Analytical Evaluation of the Temporal Focal Shift for Arbitrary Pulse Shapes

Jesús Lancis, Vicent Climent, Jose Caraquitena, Genaro Saavedra, Manuel Martínez-Corral, and Pedro Andrés

***Abstract*—In this letter, we analyze the propagation of linearly chirped arbitrary-shaped light pulses through a parabolic dispersive medium to derive an analytical formula of assessing the location of the transverse plane where the pulse root-mean-square width is minimum. Closed form expressions for compressed pulses, which are independent of the input pulse shape, are demonstrated. In this way, we demonstrate that both the relative temporal focal shift and the minimum pulsewidth are solely determined by two factors, the temporal equivalent of the Fresnel number of the geometry and the pulse quality factor, i.e., the temporal analogue of the spatial M^2 beam quality factor. Some examples are discussed.**

***Index Terms*—Optical beam focusing, optical pulse compression, optical pulses, optical transient propagation.**

SECOND-ORDER dispersion is the relevant physical mechanism that controls short pulse propagation if higher order dispersion and nonlinear effects are negligible. Under the above assumption, pulse evolution along the z axis is described by the linear Schrödinger-like equation [1], [2]

$$-i\frac{\partial \psi}{\partial z} = \frac{\beta_2}{2}\frac{\partial^2 \psi}{\partial \tau^2}. \tag{1}$$

In the above expression, $\psi(z,\tau)$ stands for the slowly varying amplitude of the pulse envelope and τ is the so-called proper time, i.e., $\tau = t - z\beta_1$. Of course, the dispersion properties of the parabolic dispersive medium are determined by the coefficients β_1 and β_2, which are defined, respectively, as the first and second derivative of the propagation constant $\beta(\omega)$, evaluated at the carrier frequency ω_o. Equation (1) is similar to the paraxial wave equation that governs diffraction of light for one-dimensional (1-D) structures provided that the second-order dispersion coefficient β_2 is replaced by $\lambda/2\pi$ [1], [2]. On the basis of this analogy, one can transfer into the temporal domain many useful concepts that were first developed in the spatial domain, such as temporal imaging [3], [4], temporal self-imaging [5], [6], and pulse propagation description by means of the pulse quality factor [7].

Manuscript received February 9, 2004; revised June 7, 2004. This work was supported by the Dirección General de Investigación Científica y Técnica, Spain, under Project PB98-1049-C02-02, and by the Generalitat Valenciana (Grant grupos03/227). The work of J. Caraquitena was supported by the Direcaión General de Investigación Cientifica y Técnica.

J. Lancis and V. Climent are with the Departament de Ciències Experimentals, Universitat Jaume I, 12080 Castelló, Spain (e-mail: lancis@exp.uji.es; climent@exp.uji.es).

J. Caraquitena, G. Saavedra, M. Martínez-Corral, and P. Andrés are with the Departamento de Óptica, Universitat de València, 46100 Burjassot, Spain (e-mail: jose.caraquitena@uv.es; genaro.saavedra@uv.es; manuel.martinez@uv.es; pedro.andres@uv.es).

Digital Object Identifier 10.1109/LPT.2004.833907

In this letter, we focus our attention on the temporal focal-shift effect. Its spatial counterpart is a diffraction phenomenon observed when dealing with weakly focused beams. In these beams, the transverse plane where the intensity root-mean-square (rms) width is minimum does not appear at the geometric focal plane [8], but is displaced toward the aperture. In the temporal domain, a similar situation emerges when a linearly chirped pulse with smooth amplitude profile propagates through a parabolic dispersive medium. In practice, there are several techniques to achieve linear chirping of pulses, such as those based on electrooptic modulation [3] or sum-frequency generation [4]. Furthermore, linearly chirped parabolic pulses, which propagate self-similarly in a Yb-doped fiber amplifier, have also been confirmed by numerical simulations and experiments [9]. Linearly chirped hyperbolic secant pulses in optical fiber amplifiers with distributed parameters have also been demonstrated [10]. From a practical point of view, the above solutions of the nonlinear Schrödinger equation with gain are well-suited for efficient pulse compression in a second linear step using a parabolic dispersive medium to compensate for the linear chirp.

