ON THE DIRECTLY AND SUBDIRECTLY IRREDUCIBLE MANY-SORTED ALGEBRAS

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Abstract. A theorem of general algebra asserts that every finite algebra can be represented as a product of a finite family of finite directly irreducible algebras. In this paper we show that the many-sorted counterpart of the above theorem is also true, but under the condition of requiring, in the definition of directly reducible many-sorted algebra, that the supports of the factors be included in the support of the many-sorted algebra. Moreover, we show that the theorem of Birkhoff according to which every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras, is also true for the many-sorted algebras.

1. Introduction

Some theorems of ordinary universal algebra can not be automatically generalized to world of many-sorted universal algebra, see e.g., [3] and [4] for the case of a representation theorem of Birkhoff-Frink, or [5] for that one of the injectivity of the insertion of the generators in the relatively free many-sorted algebras.

Our main aim in this paper is to prove, in the third section, that, under a mild condition on the supports of the factors in the definition of the concept of directly reducible many-sorted algebra, every finite many-sorted algebra can also be represented as a product of a finite family of finite directly irreducible many-sorted algebras. In addition, in the fourth section, for completeness, we show that the many-sorted counterpart of the well-known theorem of Birkhoff about the representation of every single-sorted algebra as a subdirect product of subdirectly irreducible single-sorted algebras, is also true for the many-sorted algebras.

In the second section we define those notions and constructions from the theory of many-sorted sets and algebras which are indispensable in order to attain the above indicated goals.

2. Many-sorted signatures, algebras, homomorphisms, subalgebras, products, congruences, and quotients.

In this section we begin by defining for an arbitrary, but fixed, set of sorts $S$, those concepts of the theory of $S$-sorted sets which we need in order to state the notions of many-sorted signature, algebra, subalgebra, homomorphism from a many-sorted algebra to another, product of a family of many-sorted algebras, and congruence on a many-sorted algebra.

Definition 1. Let $S$ be a set of sorts.

(1) A word on $S$ is a mapping $w: n \rightarrow S$, for some $n \in \mathbb{N}$. We denote by $S^*$ the underlying set of the free monoid on $S$, i.e., the set $\bigcup_{n \in \mathbb{N}} S^n$ of all mappings from the finite ordinals to $S$. Moreover, we call the unique mapping $\lambda: \emptyset \rightarrow S$, the empty word on $S$.

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(2) An S-sorted set \( A \) is a function \((A_s)_{s \in S}\) from \( S \) to \( \mathcal{U} \), where \( \mathcal{U} \) is a Grothendieck universe, fixed once and for all, and the support of \( A \), denoted by \( \text{supp}(A) \), is the set \( \{ s \in S \mid A_s \neq \varnothing \} \). An S-sorted set \( A \) is finite if \( \text{supp}(A) \) is finite and, for every \( s \in \text{supp}(A) \), \( A_s \) is finite, or, what is equivalent, if \( \prod_i A_i \) is finite. If \( A \) and \( B \) are S-sorted sets, then \( A \subseteq B \) if, for every \( s \in S \), \( A_s \subseteq B_s \) and \( A \subseteq B \) if \( A \) is finite and \( A \subseteq B \). Moreover, we denote by \( \text{Sub}(A) \) the set of all S-sorted sets \( X \) such that, for every \( s \in S \), \( X_s \subseteq A_s \). Finally, given a set \( I \) and an \( I \)-indexed family \((A_i)_{i \in I}\) of S-sorted sets, we denote by \( \prod_{i \in I} A_i \) the S-sorted set such that, for every \( s \in S \),

\[
(\prod_{i \in I} A_i)_s = \prod_{i \in I} A_i^s,
\]

by \( \bigcup_{i \in I} A_i \) the S-sorted set such that, for every \( s \in S \),

\[
(\bigcup_{i \in I} A_i)_s = \bigcap_{i \in I} A_i^s,
\]

and if \( I \) is nonempty, by \( \bigcap_{i \in I} A_i \) the S-sorted set such that, for every \( s \in S \),

\[
(\bigcap_{i \in I} A_i)_s = \bigcap_{i \in I} A_i^s.
\]

