.^{14,15} It has also been shown that this effect appears not only along the optical axis but also on any line directed toward the geometrical focus of the incident spherical wave front.¹⁶ Specifically, in Ref. 16 it is shown that the irradiance distribution along a line is determined by the projection of the pupil function onto the line. So both the irradiance profile and the magnitude of the focal shift depend on the selected line.

On the other hand, it is also well established that the focal-shift effect appears not only for uniformly illuminated diffracting screens but also when some nonuniform focused beams illuminate a circular aperture. In this sense it was found that when a monochromatic Gaussian beam is focused by a thin lens, the point of maximum irradiance is not located at the geometrical focus but is displaced toward the lens.^{17–19} More recent studies have shown that this focal-shift effect is governed by both the so-called Gaussian Fresnel number,²⁰ which is associated with the width of the incident beam, and the truncation parameter,^{21,22} which evaluates the ratio between the radius of the lens and the beam width.

The goal of this paper is to report on an analytical formulation to evaluate the relative focal shift for the general class of rotationally nonsymmetric scalar fields that have focus in the sense of geometrical optics. The formulation, which is based on an extension of the concept of effective Fresnel number applied to any focused beam, permits us to evaluate in a quite simple way the focal shift that appears in both converging and diverging beams and for truncated and nontruncated focusing geometries.

For describing our approach, in Section 2 we formulate the basic theory for evaluating the axial irradiance distribution corresponding to a focused beam. In Section 3 we define the effective Fresnel number over the exit plane associated with any focused beam, and we obtain a simple analytical formula, which explicitly depends on this parameter, for evaluating the relative focal shift. In Section 4 we analyze the role of the effective Fresnel number of a focused beam, which is shown to be related to the effective width of the beam. Finally, in Section 5 we illustrate our approach by examining the focal shift in some highly exemplifying focusing geometries.

2. AXIAL IRRADIANCE DISTRIBUTION IN FOCUSED BEAMS

Let us start by considering a focused beam, i.e., a monochromatic scalar wave field that has focus in the sense of geometrical optics. Therefore its amplitude distribution $U(r, \theta)$ in a given plane transversal to the propagation direction, referred to as the reference plane for the remainder of the present paper, can be expressed as the product of a spherical wave of focal length f and a real nonnegative two-dimensional function $t(r, \theta)$. In mathematical terms,

$$U(r, \theta) = \frac{\exp(-ikf)}{f} \exp\left(-i\frac{k}{2f}r^2\right)t(r, \theta), \quad (1)$$

where $k = 2\pi/\lambda$ represents the wave number of the light.

Note that this rather general formalism corresponds to the beams that provide an irradiance peak in the focus of a spherical wave and describes, among others, two quite typical focusing geometries in optics: (1) the case of a purely absorbing diffracting screen, with or without radial symmetry, illuminated by a monochromatic spherical wave and (2) the situation corresponding to a truncated or nontruncated spherical Gaussian beam. Two examples of these situations are illustrated in Fig. 1.

To evaluate the axial amplitude distribution of the focused beam, we particularize the Fresnel-Kirchhoff diffraction formula for the axial points,^{7,9} i.e.,

$$h(z) = \frac{\exp(ikz)}{i\lambda f(f+z)} \int_0^{2\pi} \int_0^{\infty} t(r, \theta) \\ \times \exp\left[-i2\pi \frac{z}{2\lambda f(f+z)} r^2\right] r \, \mathrm{d}r \mathrm{d}\theta, \qquad (2)$$

where z denotes the axial coordinate as measured from the paraxial focal point F.

At this point it is convenient to point out that the axial distances involved in Eq. (2) are directed. Their direction is determined by the point of the arrow (see Fig. 1). Note that both f and z are positive distances in Fig. 1. This fact allows us to deal with diverging focusing beams (f < 0) and to describe the virtual diffraction region (f + z < 0).



Fig. 1. Two typical examples of scalar fields that have geometrical focus: (a) radially nonsymmetric diffracting screen illuminated by a monochromatic spherical wave with focal length f, (b) truncated spherical Gaussian beam.

