GALOIS ACTION ON THE PRINCIPAL BLOCK AND GENERATION OF SYLOW 3-SUBGROUPS

EDEN KETCHUM, J. MIQUEL MARTÍNEZ, NOELIA RIZO, AND A. A. SCHAEFFER FRY

Abstract. In this paper, we prove one direction of a conjecture of Navarro–Rizo–Schaeffer Fry–Vallejo positing an algorithm to determine from the character table whether a finite group has 2-generated Sylow 3-subgroups. This gives further evidence of the blockwise version of the Galois–McKay conjecture (also known as the Alperin–McKay–Navarro conjecture). A key step involves proving the Isaacs–Navarro Galois conjecture for principal blocks for finite groups with a certain structure.

1. Introduction

In Problem 12 of his famous list of problems from 1963, Brauer asked what information could be gleaned about a Sylow p-subgroup P of a finite group G from its character table. In particular, the question of whether the number of generators of P could be obtained in this way has been studied in recent years, see e.g. [RSV20, NRSV21, MS23, Va23]. Many questions in this realm seem linked to the action of the Galois automorphism $\sigma \in \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ that fixes p'-roots of unity and sends p-power roots of unity to their $1+p$ power. For any cyclotomic extension $\mathbb{Q}_n := \mathbb{Q}(e^{2\pi i/n})$ of $\mathbb{Q}$, we keep the notation σ for the restriction of this automorphism to $\mathbb{Q}_n$. Here, we contribute to this line of problems by proving the following direction of the main conjecture from [NRSV21].

Theorem A. *Let G be a finite group and $P \in \mathrm{Syl}_3(G)$. Then $|P:\Phi(P)| = 9$ if the principal 3-block $B_0(G)$ contains exactly 6 or 9 σ-invariant characters of degree coprime to 3.*

As noted in [NRSV21], Theorem A (and its converse) would follow from the Alperin–McKay–Navarro conjecture [Nav04, Conj. B], or even the more restrictive version [IN02, Conj. D]. Hence, this result gives further evidence for these elusive conjectures. We remark that although the block-free version [IN02, Conj. C] was recently established in [RS25], the blockwise version (even for principal blocks) is much further from completion and has not yet been reduced to a problem on simple groups. For this reason, finding additional evidence of this blockwise version remains pertinent.

Theorem A can be thought of simultaneously as an extension of the main result of [NRSV21], which addressed the analogous question for $p = 2$, and of the main result of [GRSV25], which showed that $[P:P'] = 9$ if and only if $B_0(G)$ contains exactly 6 or 9 irreducible characters of degree coprime to 3.

Our proof of Theorem A uses the recent work of Ketchum [Ket25], which shows that the statement of Theorem A (and its converse) hold for almost simple groups. It also uses a reduction theorem and additional results on almost simple groups analogous to those in [GRSV25]. Moreover, the reduction relies on beautiful work of R. Brauer [Bra76] on the structure of groups having cyclic Sylow p-subgroups.

2020 *Mathematics Subject Classification.* 20C15, 20D20.

Key words and phrases. Galois action, principal block, Sylow subgroup.

The first and fourth-named authors acknowledge support of a CAREER grant from the U.S. National Science Foundation, Award No. DMS-2439897; the second and third-named authors acknowledge support from grant PID2022-137612NB-I00 funded by MCIN/AEI/ 10.13039/501100011033 and ERDF "A way of making Europe" and grant CIDEIG/2022/29 funded by Generalitat Valenciana. Part of this work was done while the second and third-named authors visited the University of Denver. They wish to thank the Department of Mathematics for their kind hospitality.

As mentioned above, [IN02, Conj. C] has been completed in [RS25]. Our next main theorem considers the principal block version, which is part of [IN02, Conj. D], and is a key step towards the proof of Theorem A. We believe it may be of independent interest. Let $\mathcal{H} \leq \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ be the Galois group considered in [Nav04].

Theorem B. *Let G be a finite group of order divisible by p, let $N \lhd G$ be an abelian p-group and assume G/N has cyclic Sylow p-subgroups and let $\tau \in \mathcal{H}$ be a p-power order element. Then τ fixes the same number of height-zero characters in $\mathrm{Irr}(B_0(G))$ as in $\mathrm{Irr}(B_0(\mathbf{N}_G(P)))$, where $P \in \mathrm{Syl}_p(G)$.*

In particular, [IN02, Conj. D] holds for principal blocks of groups satisfying the hypotheses of Theorem B. In fact, as explained in Remark 3.4, the same result holds by assuming only that every block of maximal defect of G/N satisfies [IN02, Conj. D], or [Nav04, Conj. B]. (See also [MMSV26, Prop. 5.4] for another related, but restrictive, case.)