Although, it is widely recognized that the shortest pulse duration occurs when the spectral phase of the field is constant, the problem to find a closed-form expression to calculate both the location of the maximum compression and the compression degree for chirped pulses with an arbitrary profile is not an easy task. Explicit expressions for the evaluation of the compression degree of frequency-chirped Gaussian [11], [12] or super-Gaussian optical pulses [13] in parabolic dispersive medium have been reported. The goal of this letter is to report on an analytical formulation to calculate both the location of the maximum compression plane and the minimum rms width achievable which are independent of the input pulse shape. At this point, we claim that it is not possible the direct transposition of the equations derived for the spatial focal-shift effect to the temporal domain, since the spatial case is, in essence, a two-dimensional problem whereas the temporal one is a 1-D matter.

We begin by considering that the slowly varying pulse amplitude at the input plane $\psi(z=0,\tau)$ can be factorized as

$$\psi(z=0,\tau) = A(\tau)\exp(-i\pi C\tau^2). \tag{2}$$

Here, $A(\tau)$ is a Fourier-transform-limited signal that describes the initial pulse shape and C is the chirp parameter. The origin of the coordinate system in the temporal domain is selected so that $A^2(\tau)$ has zero mean time. Note that such an assumption does not imply any loss of generality. To our purpose, it is also convenient to express the chirp parameter as $C = 1/2\pi\beta_2 f$. In this way, the quantity f can be understood as the propagation distance in the quadratic medium such that $\psi(z=f,\tau)$ is equal to a Dirac delta function when $A(\tau)$ is a constant. Accordingly, we will refer to that point as the *temporal focus* of the chirped pulse, and to f as the *pulse focal length* [3]. Obviously, for the compressing geometry one must assume $f>0$, or, equivalently, C and β_2 have the same sign.

Next, we look for the transverse plane, located at the distance $z_{\min}=f+\Delta f$, where the rms width of the propagated pulse is minimum. We define Δf (positive or negative) as the oriented distance from the temporal focus to the above-mentioned plane. Moreover, we will refer to the quotient $\Delta f/f$ as the relative temporal focal shift by analogy with the spatial case. Note that, as in [1] and [2], we conduct the analysis by using the rms width rather than the full-width at half-maximum for the characterization of pulses.

The squared rms width in intensity of the propagated pulse $\sigma^2(z)$ is defined as $\sigma^2(z)=\langle\tau^2\rangle_z-\langle\tau\rangle_z^2$, where

$$\langle\tau^n\rangle_z=\frac{1}{I_o}\int_{-\infty}^{+\infty}\tau^n\,|\psi(z,\tau)|^2\,d\tau \tag{3}$$

is the normalized n-order moment of the instantaneous intensity field. In the above equation, I_o denotes the zeroth-order moment of the function $|\psi(z,\tau)|^2$, i.e., the total power associated with the pulse, which can be assumed to be unity.

Under the above assumptions, the evolution of the parameter $\sigma^2(z)$ through a quadratic dispersive medium is determined by the following parabolic law [13]:

$$\frac{\sigma^2(z)}{\sigma^2(0)}=1+A_1z+A_2z^2 \tag{4}$$

where $\sigma(0)$ is the pulsewidth at the input plane. The coefficients A_1 and A_2 are given by [13]

$$\begin{aligned}A_1&=\frac{-2\beta_2}{\sigma^2(0)}\mathrm{Im}\int_{-\infty}^{+\infty}\psi(z=0,\tau)\tau\frac{\partial\psi^*(z=0,\tau)}{\partial\tau}d\tau\\ A_2&=\frac{\beta_2^2}{\sigma^2(0)}\left\{\int_{-\infty}^{+\infty}\left|\frac{\partial\psi(z=0,\tau)}{\partial\tau}\right|^2d\tau\right.\\&\quad-\left[\mathrm{Im}\left(\int_{-\infty}^{+\infty}\psi(z=0,\tau)\right.\right.\\&\qquad\left.\left.\left.\times\frac{\partial\psi^*(z=0,\tau)}{\partial\tau}d\tau\right)\right]^2\right\}.\end{aligned} \tag{5}$$

Equation (4) is valid independently of the original pulse form. Note that (5) only involves the pulse amplitude at the input plane.