It is convenient to perform two mathematical manipulations of Eq. (2). First, we carry out the angular integration to obtain

$$h(z) = 2\pi \frac{\exp(ikz)}{i\lambda f(f+z)} \int_0^\infty t_0(r)$$
$$\times \exp\left[-i2\pi \frac{z}{2\lambda f(f+z)} r^2\right] r \, \mathrm{d}r, \qquad (3)$$

where

$$t_0(r) = \frac{1}{2\pi} \int_0^{2\pi} t(r, \theta) \mathrm{d}\theta \tag{4}$$

is a radially symmetric function that stands for the azimuthal average of $t(r, \theta)$. Of course, any pair of focused beams that yield the same azimuthally averaged function $t_0(r)$ generate an identical axial irradiance distribution.^{23,24}

The second manipulation consists in the geometrical mapping

$$\zeta = r^2, \qquad t_0(r) = q_0(\zeta),$$
 (5)

which explicitly converts the integral in Eq. (3) into a onedimensional (1-D) Fourier transform, namely,

$$h(z) = \frac{\pi}{\lambda f(f+z)} \int_{-\infty}^{\infty} q_0(\zeta) \exp\left[-i2\pi \frac{z}{2\lambda f(f+z)} \zeta\right] d\zeta,$$
(6)

where some irrelevant premultiplying constant phase factors have been omitted. Note that the lower limit in the integral has been extended to $-\infty$, since the function $q_0(\zeta)$ is identically zero for $\zeta < 0$.

Next, we recognize that the scale factor of the 1-D Fourier transform is related to the axial variable z through the relation

$$u = \frac{z}{2\lambda f(z+f)},\tag{7}$$

which is closely related to the axial coordinate defined by Lommel.² Thus the axial amplitude distribution of the focused beam can be expressed as

$$h(z) = Q(u)$$

= $\frac{\pi}{\lambda f^2} (1 - 2\lambda f u) \int_{-\infty}^{\infty} q_0(\zeta) \exp(-i2\pi u \zeta) d\zeta.$ (8)

Finally, since our aim in this work is to calculate the position of the axial point of maximum irradiance, we express the axial behavior of the focused beam in terms of the normalized axial irradiance distribution, which is obtained by simply dividing the squared modulus of Eq. (8) by the irradiance at the paraxial focal point u = 0. Mathematically,

$$\mathscr{T}_{N}(u) = (1 - 2\lambda f u)^{2} \frac{\left| \int_{-\infty}^{\infty} q_{0}(\zeta) \exp(-i2\pi u \zeta) \mathrm{d}\zeta \right|^{2}}{\left[\int_{-\infty}^{\infty} q_{0}(\zeta) \mathrm{d}\zeta \right]^{2}}.$$
(9)

From Eq. (9) it follows that the axial irradiance distribution of the focused beam is governed by the product of two terms that can be understood, from the point of view of the Huygens-Fresnel principle,^{9,13} in the following way. The first term, which involves the 1-D Fourier transform of $q_0(\zeta)$, describes at any axial point the interference by the different Huygens spherical wavelets proceeding from all points of the reference plane. Since $t(r, \theta)$, and consequently $q_0(\zeta)$, are real and positive functions, the Huygens wavelets arrive in phase at the geometrical focus, and then the maximum of this term is achieved at the origin. However, as the secondary wavelets propagate, their amplitude suffers an attenuation that is proportional to the inverse covered distance. This attenuation is described in Eq. (9) by the term $(1 - 2\lambda f u)^2$.

It is then apparent that the competition between these two terms produces a displacement of the axial irradiance peak toward the reference plane, resulting in the wellknown focal-shift effect, whose magnitude we evaluate in Section 3.

3. RELATIVE FOCAL-SHIFT FORMULA

To obtain the relative position of the greatest value of the axial irradiance distribution, one should obtain the roots of the equation

$$\mathrm{d}\,\mathscr{T}_N/\mathrm{d}u\,=\,0\tag{10}$$

and select from among them the absolute maximum. However, in many cases of interest an exact analytical result cannot be obtained by this method. To avoid this drawback, we propose to expand the normalized irradiance distribution into a Taylor series. Taking into account that the region of interest is located in the vicinities of the geometrical focus (u = 0), we may neglect the third- and higher-order terms and restrict the series to a quadratic approximation, i.e.,

$$I_N(u) = \mathscr{T}_N(0) + \mathscr{T}'_N(0)u + \frac{\mathscr{T}''_N(0)}{2}u^2.$$
(11)