The paper is structured as follows: Section 2 contains preliminary lemmas and results, as well as a Galois version of [GRSV25, Thm. B]. Theorem B is proved in Section 3. Theorem A is reduced to simple groups in Section 4 and finally completed in Section 5, where we also provide some evidence for the version of Theorem A for arbitrary blocks.

2. Auxiliary results

2.1. **Galois action and principal blocks.** We begin by collecting some basic facts about principal blocks. Throughout, given a prime p, $B_0(G)$ denotes the principal p-block of the finite group G. We denote by $\mathrm{Irr}(B_0(G))$ the set of irreducible characters in $B_0(G)$ and by $\mathrm{Irr}_0(B_0(G))$ the subset of those with height zero (that is, p'-degree). The set of σ-invariant characters in $\mathrm{Irr}_0(B_0(G))$ will be denoted $\mathrm{Irr}_{0,\sigma}(B_0(G))$, and we set $k_{0,\sigma}(B_0(G)) := |\mathrm{Irr}_{0,\sigma}(B_0(G))|$.

Lemma 2.1. *Let G be a finite group, let $N \lhd G$ and let p be a prime.*

- (i) $\mathrm{Irr}(B_0(G/N)) \subseteq \mathrm{Irr}(B_0(G))$, *and if N has order not divisible by p then* $\mathrm{Irr}(B_0(G/N)) = \mathrm{Irr}(B_0(G))$.
- (ii) *For any $\theta \in \mathrm{Irr}(B_0(N))$, there is some $\chi \in \mathrm{Irr}(B_0(G))$ lying over θ.*
- (iii) *For any $\chi \in \mathrm{Irr}(B_0(G))$, every constituent of χ_N lies in $B_0(N)$.*
- (iv) *If G/N is a p-group, then $B_0(G)$ is the unique block covering $B_0(N)$.*
- (v) *Assume $G = H_1 \times \cdots \times H_t$. Then*

$$\mathrm{Irr}(B_0(G)) = \{\theta_1 \times \cdots \times \theta_t \mid \theta_i \in \mathrm{Irr}(B_0(H_i))\}.$$

Proof. The first part of (i) follows from the discussion before [Nav98, Thm. 7.6], and the second part is [Nav98, Thm. 9.9(c)]. Part (ii) is [Nav98, Thm. 9.4], part (iii) is [Nav98, Thm. 9.2, Cor. 9.3] and part (iv) is [Nav98, Cor. 9.6]. Part (v) is [NT16, Lem. 2.6(b)]. □

Lemma 2.2. *Let G be a finite group and let $N = S_1 \times \cdots \times S_t$ be a minimal normal subgroup of G, where the S_i are transitively permuted by G. Suppose that $G/N = \langle xN \rangle$ is cyclic, $\mathbf{C}_G(N) = 1$ and $\mathbf{N}_G(S_i) = N$. Then $G/N \lesssim \mathrm{Out}(N) = \mathrm{Out}(S_1) \wr \mathsf{S}_t$. In particular, after a suitable reordering of the S_i's, xN acts on N as $(\alpha_1, \ldots, \alpha_t)\tau \in \mathrm{Out}(S_1) \wr \mathsf{S}_t$ where $\tau = (1...t)$, and we have*

- (i) $\alpha_1 \cdots \alpha_t = 1$*;*
- (ii) *if $\eta \in \mathrm{Irr}(S_1)$, then the character $\theta = \eta \times \eta^{\alpha_1} \times \eta^{\alpha_1\alpha_2} \times \cdots \times \eta^{\alpha_1 \cdots \alpha_t}$ is G-invariant.*

Proof. Since G acts transitively on the components of N, $G/N \leq \mathrm{Out}(N) \cong \mathrm{Out}(S_1) \wr \mathsf{S}_t$. Thus, xN seen as an element of $\mathrm{Out}(N)$ acts on characters of N like some $(\alpha_1, \alpha_2, \ldots, \alpha_r)\tau$ with $\alpha_i \in \mathrm{Out}(S)$ and $\tau \in \mathsf{S}_t$ of order t and after a suitable reordering of the elements we may assume $\tau = (1...t)$. Moreover, since $\mathbf{N}_G(S_1) = N$, then $o(xN) = |G : N| = t$ and we have that

$$1 = ((\alpha_1, \ldots, \alpha_t)\tau)^t = (\prod_{j=0}^{t-1} \alpha_{\tau^j(1)}, \ldots, \prod_{j=0}^{t-1} \alpha_{\tau^j(t)})\tau^t.$$

Therefore,

$$\prod_{j=0}^{t-1} \alpha_{\tau^j(1)} = \alpha_1 \cdots \alpha_t = 1,$$

proving part (i). Now, if $\eta \in \mathrm{Irr}(S)$, then let $\theta = \eta \times \eta^{\alpha_1} \times \eta^{\alpha_1\alpha_2} \times \cdots \times \eta^{\alpha_1\cdots\alpha_t}$ and notice that $\theta^x = \theta$ by part (ii). Since $G/N = \langle xN \rangle$, θ is G-invariant. $\square$