(3) Given a sort \( t \in S \), we call delta of Kronecker in \( t \), the S-sorted set \( \delta^t = (\delta^t_s)_{s \in S} \) defined, for every \( s \in S \), as:

\[
\delta^t_s = \begin{cases} 1, & \text{if } s = t; \\ \varnothing, & \text{otherwise.} \end{cases}
\]

(4) An S-sorted set \( A \) is subfinal if, for every \( s \in S \), \( \text{card}(A_s) \leq 1 \).

(5) If \( A \) and \( B \) are S-sorted sets, an S-sorted mapping from \( A \) to \( B \) is an \( s \)-indexed family \( f = (f_s)_{s \in S} \), where, for every \( s \in S \), \( f_s \) is a mapping from \( A_s \) to \( B_s \).

(6) An S-sorted equivalence on \( A \) is a subset \( \Phi \) of \( A \times A \) such that, for every \( s \in S \), \( \Phi_s \) is an equivalence on \( A_s \). We denote by \( \text{Eqv}(A) \) the set of S-sorted equivalences on the S-sorted set \( A \) and by \( \text{Eqv}(A, \subseteq) \) the ordered set \( (\text{Eqv}(A), \subseteq) \). Moreover, \( A/\Phi \), the S-sorted quotient set of \( A \) modulus \( \Phi \), is \((A_s/\Phi_s)_{s \in S}\).

For every set of sorts \( S \), the support of an S-sorted set \( A \) is a subset of \( S \), hence it really a mapping \( \text{supp}: \mathcal{U}^S \rightarrow \text{Sub}(S) \). In the following proposition we gather together some useful properties of the mapping \( \text{supp} \).

**Proposition 1.** Let \( S \) be a set of sorts, \( A, B \) be two S-sorted sets, \((A^i)_{i \in I}\) a family of S-sorted sets, and \( \Phi \) an S-sorted equivalence on an \( A \). Then the following properties hold:

1. If \( A \subseteq B \), then \( \text{supp}(A) \subseteq \text{supp}(B) \).
2. \( \text{supp}(\varnothing_{s \in S}) = \varnothing \).
3. \( \text{supp}(\bigcup_{i \in I} A^i) = \bigcup_{i \in I} \text{supp}(A^i) \).
4. If \( I \) is nonempty, \( \text{supp}(\bigcap_{i \in I} A^i) = \bigcap_{i \in I} \text{supp}(A^i) \).
5. \( \text{supp}(\prod_{i \in I} A^i) = \bigcap_{i \in I} \text{supp}(A^i) \).
6. \( \text{supp}(A) - \text{supp}(B) \subseteq \text{supp}(A - B) \).
7. \( \text{Hom}(A, B) \neq \varnothing \iff \text{supp}(A) \subseteq \text{supp}(B) \).
8. \( \text{supp}(A) = \text{supp}(A/\Phi) \).

Following this we define the concepts of many-sorted signature, algebra, and homomorphism.

**Definition 2.** A many-sorted signature is a pair \((S, \Sigma)\), where \( S \) is a set of sorts and \( \Sigma \) an S-sorted signature, i.e., a function from \( S^* \times S \) to \( \mathcal{U} \) which sends a pair \((w, s) \in S^* \times S \) to the set \( \Sigma_{w,s} \) of the formal operations of arity \( w \), sort (or coarity)
s, and rank (or biarity) \((w, s)\). Sometimes we will write \(\sigma : w \longrightarrow s\) to indicate that the formal operation \(\sigma\) belongs to \(\Sigma_{w, s}\). From now on, to shorten notation, we will write \(\Sigma\) instead of \((\Sigma, \Sigma)\).