Now, by straightforward (albeit cumbersome) calculation, we obtain by making use of the moment theorem²⁵ the following approximated parabolic expression for the normalized axial irradiance distribution:

$$I_N(u) = 1 - 4\lambda f u + 4(\lambda^2 f^2 - \pi^2 \sigma^2) u^2, \qquad (12)$$

where

$$\sigma = \left[\frac{m_2}{m_0} - \left(\frac{m_1}{m_0}\right)^2\right]^{1/2}$$
(13)

stands for the standard deviation of the mapped function $q_0(\zeta)$ and

$$m_n = \int_{-\infty}^{\infty} q_0(\zeta) \zeta^n \, \mathrm{d}\zeta, \qquad (14)$$

denotes the *n*th moment of $q_0(\zeta)$.

By virtue of Eq. (12), we find that the position and the height of the maximum of the quadratic axial irradiance distribution are given, respectively, by

$$u_{\rm max} = \frac{1}{2\lambda f} \frac{1}{1 - \pi^2 N_{\rm eff}^2},$$
 (15)

$$I_N(u_{\rm max}) = \frac{\pi^2 N_{\rm eff}^2}{\pi^2 N_{\rm eff}^2 - 1},$$
 (16)

where we have introduced a new parameter, named here the effective Fresnel number of the focused beam, defined as

$$N_{\rm eff} = \sigma / \lambda f. \tag{17}$$

This parameter characterizes any focused beam at the reference plane and is proportional to the standard deviation of $q_0(\zeta)$, which can be interpreted as a measure of the effective width of the beam.

Now, by combining Eqs. (7) and (15), we find that the value of the relative focal shift is given by

$$\frac{z_{\max}}{f} = -\frac{1}{\pi^2 N_{\text{eff}}^2}.$$
 (18)

This relevant formula, which is a generalization of the results obtained for both the circular aperture^{7,9} and the Gaussian beam,^{20,21} indicates that any scalar field belonging to the general class of focused beams suffers a relative focal shift that, within the second-order approximation, is proportional to the inverse square of the effective Fresnel number of the beam. So it is apparent that, independently of both the profile and the scale of the function $t(r, \theta)$ and the sign of the focal length f, any pair of focused beams having the same value for $N_{\rm eff}$, and consequently the same effective width, exhibit the same relative focal shift.

The minus sign in Eq. (18) indicates that the distances z_{\max} and f have opposite sign. Therefore, either in a converging (f > 0) or in a diverging (f < 0) focusing geometry, the displacement of the axial irradiance peak is always toward the reference plane. Note that in the diverging case the virtual focal shift can be viewed by focusing a microscope with low magnifying power behind the reference plane, assuming that the optical instrument has a numerical aperture large enough not to introduce significant diffraction effects on its own account. It is important to remark that, to the best of our knowledge, the virtual focal shift has been referred to only by Nye²⁶ when he studied the light diffracted by small unstopped lenses.

For illustrating the variation of the focal-shift effect with the value of the effective Fresnel number of the focused beam, in Fig. 2 we have plotted the relative focal shift against $N_{\rm eff}$. From this figure we infer that the modulus of the relative focal shift greatly increases as $N_{\rm eff}$ decreases, being, for example, of the order of 0.1% when $N_{\rm eff} = 10$ and 2.5% when $N_{\rm eff} = 2$.

Concerning the accuracy of Eqs. (16) and (18), it can be easily shown (see Appendix A) that the relative error in the evaluation of the relative focal shift, resulting from the parabolic approximation, is given by

$$\frac{\Delta(z_{\rm max})}{|z_{\rm max}|} = \frac{2}{\pi^2 N_{\rm eff}^2 - 1},$$
(19)

whereas the error in the estimation of the height of the irradiance peak is



Fig. 2. Relative focal shift as a function of the effective Fresnel number of the focused beam.



