ALGEBRA OF THE LINEAR CITY MODEL WITH QUADRATIC TRANSPORT COSTS

1) Obtaining the demand functions:

We have two unknowns (x_1 and x_2), therefore we need two equations relating x_1 and x_2:

1. We know that: $x_1 + x_2 = L - a - b$ (1)
2. We use the indifferent consumer condition

$$U_{x,1} = U_{x,2}$$
$$r - p_1 - tx_1^2 = r - p_2 - tx_2^2$$
$$p_1 + tx_1^2 = p_2 + tx_2^2$$
$$p_1 - p_2 = t\left(x_2^2 - x_1^2\right)$$
$$p_1 - p_2 = t\left(x_2 - x_1\right)\left(x_2 + x_1\right)$$
$$p_1 - p_2 = \left(L - a - b\right)t\left(x_2 - x_1\right)$$
$$\frac{p_1 - p_2}{\left(L - a - b\right)t} = \left(x_2 - x_1\right) \quad (2)$$

Now, we can solve the system of equations given by (1) and (2) and obtain x_1 and x_2. We solve by reduction:

$$x_1 + x_2 = L - a - b$$
$$\left(x_2 - x_1\right) = \frac{p_1 - p_2}{\left(L - a - b\right)t}$$
$$\overline{}$$
$$2x_2 = L - a - b + \frac{p_1 - p_2}{\left(L - a - b\right)t}$$
$$x_2 = \frac{L - a - b}{2} + \frac{p_1 - p_2}{2t\left(L - a - b\right)}$$

and

$$x_1 = \left(L - a - b\right) - x_2 = \left(L - a - b\right) - \left[\frac{L - a - b}{2} + \frac{p_1 - p_2}{2t\left(L - a - b\right)}\right] =$$
$$= \frac{L - a - b}{2} + \frac{p_2 - p_1}{2t\left(L - a - b\right)}$$

Therefore,

$$d_1 = a + x_1 = a + \frac{L - a - b}{2} + \frac{p_2 - p_1}{2t\left(L - a - b\right)}$$

$$d_2 = b + x_2 = b + \frac{L-a-b}{2} + \frac{p_1 - p_2}{2t(L-a-b)}$$

2) Obtaining the Nash equilibrium in prices:

a) Solve the maximization problem of firm 1:

$$\underset{p_1}{Max}\, \pi_1 = (p_1 - c)d_1 = (p_1 - c)\left[a + \frac{L-a-b}{2} + \frac{p_2 - p_1}{2t(L-a-b)}\right]$$

$$F.O.C. \frac{d\pi_1}{dp_1} = a + \frac{L-a-b}{2} + \frac{p_2 - p_1}{2t(L-a-b)} - \frac{1}{2t(L-a-b)}(p_1 - c) = 0$$

$$a + \frac{L-a-b}{2} + \frac{p_2 - 2p_1 + c}{2t(L-a-b)} = 0$$

$$\frac{(2a + L - a - b)t(L-a-b)}{2t(L-a-b)} + \frac{p_2 - 2p_1 + c}{2t(L-a-b)} = 0$$

$$(L + a - b)t(L-a-b) + p_2 - 2p_1 + c = 0$$

$$p_1^*(p_2) = \frac{(L+a-b)t(L-a-b)}{2} + \frac{p_2}{2} + \frac{c}{2}$$ → Firm 1 reaction function

b) Solve the maximization problem of firm 2:

$$\underset{p_2}{Max}\, \pi_1 = (p_2 - c)d_2 = (p_2 - c)\left[b + \frac{L-a-b}{2} + \frac{p_1 - p_2}{2t(L-a-b)}\right]$$

$$F.O.C. \frac{d\pi_2}{dp_2} = b + \frac{L-a-b}{2} + \frac{p_1 - p_2}{2t(L-a-b)} - \frac{1}{2t(L-a-b)}(p_2 - c) = 0$$

$$b + \frac{L-a-b}{2} + \frac{p_1 - 2p_2 + c}{2t(L-a-b)} = 0$$

$$\frac{(2b + L - a - b)t(L-a-b)}{2t(L-a-b)} + \frac{p_1 - 2p_2 + c}{2t(L-a-b)} = 0$$

$$(L - a + b)t(L-a-b) + p_1 - 2p_2 + c = 0$$

$$p_2^*(p_1) = \frac{(L-a+b)t(L-a-b)}{2} + \frac{p_1}{2} + \frac{c}{2}$$ → Firm 2 reaction function

c) Solve the system of equation formed by the two reaction functions: we substitute p_2 by $p_2^*(p_1)$ in the reaction function of firm 1.

$$p_1^*(p_2) = \frac{(L+a-b)t(L-a-b)}{2} + \left[\frac{(L-a+b)t(L-a-b)}{4} + \frac{p_1}{4} + \frac{c}{4}\right] + \frac{c}{2}$$

$$p_1 = \frac{t\left(L-a-b\right)}{2}\left[\frac{\left(L+a-b\right)}{2}+\frac{\left(L-a+b\right)}{4}\right]+\frac{p_1}{4}+\frac{3c}{4}$$

$$p_1\left(1-\frac{1}{4}\right)=t\left(L-a-b\right)\left[\frac{2L+2a-2b+L-a+b}{4}\right]+\frac{3}{4}c$$

$$\frac{3}{4}p_1 = t\left(L-a-b\right)\left[\frac{3L+a-b}{4}\right]+\frac{3}{4}c$$

$$p_1 = t\left(L-a-b\right)\left[\frac{3L+a-b}{3}\right]+c \rightarrow p_1 = t\left(L-a-b\right)\left[L+\frac{a-b}{3}\right]+c$$

And now substituting in firm 2 reaction function we obtain p_2:

$$p_2^*\left(p_1\right)=\frac{\left(L-a+b\right)t\left(L-a-b\right)}{2}+\frac{t\left(L-a-b\right)\left[L+\frac{a-b}{3}\right]+c}{2}+\frac{c}{2}$$

$$p_2 = \frac{\left(L-a+b\right)t\left(L-a-b\right)}{2}+t\left(L-a-b\right)\left[\frac{3L+a-b}{6}\right]+\frac{c}{2}+\frac{c}{2}$$

$$p_2 = t\left(L-a-b\right)\left[\frac{\left(L-a+b\right)}{2}+\frac{3L+a-b}{6}\right]+c$$

$$p_2 = t\left(L-a-b\right)\left[\frac{3L-3a+3b}{6}+\frac{3L+a-b}{6}\right]+c$$

$$p_2 = t\left(L-a-b\right)\left[\frac{6L+2b-2a}{6}\right]+c \rightarrow p_2 = t\left(L-a-b\right)\left[L+\frac{b-a}{3}\right]+c$$