JOSA COMMUNICATIONS

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Sampling expansions for three-dimensional light amplitude distribution in the vicinity of an axial image point: comment

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Landgrave and Berriel-Valdos presented axial and radial sampling expansions for three-dimensional light amplitude distribution around the Gaussian focal point. [J. Opt. Soc. Am. A 14, 2962 (1997)]. The expansions were obtained under the assumption that the pupil function was rotationally symmetric. We present a new derivation of the axial expansion that does not make use of arbitrary formal assumptions used by Landgrave and Berriel-Valdos and eliminates some faults of the derivation given by Arsenault and Boivin, who published this expansion in 1967 [J. Appl. Phys. 38, 3988 (1967)]. We also discuss generalizations of the axial expansion to the case of pupils that exhibit no symmetry with respect to the axis considered. © 2003 Optical Society of America

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When searching the reference material for our recent paper on diffractional behavior of focused light beams modulated by an annular binary filter,¹ we found several papers on sampling of three-dimensional (3D) fields. Among them was the work of Landgrave and Berriel-Valdos,² in which the authors rediscovered a well-known result, usually referred to as the axial sampling theorem (AST). To our knowledge Arsenault and Boivin originally published the AST in 1967.³

The authors of Refs. 2 and 3 dealt with coherently illuminated optical systems in which axially symmetric spherical converging light waves were diffracted (Fig. 1). Specifically, in Ref. 2 the exit-pupil filter of an optical imaging system diffracted the light wave, whereas in Ref. 3 there was just a diffracting aperture. Both the pupil filter and the diffracting aperture were axially symmetric. The subject of interest of the authors of both papers was the 3D distribution of the complex scalar amplitude G of the diffracted waves around the Gaussian focus of the converging beam. Calculation of the diffraction integrals were performed, in both Ref. 2 and Ref. 3, by means of the sampling expansion, yielding the following results,

$$G(\delta, v) = \sum_{m = -\infty} G(m, 0)L(\delta - m, v), \quad (1) \quad [2-3.7]$$
$$G(y, z) = \sum_{n = -\infty}^{\infty} G(-4n\pi, 0)O(y + 4n\pi, z),$$

(2) [3-14]

where both $L(\delta, v)$ and O(y, z) stand for the Lommel diffraction integral, δ and y for the amount of defocusing, and v and z for the reduced lateral coordinate in the focal region. The equation number [X-Y] denotes Eq. Y in Ref. X referenced in this comment. Thus Eqs. (1) and (2) that appear in Refs. 2 and 3, respectively, are essentially the same. The apparent difference results from the different definitions used in those papers for the direct and the inverse Fourier transforms (FTs).⁴ Both Eq. (1) and Eq. (2) are referred to as the AST.

The derivation of the AST proposed by Arsenault and Boivin has some faults. On the one hand, the window function,

$$p_T(r) = \begin{cases} 1, & -T/2 < r < T/2\\ 0, & \text{otherwise} \end{cases}, \tag{3}$$

was defined for both positive and negative r. Note, however, that since the variable r was defined as the square of the actual radial coordinate, it must be a positive number. Moreover, Arsenault and Boivin used the function $p_{T/2}(r - T/2)$, which, by definition of $p_T(r)$, equals unity for T/4 < r < 3T/4 and zero otherwise. Nevertheless, it seems evident that the authors meant the function that was equal to unity in the interval [0, T].

In the derivation of the AST presented by Landgrave and Berriel-Valdos, the authors deliberately rejected the method of Fourier expansion of the pupil function $F(\rho^2)$, which is one of the factors in the integrand of the integral



Fig. 1. Geometry of the problem. A pupil filter, or diffracting aperture, $F(\rho)$, is illuminated by a spherical wave that converges to the focal point F. In Ref. 6 it is shown that the amplitude distribution along the *AB* segment is proportional to the one-dimensional FT of the zero-order circular-harmonic component of $F(\rho, \theta)$, calculated for the point A, which is the center of the circular-harmonic expansion. Point B is located in the observation plane, which is at a distance z from the image or focal plane.

$$G(\delta, v) = 2 \int_0^1 F(\rho^2) \exp(-i2\pi\delta\rho^2) J_0(\rho v) \rho d\rho.$$

(4) [2-2.11]

where

$$C_m = \int_0^1 F(\rho) \exp(-2\pi i m \rho) d\rho = G(m, 0).$$
(9)

(8)

The series in Eq. (8) converges to a periodic function whose unit cell is precisely $F_p(\rho)$. When Eq. (9) is substituted into Eq. (8), Eq. (8) into Eq. (7), and the inverse change of variable $\rho \rightarrow \rho^2$ is made, the 3D distribution of the amplitude *G* in the vicinity of an axial image point takes its final form given by Eq. [2-3.7]:

 $F_p(\rho) = p(\rho) \sum_{m=-\infty}^{+\infty} C_m \exp(i2\pi m\rho),$

$$G(\delta, v) = \int_{0}^{\infty} p(\rho) \left[\sum_{m=-\infty}^{\infty} G(m, 0) \exp(2\pi i m \rho) \right]$$

$$\times \exp(-i2\pi\delta\rho) J_{0}(v \rho^{1/2}) d\rho$$

$$= \sum_{m=-\infty}^{\infty} G(m, 0) \int_{0}^{1} \exp[-2\pi i (\delta - m) \rho]$$

$$\times J_{0}(v \rho^{1/2}) d\rho$$

$$= 2 \sum_{m=-\infty}^{\infty} G(m, 0) \int_{0}^{1} \exp[-2\pi i (\delta - m) \rho^{2}]$$

$$\times J_{0}(v \rho) \rho d\rho$$

$$= \sum_{m=-\infty}^{\infty} G(m, 0) L(\delta - m, v), \qquad (10)$$

where the definition of L given by Eq. [2-12.a] has been used. The above derivation eliminates the faults of those previously offered, is straightforward and remains in a well-established general scheme of derivation of sampling theorems.

Colombeau *et al.*⁵ presented a generalization of the AST. They extended the original result of Arsenault and

The reason was of a rather esthetic nature: "... to give a sense of unity to our treatment,...." Instead, they proposed an expansion of the product of the other two factors in the integrand above:

$$\exp(-i2\,\pi\,\delta\rho^2)J_0(v\,\rho) = \sum_{m=-\infty}^{\infty} f_m^{(a)}(\delta,\,v)\exp(-i2\,\pi m\,\rho^2),$$
(5)

and they did not present any justification of this quitespecific proposition. The derivation of the AST based on Eq. (5) is not particularly straightforward.

In the following, we present a new derivation of the AST that is equivalent to that of Arsenault and Boivin. The equivalence comes exactly from the fact that we apply here the Fourier-series expansion of the pupil function $F(\rho^2)$, which does not converge to this function but converges to a periodic function. Arsenault and Boivin effectively obtained the same Fourier series by taking the convolution of the pupil function with a comb function. In our derivation we use the terms and the notation of Landgrave and Berriel-Valdos. With the change of variable $\rho^2 \rightarrow \rho$ and using the window function

$$p(\rho) = \begin{cases} 1, & 0 < \rho < 1\\ 0, & \text{otherwise'} \end{cases}$$
(6)

we can rewrite Eq. [2-2.11] as

$$G(\delta, v) = \int_{0}^{\infty} p(\rho) F(\rho) \exp(-i2\pi\delta\rho) J_{0}(v\rho^{1/2}) d\rho.$$
(7)

By means of the Fourier-series expansion of the function $F(\rho)$ within the interval [0, 1], the pupil function $F_p(\rho) \equiv p(\rho)F(\rho)$ can be written as

Boivin to illuminating wave fronts of arbitrary form (not necessarily a converging spherical wave) and to pupils possessing no symmetry. They showed that in such cases there exists a FT relation between the first derivative, with respect to r^2 , of the zero-order circular-harmonic component of the wave amplitude emerging just behind the aperture and the amplitude distribution along any longitudinal axis Oz perpendicular to the diffractingaperture plane. The variables r and z are radial and axial cylindrical coordinates of the system originating at the point O at which the axis pierces the diffractingaperture plane and the circular-harmonic decomposition is performed. A similar FT relation was found by Andrés et al.⁶ They considered the classical configuration of Fig. 1 in the context of the focal-shift phenomenon and analyzed complex-amplitude distribution along any line directed toward the geometrical focus of a spherical wave front that passes a rotationally nonsymmetrical diffracting screen. The results of Refs. 2, 3, and 6 should be compared with that of Ref. 5 with caution. This is because the axial distance to the observation point is taken in Refs. 2, 3, and 6 from the focal plane or focal point, whereas in Ref. 5 it is taken from the diffracting-screen plane.

Neither Colombeau *et al.*⁵ nor Andrés *et al.*⁶ presented an explicit axial expansion. However, this is not strictly necessary because the FT relation always underlies a sampling expansion of a function defined in one domain provided that the Fourier-conjugate function in the other domain has compact support. This was the case in the problems considered by Colombeau *et al.* and Andrés *et al.* Further generalization along this line was presented by Piestun *et al.*,⁷ who presented 3D sampling expansions for generalized diffracting apertures and arbitrarily directed axes in the diffracted field.

We found ten papers,^{1,5,7-14} among them five papers published in OSA journals, that cite Ref. 3. Thus one may conclude that the pioneering result published by Arsenault and Boivin in 1967 was not forgotten and is still an inspiration for today's researchers, e.g., the authors of Ref. 1. The 3D radial sampling expansion remains an original result of Landgrave and Berriel-Valdos; nevertheless some results of Dragoman¹⁵ are worth mentioning in this context.

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