Fig. 3. Relative error associated with the focal shift determined by Eq. (18) versus the effective Fresnel number of the focused beam.

$$\Delta I_{\rm max} = \frac{2\,\pi^2 N_{\rm eff}^2}{(\pi^2 N_{\rm eff}^2 - 1)^3}.$$
 (20)

We conclude from Eqs. (19) and (20) that both errors depend solely on the effective Fresnel number of the focused beam. In particular, the variation of the ratio $\Delta(z_{\rm max})/|z_{\rm max}|$ with $N_{\rm eff}$ is as indicated in Fig. 3. The examination of the curve in Fig. 3 reveals that the error is low for moderate values of $N_{\rm eff}$ (for example, when $N_{\rm eff} \ge 2.0$, the error is less than 5%; and it is even less than 1% if $N_{\rm eff} \ge 4.5$). On the contrary, for N < 1, $\Delta(z_{\rm max})/|z_{\rm max}|$ increases dramatically as N decreases.

4. EFFECTIVE FRESNEL-NUMBER ANALYSIS

In Section 3 we established that the position and the height of the axial irradiance peak for a general focused beam are determined by a single parameter, the effective Fresnel number of the focusing geometry. Hence it seems to be convenient to investigate more the role of the effective Fresnel-number parameter. To this end we now recall the interpretation of the axial irradiance distribution in terms of the Huygens–Fresnel superposition principle, as was done in Section 2. Note that, arising from this principle, when we deal with a large pupil function $t(r, \theta)$ that involves a huge number of Fresnel zones in the process of Huygens wavelet interference, the 1-D Fourier transform in Eq. (9) provides a very sharp function centered at the geometrical focus u = 0. So its value is negligible unless u is very small. In this way the factor $(1 - 2\lambda f u)^2$ is approximately unity in the region where the 1-D Fourier transform is nonzero, and consequently it can be ignored. In this case the axial irradiance distribution is fixed only by the interference term and is symmetric around the geometrical focus, since the Fourier transform of a real function is Hermitian. However, when $t(r, \theta)$, and consequently $q_0(\zeta)$, are narrow, which implies that a small number of Fresnel zones are involved in the interference process, the Fourier transform gives rise to a function that decreases smoothly in the neighborhood of the paraxial focal point. Now the attenuation term $(1 - 2\lambda f u)^2$ is not unity in the whole region in which the Fourier transform is clearly different from zero. The presence of this term, whose value increases with negative values of the product fu, shifts the irradiance peak toward the reference plane, resulting in the focalshift effect.

It is then clear that the axial irradiance distribution of the focused beam, and thus the relative focal shift, are determined by both the number of Fresnel zones involved in the interference process and their relative contribution. In other words, the axial pattern is governed by the effective width of the beam at the reference plane. In this context we conclude that the effective Fresnel number, which is proportional to the standard deviation of the function $q_0(\zeta)$, evaluates in a certain way the effective number of Fresnel zones in the focusing geometry involved in the above process, and therefore it also takes into account the importance of the relative contribution of each individual zone in the final result.

5. APPLICATION TO SEVERAL FOCUSING GEOMETRIES

We start by considering the case of an annular aperture of inner and outer radii ϵa ($0 \leq \epsilon < 1$) and a, respectively, illuminated by a monochromatic spherical wave of focal length f. According to Eq. (17), the effective Fresnel number for this focusing geometry is

$$N_{\rm eff} = \frac{(1-\epsilon^2)}{\sqrt{12}} \frac{a^2}{\lambda f}.$$
 (21)

Note that the above effective Fresnel number is indeed, apart from a proportionality factor $1/\sqrt{12}$, equal to the classical Fresnel number for annular focusing setups,^{11,15} which is given by

$$N = \frac{(1 - \epsilon^2)a^2}{\lambda f},\tag{22}$$

which implies that, because of the close connection between these concepts, our formalism will allow us to reproduce the classical results, as we show next.

By substituting the value of $N_{\rm eff}$ into Eqs. (16) and (18), we find that the relative position and the height of the axial irradiance peak are

$$\frac{z_{\max}}{f} = -\frac{12}{\pi^2 N^2},$$
 (23a)

$$I_N(z_{\rm max}) = \frac{\pi^2 N^2}{\pi^2 N^2 - 12},$$
 (23b)

which are expressed in terms of the classical Fresnel number.