**Definition 3.** Let \(\Sigma\) be a many-sorted signature. Then

1. The \(S^* \times S\)-sorted set of the finitary operations on an \(S\)-sorted set \(A\), denoted by \(\text{HOp}_S(A)\), is
   \[
   (\text{Hom}(A_w, A_s))_{(w, s) \in S^* \times S},
   \]
   where \(A_w = \prod_{i \in |w|} A_{w_i}\), with \(|w|\) denoting the length of the word \(w\).
2. A structure of \(\Sigma\)-algebra on an \(S\)-sorted set \(A\) is a family \(F = (F_{w,s})_{(w,s) \in S^* \times S}\), where, for \((w,s) \in S^* \times S\), \(F_{w,s}\) is a mapping from \(\Sigma_{w,s}\) to \(\text{Hom}(A_w, A_s)\).
   For a pair \((w,s) \in S^* \times S\) and a formal operation \(\sigma \in \Sigma_{w,s}\), in order to simplify the notation, the operation from \(A_w\) to \(A_s\) corresponding to \(\sigma\) under \(F_{w,s}\) will be written as \(F_{\sigma}\) instead of \(F_{w,s}(\sigma)\).
3. A \(\Sigma\)-algebra is a pair \((A, F)\), abbreviated to \(A\), where \(A\) is an \(S\)-sorted set and \(F\) a structure of \(\Sigma\)-algebra on \(A\).
4. A \(\Sigma\)-homomorphism from \(A\) to \(B\), where \(B = (B, G)\), is a triple \((A, f, B)\), abbreviated to \(f : A \longrightarrow B\), where \(f\) is an \(S\)-sorted mapping from \(A\) to \(B\) such that, for every \((w,s) \in S^* \times S\), \(\sigma \in \Sigma_{w,s}\), and \((a_i)_{i \in |w|} \in A_w\) we have that
   \[
   f_s(F_{\sigma}((a_i)_{i \in |w|})) = G_{\sigma}(f_w((a_i)_{i \in |w|})),
   \]
   where \(f_w\) is the mapping \(\prod_{i \in |w|} f_{w_i}\) from \(A_w\) to \(B_w\) which sends \((a_i)_{i \in |w|}\) in \(A_w\) to \((f_{w_i}(a_i))_{i \in |w|}\) in \(B_w\).

We denote by \(\text{Alg}(\Sigma)\) the category of \(\Sigma\)-algebras.

Sometimes, to avoid any confusion, we will denote the structures of \(\Sigma\)-algebra of the \(\Sigma\)-algebras \(A, B, \ldots\), by \(F^A, F^B, \ldots\), respectively, and the components of \(F^A, F^B, \ldots\), as \(F^A_\sigma, F^B_\sigma, \ldots\), respectively.

Next we define the concept of subalgebra of a many-sorted algebra.

**Definition 4.** Let \(A\) be a \(\Sigma\)-algebra and \(X \subseteq A\).

1. Let \(\sigma\) be such that \(\sigma : w \longrightarrow s\), i.e., a formal operation in \(\Sigma_{w,s}\). We say that \(X\) is closed under the operation \(F_\sigma : A_w \longrightarrow A_s\) if, for every \(a \in X_w\), \(F_\sigma(a) \in X_s\).
2. We say that \(X\) is a subalgebra of \(A\) if \(X\) is closed under the operations of \(A\). We denote by \(\text{Sub}(A)\) the set of all subalgebras of \(A\).

Following this we recall the concept of product of a family of many-sorted algebras.

**Definition 5.** Let \((A^i)_{i \in I}\) be a family of \(\Sigma\)-algebras, where, for \(i \in I\), \(A^i = (A^i, F^i)\).