We then proceed to evaluate the relative temporal focal shift. We find from (4) that the minimum pulsewidth $\sigma_{\min}$ occurs for $z_{\min}=-(A_1/2A_2)$, and, by inserting (2) into (5), we finally find

$$\frac{\Delta f}{f}=-\frac{1}{1+16\pi^2\left(\frac{N_{\mathrm{eff}}}{P^2}\right)^2},\quad \frac{\sigma_{\min}}{\sigma(0)}=\left(-\frac{\Delta f}{f}\right)^{\frac{1}{2}}. \tag{6}$$

The derivation of both formulae is not difficult to be carried out and will be omitted here due to limited space. Note that in order to write the above equation, first, we have introduced the effective temporal Fresnel number of the chirped pulse $N_{\mathrm{eff}}=\sigma^2(0)/(2\pi\beta_2 f)$. We recognize N_{eff} as the number of semioscillations of the chirp function in (2) within the effective rms width of the pulse envelope $A(\tau)$. Furthermore, we have also considered the quality factor of the unchirped pulse, as defined in [7], which can be written as $P^2=2\sigma(0)\sigma_\omega$. Of course, σ_ω stands for the rms width of the Fourier transform of $A(\tau)$, i.e., $\sigma_\omega=\langle\omega^2\rangle$.

Equation (6) is the key result of this letter. Of course, when it is applied to the particular case of chirped Gaussian or super-Gaussian pulses, we reobtain previously reported results [11]–[13]. It is apparent that any chirped pulse suffers, in principle, from focal shift. It should be emphasized that the negative value of the ratio $\Delta f/f$ indicates that $\Delta f<0$ for the converging focusing geometry. Therefore, $z_{\min}<f$ and the temporal focus is always shifted toward the input plane.

Furthermore, the amount of the displacement is fully controlled by two parameters, N_{eff} and P^2. On the one hand, note that the modulus of the relative focal shift increases as N_{eff} decreases. On the other hand, and taking into account that the value of the time-bandwidth product for a Gaussian envelope is 1/2, the pulse quality factor P^2 can be rewritten as $P^2=\sigma(0)\sigma_\omega/(\sigma(0)\sigma_\omega)_{\mathrm{Gauss}}$. In this way, it is possible to infer that $P^2>1$. As its spatial homologue M^2, P^2 can be estimated from experimental autocorrelation measurements, even though these measurements do not give us access to the real pulse shape, and then provides a practical measurement of its quality, in comparison with that of the ideal Gaussian envelope. We note from (6) that, for a fixed N_{eff}, the bigger the deviation from the Gaussian envelope, the greater the shift of the temporal focus. Finally, (6) shows that the compression ratio of the pulse $\sigma_{\min}/\sigma(0)$ is directly related to the relative focal shift. In fact, the bigger the focal shift, the lower the compression ratio.

Now we apply our formulation for the evaluation of the focal shift for two envelopes. The first case is devoted to the super-Gaussian envelope family given by

$$A_m(\tau)=k_m\exp\left(-\left(\frac{\tau}{w_m}\right)^{2m}\right). \tag{7}$$

Here the order m is a natural number that fixes the degree of rectangularity of the super-Gaussian function. Note that $m=1$ corresponds to the Gaussian profile. We select the value of the peak amplitude k_m so that I_o in (3) is unity. Furthermore, the $1/e$ intensity half-width w_m is set as $w_m=\sqrt{\Gamma(1/2m)/\Gamma(3/2m)}\sigma(0)$, with $\Gamma(x)$ the gamma function. In this way, a fair comparison of compression of signals with equal initial rms intensity width is considered. In Fig. 1(a), we have plotted three elements of the above-described family. After some algebraical operations, we find that the value of the parameter P^2 is

$$P^2(m)=\frac{2m\sqrt{\Gamma\left(2-\frac{1}{2m}\right)\Gamma\left(\frac{3}{2m}\right)}}{\Gamma\left(\frac{1}{2m}\right)}. \tag{8}$$