We emphasize that, if we simply set $\epsilon = 0$, the above equations are valid, in particular, for describing the focal shift for the quite typical case of the circular aperture. However, although equivalent equations have been derived elsewhere^{7,9} for converging illumination, it was not recognized that they are also valid for describing the virtual focal-shift effect when the circular aperture is illuminated by a diverging monochromatic spherical wave.

As a second example we consider the case of a spherical Gaussian beam that illuminates a circular aperture of radius a, f being the radius of curvature of the beam at the aperture plane. To carry out this study, it is convenient to introduce the truncation parameter of the focusing geometry, defined by

$$\alpha = (a/\omega)^2, \tag{24}$$

where ω stands for the width of the Gaussian beam at the reference plane. The expression for $N_{\rm eff}$ in this case is

$$N_{\rm eff} = \frac{\omega^2}{\lambda f} \left\{ 1 - \frac{\alpha^2 \, \exp(\alpha)}{\left[1 - \exp(\alpha)\right]^2} \right\}^{1/2}.$$
 (25)

Now, by substituting Eq. (25) into Eq. (18), we find that the relative focal shift corresponding to a truncated spherical Gaussian beam is given, in terms of the truncation parameter, by

$$\frac{z_{\max}}{f} = -\frac{\lambda f}{\pi^2 \omega^2 \{1 - \alpha^2 \exp(\alpha) / [1 - \exp(\alpha)]^2\}}.$$
 (26)

As in the previous example, next we analyze some particular case. First, for strong truncation, the width of the incident beam is much higher than the radius of the circular aperture. In this case the value of the truncation parameter α is almost zero. By performing the limit when α tends to zero in Eqs. (25) and (26), we achieve the effective Fresnel number and the relative focal shift corresponding to the uniformly illuminated circular aperture.

In the other extreme case, corresponding to an unapertured Gaussian beam (weak truncation), the coefficient α tends to infinity. Here the limit value for the effective Fresnel number is

$$N_{\rm eff} = \omega^2 / \lambda f. \tag{27}$$

This expression is just the same as that of the classical Fresnel number defined for focused Gaussian beams.^{20–22} By substitution of this value into Eq. (18), the relative focal shift for unapertured Gaussian beams is

$$\frac{z_{\max}}{f} = -\frac{\lambda^2 f^2}{\pi^2 \omega^4},\tag{28}$$

which reproduces in a quite good approximation the previously known result. $^{\rm 27}$

Up to now we have illustrated our approach by reexamining, with our approach, the general case of the annular aperture or the truncated Gaussian beam, which have indeed been studied in the literature. Now we go one step further and investigate the axial behavior for two diffracting screens, reported in the optical literature for certain applications, when they are illuminated with a monochromatic spherical wave front of focal length f.

The first screen under study is that of mapped transmittance

$$q_0^s(\zeta) = \begin{cases} 4[(\zeta/a^2) - 0.5]^2 & \text{if } 0 \le \zeta \le a^2 \\ 0 & \text{otherwise} \end{cases}, (29)$$

which produces a superresolving axial irradiance pattern.²⁸ The mapped transmittance for the second selected diffracting screen is

$$q_0^a(\zeta) = \begin{cases} 1 - q_0^s(\zeta) & \text{if } 0 \le \zeta \le a^2 \\ 0 & \text{otherwise} \end{cases}, \quad (30)$$

which produces an axial apodization effect.²⁹ The profiles of both screens in the ζ coordinate and in the radial coordinate are shown in Fig. 4. It is straightforward to obtain analytically the standard deviation, and conse-



Fig. 4. Amplitude transmittance of the axially superresolving diffracting screen (solid curves) and the axially apodizing parabolic filter (dashed curves): (a) ζ -space representation, (b) radial coordinate representation.

quently the corresponding effective Fresnel number, for these quadratic functions. We have

$$N_{\rm eff}^{\rm s} = \sqrt{\frac{3}{20}} \frac{a^2}{\lambda f}, \qquad (31a)$$

$$N_{\rm eff}^a = \frac{1}{\sqrt{20}} \ \frac{a^2}{\lambda f}.$$
 (31b)

When we take into account Eq. (18), the relative position of the axial irradiance peak is given, respectively, by

$$\frac{z_{\max}^s}{f} = \frac{-20}{3\pi^2} \left(\frac{\lambda f}{a^2}\right)^2, \qquad (32a)$$

$$\frac{z_{\max}^{2}}{f} = \frac{-20}{\pi^{2}} \left(\frac{\lambda f}{a^{2}}\right)^{2}.$$
 (32b)