1. The product of \((A^i)_{i \in I}\), denoted by \(\prod_{i \in I} A^i\), is the \(\Sigma\)-algebra \((\prod_{i \in I} A^i, F)\) where, for every \(\sigma : w \longrightarrow s\) in \(\Sigma\), \(F_\sigma\) is defined as
   \[
   F_\sigma \left\{ (\prod_{i \in I} A^i)_{w} \longrightarrow \prod_{i \in I} A^i_{\sigma} \left\{ (a_{\alpha} : \alpha \in |w|) \longmapsto (F^i_{\sigma}(a_{\alpha}) : \alpha \in |w|) \right\} \right\} \in I
   \]
2. The \(i\)-th canonical projection, \(pr^i\), is the homomorphism from \(\prod_{i \in I} A^i\) to \(A^i\) defined, for every \(s \in S\), as follows
   \[
   pr^i : \left\{ \prod_{i \in I} A^i \longrightarrow A^i \right\} (a_i : i \in I) \longmapsto a_i
   \]

We define next the concept of subfinal many-sorted algebra, since it will be used in the following section in an essential way.
Definition 6. A $\Sigma$-algebra $A$ is subfinal if, for every $\Sigma$-algebra $B$, there is at most a homomorphism from $B$ to $A$.

We point out that the subfinal many-sorted algebras are subobjects of the final many-sorted algebra, therefore their underlying many-sorted sets are subfinal.

We define now the concepts of many-sorted congruence on a many-sorted algebra and of many-sorted quotient algebra of a many-sorted algebra modulo a many-sorted congruence.

Definition 7. Let $A$ be a $\Sigma$-algebra and $\Phi$ an $S$-sorted equivalence on $A$. We say that $\Phi$ is an $S$-sorted congruence on $A$ if, for every $(w, s) \in (S^* - \{\lambda\}) \times S$, $\sigma: w \longrightarrow s$, and $a, b \in A_w$ we have that

$$\forall i \in \pi \cdot a_i \equiv_{\Phi_w} b_i$$

$$F_\sigma(a) \equiv_{\Phi_s} F_\sigma(b)$$

We denote by $\text{Cgr}(A)$ the set of $S$-sorted congruences on $A$ and by $\text{Cgr}(A)$ the ordered set $(\text{Cgr}(A), \subseteq)$.

Definition 8. Let $A$ be a $\Sigma$-algebra and $\Phi \in \text{Cgr}(A)$. The many-sorted quotient algebra of $A$ modulus $\Phi$, $A/\Phi$, is the $\Sigma$-algebra $((A_s/\Phi_s)_{s \in S}, F')$ where, for every $\sigma: w \longrightarrow s$ in $\Sigma$, the operation $F_\sigma: (A/\Phi)_w \longrightarrow A_s/\Phi_s$ is defined, for every $([a_i]_{\Phi_{w(i)}})_{i \in \pi} \in (A/\Phi)_w$, as follows

$$F_\sigma \left\{ \begin{array}{c} (A/\Phi)_w \longrightarrow A_s/\Phi_s \\ ([a_i]_{\Phi_{w(i)}})_{i \in \pi} \longrightarrow [F_\sigma(a_i) \mid i \in \pi]_{\Phi_s} \end{array} \right\}$$

3. Directly irreducible many-sorted algebras.

In this section we show that every finite many-sorted algebra is isomorphic to a finite product of finite directly irreducible many-sorted algebras.

Unlike that which happens for single-sorted algebras, there exists subfinal, but not final, many-sorted algebras that are isomorphic to products of nonempty families of nonsubfinal many-sorted algebras, and this is so because the supports of the factors can strictly contain the support of the product. This suggest that in the definition of directly reducible many-sorted algebra we should require that the supports of the factors of the product be included in the support of the many-sorted algebra under consideration. This additional condition will allow us to obtain the theorem about the representation of a finite many-sorted algebra as a product of a finite family of finite directly irreducible many-sorted algebras.

Definition 9. Let $A$ be a $\Sigma$-algebra. We say that $A$ is directly reducible if $A$ is isomorphic to a product of two nonsubfinal $\Sigma$-algebras such that their supports are included in that of $A$. If $A$ is not directly reducible, then we will say that $A$ is directly irreducible.

Obviously, every subfinal $\Sigma$-algebra is directly irreducible. Moreover, every finite $\Sigma$-algebra $A$ such that, for some $S \in S$, $\text{card}(A_s)$ is a prime number is also directly irreducible.