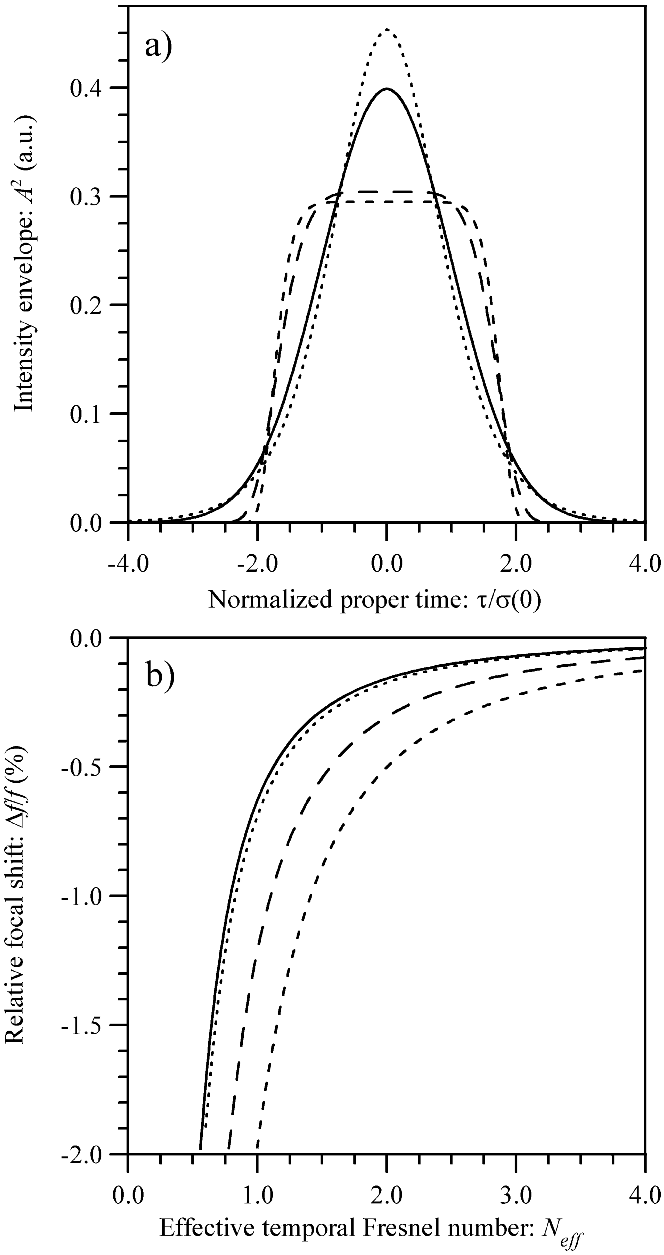

Fig. 1. (a) Plot of the pulse-envelope intensity at the input plane, and (b) relative temporal focal shift as a function of the effective Fresnel number for the above chirped pulses. Super-Gaussian profiles of order $m = 1$ (solid curve), $m = 3$ (long-dashed curve), and $m = 5$ (short-dashed curve), together with the sech-type amplitude profile (dotted curve) are dealt with.

From the above equation, we recognize that the quality factor increases with the order m, as is expected. In order to investigate the magnitude of the focal-shift effect predicted by (6), in Fig. 1(b), we have plotted the relative temporal focal shift against N_{eff}. In this plot, $P^2 = 1$ for $m = 1$, $P^2 = 1.39$ for $m = 3$, and $P^2 = 1.78$ for $m = 5$. We recognize that a higher focal shift, and thus, a lower compression ratio, is obtained as m increases.