Note that if all geometric parameters in both optical setups are equal, it is apparent that $z^a_{\max} = 3z^s_{\max}$. Therefore the axially apodizing diffracting screen is three times more sensitive to the focal-shift effect than that which produces axial superresolution. This is what one would expect, since, as we mentioned in Section 4, the wider the central lobe of the axial-distribution irradiance, the greater the amount of the focal shift. In this work we recognize that this general statement can be quantified by means of the effective Fresnel-number parameter.

6. CONCLUSIONS

We have presented a quite simple analytical formulation for evaluating, in a second-order approximation, the relative focal shift for any focused beam with or without radial symmetry. The proposed formula depends solely on a beam parameter named the effective Fresnel number of the focused beam. This important result means that any pair of focused beams with the same effective Fresnel number show the same relative focal shift, independently of their geometric parameters and their transverse profile. Moreover, our formalism also permits us to quantify the virtual focal-shift effect for any diverging focused beam. Finally, we have illustrated our approach by discussing the focal-shift effect shown by different focusing geometries.

APPENDIX A

The error involved in the approximation carried out in Eq. (11), i.e., the modulus of the difference between the exact axial irradiance and the value provided by the quadratic approximation, is given by the absolute value of the so-called Lagrange remainder, namely,

$$R(\mathscr{T}_N; 0, u) = \frac{\mathscr{T}_N^{''}(u_0)}{6} u^3, \qquad (A1)$$

where u_0 is an axial-coordinate value between 0 and u.

If we assume that the normalized irradiance distribution is a slowly varying function and that the axial coordinate u under investigation is close to the paraxial focus, the approximation

$$\mathscr{T}_{N}^{'''}(u_{0}) \approx \mathscr{T}_{N}^{'''}(0) = 96\pi^{2}\sigma^{2}\lambda f.$$
(A2)

may be applied. If we particularize under this assumption the value of the Lagrange remainder for $u = u_{\text{max}}$, we obtain that the error in the estimation of the height of the irradiance peak, which is derived by inserting Eq. (15) and relation (A2) into Eq. (A1), is

$$\Delta I_{\rm max} = \Delta [I_N(u_{\rm max})] = \frac{2\pi^2 N_{\rm eff}^2}{(\pi^2 N_{\rm eff}^2 - 1)^3}.$$
 (A3)

For obtaining the error in the evaluation of the relative focal shift, first we express this parameter in terms of $I_N(u_{\text{max}})$ by combining Eqs. (16) and (18). In this way Eq. (18) can be written as

$$\frac{z_{\max}}{f} = \frac{1}{I_N(u_{\max})} - 1.$$
 (A4)

Now, by merely applying standard error propagation techniques, we obtain

$$\Delta \left(\frac{z_{\text{max}}}{f} \right) = \frac{2}{\pi^2 N_{\text{eff}}^2} \frac{1}{\pi^2 N_{\text{eff}}^2 - 1}.$$
 (A5)

Finally, the relative error in the determination of the relative focal shift is achieved by simply dividing Eq. (A5) by the absolute value of Eq. (18). Thus

$$\frac{\Delta(z_{\rm max})}{|z_{\rm max}|} = \frac{2}{\pi^2 N_{\rm eff}^2 - 1}.$$
 (A6)

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REFERENCES AND NOTES

- E. Collet and E. Wolf, "Symmetry properties of focused fields," Opt. Lett. 5, 264–266 (1980).
- M. Born and E. Wolf, *Principles of Optics*, 6th ed. (Pergamon, Oxford, 1980), Sec. 8.8.1.
- E. Wolf and Y. Li, "Conditions for the validity of Debye integral representation of focused fields," Opt. Commun. 39, 205-209 (1981).
- D. A. Holmes, J. E. Korka, and P. V. Avizonis, "Parametric study of apertured focused Gaussian beams," Appl. Opt. 11, 565–574 (1972).
- A. Arimoto, "Intensity distribution of aberration-free diffraction patterns due to a circular aperture in large *F*-number optical systems," Opt. Acta 23, 245-250 (1976).
- M. A. Gusinow, M. E. Riley, and M. A. Palmer, "Focusing in a large *F*-number optical system," Opt. Quantum Electron. 9, 465–471 (1977).
- 7. Y. Li and E. Wolf, "Focal shifts in diffracted converging spherical waves," Opt. Commun. **39**, 211–215 (1981).
- J. J. Stamnes and B. Spjelkavik, "Focusing at small angular apertures in the Debye and Kirchhoff approximations," Opt. Commun. 40, 81–85 (1981).
- 9. M. P. Givens, "Focal shifts in diffracted converging spherical waves," Opt. Commun. **41**, 145–148 (1982).
- 10. Y. Li and H. Platzer, "An experimental investigation of dif-