As for single-sorted algebras, we define the factorial congruences on a many-sorted algebra, from which we will obtain a characterization of the directly irreducible many-sorted algebras.

Definition 10. Let $\Phi$ and $\Psi$ be two congruences on a $\Sigma$-algebra $A$. We say that $\Phi$ and $\Psi$ are a pair of factorial congruences on $A$ if they satisfy the following
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conditions:

\[ \Phi \land \Psi = \Delta_A, \]
\[ \Phi \circ \Psi = \Psi \circ \Phi, \]
\[ \Phi \lor \Psi = \nabla_A. \]

**Proposition 2.** Let \( A \) and \( B \) be two \( \Sigma \)-algebras. Then the kernels of the canonical projections from \( A \times B \) to \( A \) and \( B \), denoted by \( \text{Ker}(pr_0) \) and \( \text{Ker}(pr_1) \), respectively, are a pair of factorial congruences on \( A \times B \).

**Proof.** Let \( f: A \longrightarrow A/\Phi \times A/\Psi \) be the \( S \)-sorted mapping defined, for every \( s \in S \) and \( a \in A_{s \in S} \), as \( f_s(a) = ([a]_\Phi, [a]_\Psi) \). It is obvious that \( f \) is a homomorphism. Moreover, if \( f_s(a) = f_s(b) \), then \( (a, b) \in \Phi_s \) and \( (a, b) \in \Psi_s \), hence \( f \) is injective. Finally, if \( a, b \in A_s \), then, because the congruences are such that \( \Phi \circ \Psi = \Psi \circ \Phi \), there exists an \( c \in A_s \) such that \( (a, c) \in \Phi_s \) and \( (c, b) \in \Psi_s \), hence \( f_s(c) = ([a]_\Phi, [a]_\Psi) \) and \( f \) is surjective. \( \square \)

**Proposition 3.** If \( \Phi \) and \( \Psi \) is a pair of factorial congruences on \( A \), then we have that \( A \cong A/\Phi \times A/\Psi \).

**Proof.** Let \( f: A \longrightarrow A/\Phi \times A/\Psi \) be the \( S \)-sorted mapping defined, for every \( s \in S \) and \( a \in A_{s \in S} \), as \( f_s(a) = ([a]_\Phi, [a]_\Psi) \). It is obvious that \( f \) is a homomorphism. Moreover, if \( f_s(a) = f_s(b) \), then \( (a, b) \in \Phi_s \) and \( (a, b) \in \Psi_s \), hence \( f \) is injective. Finally, if \( a, b \in A_s \), then, because the congruences are such that \( \Phi \circ \Psi = \Psi \circ \Phi \), there exists an \( c \in A_s \) such that \( (a, c) \in \Phi_s \) and \( (c, b) \in \Psi_s \), hence \( f_s(c) = ([a]_\Phi, [a]_\Psi) \) and \( f \) is surjective. \( \square \)

**Proposition 4.** Let \( A \) be a \( \Sigma \)-algebra. Then \( A \) is directly irreducible if and only if \( \Delta_A \) and \( \nabla_A \) is the only pair of factorial congruences on \( A \).

**Theorem 1.** Every finite \( \Sigma \)-algebra is isomorphic to a product of a finite family of finite directly irreducible \( \Sigma \)-algebras.

**Proof.** Let \( A \) be a finite \( \Sigma \)-algebra. If \( \text{card}(\prod_{s \in S} A_s) = 0 \), then \( A \) is irreducible. Let \( A \) be such that \( \text{card}(\prod_{s \in S} A_s) = n + 1 \), with \( n \geq 0 \), and let us assume the theorem for every finite \( \Sigma \)-algebra \( B \) such that \( \text{card}(\prod_{s \in S} B_s) \leq n \). If \( A \) is directly irreducible, then we are finished. Otherwise, we have that \( A \cong A^0 \times A^1 \), with \( A^0 \) and \( A^1 \) nonsubfinal \( \Sigma \)-algebras and such that, for \( i = 0, 1 \), \( \text{supp}(A^i) \subseteq \text{supp}(A) \).