Theorem 2.3 (Alperin–Dade)**.** *Let $N \triangleleft G$ and $P \in \mathrm{Syl}_p(G)$. If p does not divide $|G : N|$ and $N\mathbf{C}_G(P) = G$, then restriction defines a bijection*

$$\mathrm{Irr}_{0,\sigma}(B_0(G)) \to \mathrm{Irr}_{0,\sigma}(B_0(N)).$$

Proof. The fact that restriction defines a bijection $\mathrm{Irr}(B_0(G)) \to \mathrm{Irr}(B_0(N))$ was proved when G/N is solvable in [Alp76] and without the solvability hypothesis in [Dad77]. Since $\chi_N^\sigma = (\chi^\sigma)_N$, the result follows. $\square$

Lemma 2.4. *Let G be a finite group $H \leq G$ with $p \nmid |G : H|$. Let $\tau \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$ have p-power order. Assume that $\mathbf{C}_G(Q) \subseteq H$ for $Q \in \mathrm{Syl}_p(H)$. If $\theta \in \mathrm{Irr}_{0,\tau}(B_0(H))$ then θ^G contains a constituent $\chi \in \mathrm{Irr}_{0,\tau}(B_0(G))$.*

Proof. This follows the proof of [MMSV26, Lem. 3.7] using that $\langle \tau \rangle$ has p-power order. $\square$

Lemma 2.5. *Let $N \triangleleft G$ be a p-subgroup contained in $P \in \mathrm{Syl}_p(G)$. Further, assume $\theta \in \mathrm{Lin}(N)$ is G-invariant. Let $\tau \in \mathrm{Gal}(\mathbb{Q}_{|G|}/\mathbb{Q})$ have p-power order and assume θ extends to $\lambda \in \mathrm{Irr}_{0,\tau}(P)$. Then θ extends to a $\langle \tau \rangle$-invariant character in $B_0(G)$.*

Proof. Let $H = P\mathbf{C}_G(P)$ and write $H = P \times X$. Letting $\hat{\lambda} = \lambda \times 1_X \in \mathrm{Irr}_{0,\tau}(B_0(H))$, we have that $\hat{\lambda}^G$ contains some $\chi \in \mathrm{Irr}_{0,\tau}(B_0(G))$ by Lemma 2.4. Now $\det(\chi^\tau) = \det(\chi)^\tau$ by [Nav18, Prob. 3.8], so $\delta := \det(\chi)^b$ (where b is as in the proof of [GRSV25, Lem. 2.5]) is $\langle \tau \rangle$-invariant, lies in $B_0(G)$ and extends θ. $\square$

The following is a principal block version of [MS23, Lem. 3].

Lemma 2.6. *Let G be a finite group with $\mathbf{O}_{p'}(G) = 1$ and let $P \in \mathrm{Syl}_p(G)$. Then*

$$K = \bigcap_{\chi \in \mathrm{Irr}_{0,\sigma}(B_0(G))} \ker(\chi) \leq \Phi(P).$$

Proof. Let $\lambda \in \mathrm{Irr}(P/\Phi(P))$. Write $P\mathbf{C}_G(P) = P \times X$ and let $\psi = \lambda \times 1_X \in \mathrm{Irr}_{0,\sigma}(B_0(P\mathbf{C}_G(P)))$. By Lemma 2.4 we have that λ^G contains a constituent $\chi \in \mathrm{Irr}_{0,\sigma}(B_0(G))$. By Frobenius reciprocity, $\chi_{P\mathbf{C}_G(P)}$ contains $\lambda \times 1_X$ so χ_P contains λ_P. Moreover $\chi_{P\cap K} = \chi(1)1_{P\cap K}$ contains $\lambda_{P\cap K}$ which shows that $\ker(\lambda)$ contains $P \cap K$. Therefore

$$P \cap K \subseteq \bigcap_{\lambda \in \mathrm{Irr}(P/\Phi(P))} \ker(\lambda) = \Phi(P).$$

By Tate's theorem [Nav18, Cor. 6.14], this implies that K is p-nilpotent. Then if K is not a p-group, $\mathbf{O}_{p'}(K) > 1$, which contradicts $\mathbf{O}_{p'}(G) = 1$. Therefore K is a normal p-subgroup so $K = P \cap K \subseteq \Phi(P)$, as desired. $\square$

The following is a variation of [GRSV25, Thm. 3.5].