fraction patterns in low-Fresnel-number focusing systems," Opt. Acta **30**, 1621–1643 (1983).

- V. N. Mahajan, "Axial irradiance and optimum focusing of laser beams," Appl. Opt. 22, 3042–3053 (1983).
- J. Ojeda-Castañeda, M. Martínez-Corral, P. Andrés, and A. Pons, "Strehl ratio versus defocus for noncentrally obscured pupils," Appl. Opt. 33, 7611-7616 (1994).
- M. Martínez-Corral and V. Climent, "Focal switch: a new effect in low-Fresnel-number systems," Appl. Opt. 35, 24-27 (1996).
- S. Szapiel, "Maréchal intensity formula for small-Fresnelnumber systems," Opt. Lett. 8, 327–329 (1983).
- M. Martínez-Corral, P. Andrés, and J. Ojeda-Castañeda, "On-axis diffractional behavior of two-dimensional pupils," Appl. Opt. 33, 2223–2229 (1994).
- P. Andrés, M. Martínez-Corral, and J. Ojeda-Castañeda, "Off-axis focal shift for rotationally nonsymmetric screens," Opt. Lett. 18, 1290-1292 (1993).
- G. Goubau, "Optical relations for coherent wave beams," in *Electromagnetic Theory and Antennas*, E. C. Jordan, ed. (Macmillan, New York, 1963), Part 2, pp. 907–918.
- A. G. van Nie, "Rigorous calculation of the electromagnetic field for wave beams," Philips Res. Rep. 19, 378–394 (1964).
- 19. H. Kogelnik, "Imaging of optical modes-resonators with internal lenses," Bell Syst. Tech. J. 44, 455-594 (1965).
- W. H. Carter, "Focal shift and concept of effective Fresnel number for a Gaussian laser beam," Appl. Opt. 21, 1989– 1994 (1982).

- Y. Li and E. Wolf, "Focal shift in focused truncated Gaussian beams," Opt. Commun. 42, 151–156 (1982).
- Y. Li, "Focal shift formula for focused, apertured Gaussian beams," J. Mod. Opt. 39, 1761–1764 (1992).
- C. W. McCutchen, "Generalized aperture and threedimensional diffraction image," J. Opt. Soc. Am. 54, 240-244 (1964).
- J. Ojeda-Castañeda, P. Andrés, and M. Martínez-Corral, "Zero axial irradiance by annular screens with angular variation," Appl. Opt. 31, 4600-4602 (1992).
- J. D. Gaskill, Linear Systems, Fourier Transforms, and Optics (Wiley, New York, 1978), pp. 212–214.
- J. F. Nye, "Diffraction by a small unstopped lens," J. Mod. Opt. 38, 743-754 (1991).
- 27. Specifically, it is shown in Refs. 17–22 that the point of maximum axial irradiance is located at a normalized distance $z_{\max}/f = -1/(1 + \pi^2 \omega^4/\lambda^2 f^2)$ from the geometrical focus. We would like to remark that this equation is equivalent to Eq. (28) in the text, except for an additive constant in the denominator. It is easy to show that the difference between the two results is very low for moderate values of $N_{\rm eff}$.
- C. J. R. Sheppard and Z. S. Hegedus, "Axial behavior of pupil-plane filters," J. Opt. Soc. Am. A 5, 643–647 (1988).
- M. J. Yzuel, J. C. Escalera, and J. Campos, "Polychromatic axial behavior of axial apodizing and hyperresolving filters," Appl. Opt. 29, 1631–1641 (1990).