Let \( A^i|T \) be, for \( i = 0, 1 \) and \( T = \text{supp}(A) = \text{supp}(A^0) \cap \text{supp}(A^1) \), the \( \Sigma \)-algebra \( (A^i|T, F^{A^i}|T) \), where \( A^i|T \), for every \( s \in S \), is defined as

\[
(A^i|T)_s = \begin{cases} A^i, & \text{if } s \in T; \\
\varnothing, & \text{otherwise,} \end{cases}
\]

and \( F^{A^i}|T \) is defined, for every \( (w, s) \in S^* \times S \), as

\[
F^{A^i}|T \begin{cases} \Sigma_{w,s} \longrightarrow \text{Hom}(A^i|T)_w, (A^i|T)_s) \\
\sigma \longmapsto \begin{cases} F^{A^i}(\sigma), & \text{if } \text{Im}(w) \subseteq T \text{ and } s \in T; \\
\alpha_{A^i} : \varnothing \longrightarrow A_s, & \text{if } \text{Im}(w) \nsubseteq T, \end{cases} \end{cases}
\]

where \( \alpha_{A^i} \) is the unique mapping from \( \varnothing \) to \( A_s \). The definition of the many-sorted structure is sound since, for \( \sigma: w \longrightarrow s \), both \( \text{Im}(w) \subseteq T \) and \( s \notin T \) can not occur.

From this it follows that \( A \cong A^0|T \times A^1|T \) and, for \( i = 0, 1 \), that \( \text{card}(A^i|T) \leq \text{card}(A) \), hence, by the induction hypothesis, we can assert that

\[
A^0|T \cong B^0 \times \cdots \times B^{p-1} \]
\[
A^1|T \cong C^0 \times \cdots \times C^{q-1} \]

where, for \( j \in p \) and \( h \in q \), \( B^j \) and \( C^k \) are directly irreducible. Therefore,

\[
A \cong B^0 \times \cdots \times B^{p-1} \times C^0 \times \cdots \times C^{q-1}. \]

\( \square \)
4. Subdirectly irreducible algebras.

In this last section we extend to the many-sorted algebras that theorem of Birkhoff according to which every algebra is isomorphic to a subdirect product of subdirectly irreducible algebras. To this end we begin by defining the concept of subdirect product of a family of many-sorted algebras.

**Definition 11.** A $\Sigma$-algebra $A$ is a subdirect product of a family of $\Sigma$-algebras $(A^i)_{i \in I}$ if it satisfies the following conditions

1. $A$ is a subalgebra of $\prod_{i \in I} A^i$.
2. For every $i \in I$, $\text{pr}^i[A]$ is surjective.

On the other hand, we will say that an embedding $f: A \to \prod_{i \in I} A^i$ is a subdirect embedding if $f[A]$ is a subdirect product of $(A^i)_{i \in I}$.

**Proposition 5.** Let $A$ be a $\Sigma$-algebra and $(\Phi^i)_{i \in I}$ a family of congruences on $A$. Then $A/\bigcap_{i \in I} \Phi^i$ can be subdirectly embedded into $\prod_{i \in I} A/\Phi^i$.

**Proof.** Let $f^i$ be, for every $i \in I$, the unique homomorphism from $A/\bigcap_{i \in I} \Phi^i$ into $A_i/\Phi^i$ such that $f^i \circ \text{pr}^i(1) = \text{pr}^i$. Then the unique homomorphism $f: A \to \prod_{i \in I} A_i/\Phi^i$ determined by the universal property of the product, is a subdirect embedding.

**Corollary 1.** Let $A$ be a $\Sigma$-algebra and $(\Phi^i)_{i \in I}$ a family of congruences on $A$ such that $\bigcap_{i \in I} \Phi^i = \Delta_A$. Then $(\text{pr}^i)_{i \in I}: A \to \prod_{i \in I} A_i/\Phi^i$ is a subdirect embedding.