We select as second example the sech-type amplitude profile

$$A(\tau) = k\,\text{sech}\left(\frac{\pi\tau}{2\sigma(0)\sqrt{3}}\right). \tag{9}$$

Its profile and the corresponding relative focal shift are also plotted, in dotted curves, in Fig. 1. Again power and width normalization are assumed. In this case, we obtain $P^2 = \pi/3$ and, thus, a nearly Gaussian behavior is achieved.

In summary, we have addressed in general terms the problem of linearly chirped pulse compression through a parabolic dispersive medium. We have expressed in an analytical manner that this effect is solely controlled by two dimensionless parameters the effective temporal Fresnel number and the pulse quality factor. These expressions are useful for applications dealing with compression of arbitrary pulses.

ACKNOWLEDGMENT

The authors would like to thank R. Martínez-Herrero of the Universidad Complutense of Madrid, Spain, for useful discussions. P. Andrés thanks D. Mendlovic from Civcom, Israel, for bringing to his attention the space–time analogy.

REFERENCES

[1] G. P. Agrawal, *Nonlinear Fiber Optics*, 2nd ed. San Diego, CA: Academic, 1995.
[2] J. C. Diels and W. Rudolph, *Ultrashort Laser Pulse Phenomena*. San Diego, CA: Academic, 1996.
[3] B. H. Kolner, "Space-time duality and the theory of temporal imaging," *IEEE J. Quantum Electron.*, vol. 30, pp. 1951–1963, Aug. 1994.
[4] C. V. Bennet and B. H. Kolner, "Principles of parametric temporal imaging—part I: systems configurations," *IEEE J. Quantum Electron.*, vol. 36, pp. 430–437, Apr. 2000.
[5] J. Azaña and M. A. Muriel, "Temporal self-imaging effects: Theory and application for multiplying pulse repetition rates," *IEEE J. Select. Topics Quantum Electron.*, vol. 7, pp. 728–744, July/Aug. 2001.
[6] C. B. Huang and Y. C. Lai, "Loss-less pulse intensity repetition-rate multiplication using optical all-pass filtering," *IEEE Photon. Technol. Lett.*, vol. 12, pp. 167–169, Feb. 2000.
[7] G. Rousseau, N. McCarthy, and M. Piché, "Description of pulse propagation in a dispersive medium by use of a pulse quality factor," *Opt. Lett.*, vol. 27, pp. 1649–1651, Sept. 2002.
[8] S. De Nicola, D. Anderson, and M. Lisak, "Focal shift effects in diffracted focused beams," *Pure Appl. Opt.*, vol. 7, pp. 1249–1259, Nov. 1998.
[9] M. E. Ferman, V. I. Kruglov, B. C. Thomsen, J. M. Dudley, and J. D. Harvey, "Self-similar propagation and amplification of parabolic pulses in optical fibers," *Phys. Rev. Lett.*, vol. 84, pp. 6010–6012, June 2000.
[10] V. I. Kruglov, A. C. Peacock, and J. D. Harvey, "Exact self-similar solutions of the generalized nonlinear Schrödinger equation with distributed coefficients," *Phys. Rev. Lett.*, vol. 90, p. 113902-1, Mar. 2003.
[11] D. Anderson, M. Lisak, and P. Anderson, "Nonlinearly enhanced chirp pulse compression in single-mode fibers," *Opt. Lett.*, vol. 10, pp. 134–136, Mar. 1985.
[12] M. Nakazawa, H. Kubota, A. Sahara, and K. Tamura, "Time-domain ABCD matrix formalism for laser mode-locking and optical pulse transmission," *IEEE J. Quantum Electron.*, vol. 34, pp. 1075–1081, July 1981.
[13] D. Anderson and M. Lisak, "Propagation characteristics of frequency-chirped super-Gaussian optical pulses," *Opt. Lett.*, vol. 11, pp. 569–571, Sept. 1986.