**Definition 12.** Let $A$ be a $\Sigma$-algebra. We say that $A$ is subdirectly irreducible if, for every subdirect embedding $f$ of $A$ into the cartesian product $\prod_{i \in I} A^i$ of a nonempty family of $\Sigma$-algebras $(A^i)_{i \in I}$, there exists an index $i \in I$ such that the homomorphism $\text{pr}^i \circ f: A \to A^i$ is injective.

**Proposition 6.** A $\Sigma$-algebra $A$ is subdirectly irreducible if and only if $A$ is subfinal or there exists a minimum congruence in $\text{Cgr}(A) - \{\Delta_A\}$.

**Proof.** If $A$ is not subfinal and $\text{Cgr}(A) - \{\Delta_A\}$ has not a minimum congruence, then $\bigcap\{\text{Cgr}(A) - \{\Delta_A\}\} = \Delta_A$. Let $I = \text{max} \text{mthrn} \text{Cgr}(A) - \{\Delta_A\}$ be, then the canonical mapping $(\text{pr}^i)_{i \in I}: A \to \prod_{i \in I} A_i/\Phi^i$ is, by the Corollary 1, a subdirect embedding and since, for every $\Phi \in I$, the canonical projections $\text{pr}^i: A \to A_i/\Phi$ are not injectives, it follows that $A$ is not subdirectly irreducible. Therefore, if $A$ is subdirectly irreducible, then $A$ is subfinal or there exists a minimum congruence in $\text{Cgr}(A) - \{\Delta_A\}$.

If $A$ is subfinal, then it is subdirectly irreducible, since if $f$ is a subdirect embedding of $A$ into the cartesian product $\prod_{i \in I} A^i$ of a nonempty family of $\Sigma$-algebras $(A^i)_{i \in I}$, then, for every $i \in I$, $\text{pr}^i$ is surjective and $\text{supp}(A) = \text{supp}(\prod_{i \in I} A^i) = \text{supp}(A^i)$, hence $A \cong \prod_{i \in I} A_i \cong A^i$, for every $i \in I$.

Finally, let us suppose that there exists a minimum congruence $\Phi$ in $\text{Cgr}(A) - \{\Delta_A\}$, hence, necessarily, $\Phi = \bigcap\{\text{Cgr}(A) - \{\Delta_A\}\}(\neq \Delta_A)$ and $A$ is not subfinal. Therefore we can choose a sort $s \in S$ and a pair $(a, b) \in \Phi_s$ such that $a \neq b$.

Let $f: A \to \prod_{i \in I} A^i$ be a subdirect embedding of $A$ into the cartesian product $\prod_{i \in I} A^i$ of the $\Sigma$-algebras $A^i$. Then there exists an index $i \in I$ such that $(\text{pr}^i_s \circ f_s)(a) \neq (\text{pr}^i_s \circ f_s)(b)$, since, otherwise, $f_s(a) = f_s(b)$ and therefore $a = b$, which is a contradiction. From this follows that $(a, b) \notin \ker(\text{pr}^i_s \circ f_s)$ and, since $(a, b) \in \Phi_s$, that $\Phi \not\subseteq \ker(\text{pr}^i_s \circ f_s)$, thus $\ker(\text{pr}^i \circ f) = \Delta_A$ and, consequently, $\text{pr}^i \circ f: A \to A^i$ is injective. From this we can assert that $A$ is subdirectly irreducible. □
Remark. If for a \( \Sigma \)-algebra \( A \) the lattice \((\text{Cgr}(A) - \{\Delta_A\}, \subseteq)\) has a minimum \( \Phi \), then the lattice \( \text{Cgr}(A) \) has the form:

\[
\Phi = \bigcap \{\text{Cgr}(A) - \{\Delta_A\}\}
\]

where \( \nabla_A \) is \( A \times A \), the maximum congruence on \( A \). The congruence \( \Phi \), called the monolith of \( A \) and denoted by \( M_A \), has the property that \( M_A = \text{Cg}_A(\delta_s, (a, b)) \), for every \( s \in S \) and every \( (a, b) \in M_s \), with \( a \neq b \), where \( \delta_s, (a, b) \) is the \( S \)-sorted set which has as \( s \)-th coordinate the set \( \{(a, b)\} \) and as \( t \)-th coordinate, for \( t \neq s \), the empty set, and \( \text{Cg}_A \) the generated congruence operator for \( A \).

We define next the simple many-sorted algebras, that are a special kind of subdirectly irreducible algebra.

**Definition 13.** Let \( A \) be a \( \Sigma \)-algebra. We say that \( A \) is simple if \( A \) is subfinal or \( \text{Cgr}(A) \) has exactly two congruences. Moreover, we say that a congruence \( \Phi \) on \( A \) is maximal if the interval \([\Phi, \nabla_A]\) in the lattice \( \text{Cgr}(A) \) has exactly two congruences.

As for single-sorted algebras, also many-sorted algebras it is true that the quotient many-sorted algebra of a many-sorted algebra by a congruence on it is simple if and only if the congruence is maximal or the congruence is the maximum congruence on the many-sorted algebra.

**Proposition 7.** Let \( A \) be a \( \Sigma \)-algebra and \( \Phi \) a congruence on \( A \). Then \( A/\Phi \) is simple if and only if \( \Phi \) is a maximal congruence on \( A \) or \( \Phi = \nabla_A \).

In the following proposition we gather together some relations between the simple, the subdirectly irreducible, and the directly irreducible many-sorted algebras.

**Proposition 8.** Every simple many-sorted algebra is subdirectly irreducible and every subdirectly irreducible many-sorted algebra is directly irreducible.

We prove next, as was announced in the introduction of this paper, the many-sorted counterpart of the well-known theorem of Birkhoff about the representation of every single-sorted algebra as a subdirect product of subdirectly irreducible single-sorted algebras, is also true for the many-sorted algebras.

**Theorem 2** (Birkhoff). Every many-sorted algebra is isomorphic to a subdirect product of a family of subdirectly irreducible many-sorted algebras.

**Proof.** Since the subfinal \( \Sigma \)-algebras are subdirectly irreducibles, it is enough to consider nonsubfinal \( \Sigma \)-algebras. Let \( A \) be a nonsubfinal \( \Sigma \)-algebra and

\[
I = \bigcup_{s \in S} \{(s, (a, b)) \times (A_s^2 - \Delta_{A_s})\}
\]

that is nonempty, because \( A \) is nonsubfinal. Then, for every \( (s, (a, b)) \in I \), making use of the lemma of Zorn, there exists a congruence \( \Phi^{(s, (a, b))} \) on \( A \) such that \( \Phi^{(s, (a, b))} \cap \delta_s, (a, b) = (2)_{s \in S} \) and maximal with that property. Moreover, the congruence \( \Phi^{(s, (a, b))} \lor \text{Cg}_A(\delta_s, (a, b)) \) is the minimum in \([\Phi^{(s, (a, b))}, \nabla_A] - \{\Phi^{(s, (a, b))}\}\). Therefore, in the lattice \( \text{Cgr}(A/\Phi^{(s, (a, b))}) \), the congruence \( \Phi^{(s, (a, b))} \lor \text{Cg}_A(\delta_s, (a, b)) \) is the monolith of \( A/\Phi^{(s, (a, b))} \), that is subdirectly irreducible.
Since $\bigcap\{(s,(a,b)) \mid (s,(a,b)) \in I\} = \Delta_A$, we have, finally, that $A$ can be subdirectly embedded in $\prod(A/\Phi(s,(a,b)))_{(s,(a,b)) \in I}$, which is a product of subdirectly irreducible $\Sigma$-algebras. □

**Corollary 2.** Every finite many-sorted algebra is isomorphic to a subdirect product of a finite family of finite subdirectly irreducible many-sorted algebras.

**References**


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