Lemma 2.7. *Assume $N \triangleleft G$ is a direct product $N = S_1 \times \cdots \times S_t$ of nonabelian simple groups of order divisible by p transitively permuted by G. Let $\theta = \theta_1 \times \cdots \times \theta_t$ be G-invariant. If θ_1 extends to some σ-invariant irreducible character of p'-degree in $B_0(T)$ for all $S_1 \leq T \leq \mathrm{Aut}(S_1)_{\theta_1}$, then θ extends to some $\chi \in \mathrm{Irr}_{0,\sigma}(B_0(G))$*

Proof. Following the proof of [GRSV25, Thm. 3.5] we have that θ_1 extends to some $\hat{\theta}_1$ in $\mathrm{Irr}_{0,\sigma}(B_0(\mathbf{N}_G(S_1)))$ by hypothesis. Let χ be the tensor induced character $\chi = \hat{\theta}_1^{\otimes}$ (see [Nav18, Sec. 10.5]), which lies in $B_0(G)$ and extends θ (again by [GRSV25, Thm. 3.5]). By the formula in [GI83, Def. 2.1], $\mathbb{Q}(\chi) \subseteq \mathbb{Q}(\hat{\theta}_1)$ and it follows that χ is σ-invariant, as desired. $\square$

The following are the only results of this section that are not valid for arbitrary primes. Next is a σ-version of [GRSV25, Thm. B].

Proposition 2.8. *Let G be a finite group, $p \in \{2,3\}$, and $P \in \mathrm{Syl}_p(G)$. Let $N \lhd G$ and assume p divides $|G:N|$. Suppose that $\theta \in \mathrm{Irr}(B_0(N))$ is $P \times \langle\sigma\rangle$-invariant. Then p divides $k_{0,\sigma}(B_0(G)|\theta)$. In particular, if θ extends to $\hat{\theta} \in \mathrm{Irr}_{0,\sigma}(B_0(PN))$ then $k_{0,\sigma}(B_0(G)|\theta) \geq p$.*

Proof. Arguing as in [GRSV25, Thm. 2.7] we get that

$$\sum_{\chi \in \mathrm{Irr}_0(B_0(G)|\theta)} \chi(1)^2 \equiv 0 \bmod p.$$

Recall that $\langle\sigma\rangle$ is a p-group and acts on $\mathrm{Irr}_0(B_0(G)|\theta)$. Write

$$\mathrm{Irr}_0(B_0(G)|\theta) = \mathrm{Irr}_{0,\sigma}(B_0(G)|\theta) \cup \Omega_1 \cup \cdots \cup \Omega_t$$

where the Ω_i's are the nontrivial $\langle\sigma\rangle$-orbits, and let χ_i be a representative of each Ω_i. Since each element of Ω_i has degree $\chi_i(1)$ and $|\Omega_i| \equiv 0 \bmod p$, we have

$$\sum_{\chi \in \mathrm{Irr}_0(B_0(G)|\theta)} \chi(1)^2 = \sum_{\chi \in \mathrm{Irr}_{0,\sigma}(B_0(G)|\theta)} \chi(1)^2 + \sum_{i=1}^{t} |\Omega_i|\chi_i(1)^2 \equiv \sum_{\chi \in \mathrm{Irr}_{0,\sigma}(B_0(G)|\theta)} \chi(1)^2 \bmod p$$

and this shows that

$$\sum_{\chi \in \mathrm{Irr}_{0,\sigma}(B_0(G)|\theta)} \chi(1)^2 \equiv 0 \bmod p.$$

Now, for every $\chi \in \mathrm{Irr}_{0,\sigma}(B_0(G)|\theta)$ we have $\chi(1)^2 \equiv 1 \bmod p$ and we conclude that p divides $k_{0,\sigma}(B_0(G)|\theta)$, as desired.

For the final part, notice that if there is some $\hat{\theta} \in \mathrm{Irr}_{0,\sigma}(B_0(PN)|\theta)$ extending θ then there is $\psi \in \mathrm{Irr}_{0,\sigma}(B_0(PN\mathbf{C}_G(P))|\theta)$ extending $\hat{\theta}$ by Theorem 2.3. By Lemma 2.4, ψ^G contains some $\chi \in \mathrm{Irr}_{0,\sigma}(B_0(G))$. Therefore $k_{0,\sigma}(B_0(G)|\theta) > 0$ by Frobenius reciprocity, so the result follows. □

Remark 2.9. *The previous result is also true for non-principal blocks of maximal defect, where one has to use that they contain at least one σ-invariant height-zero character (this was proved in* [BP80, Prop. 2.6]*).*

Lemma 2.10. *Assume $p \leq 3$ and that a p-group P acts by automorphisms on a group K of order divisible by p. Then the set of P-invariant characters in $\mathrm{Irr}_{0,\sigma}(B_0(K))$ is nonempty and has size divisible by p.*

Proof. This follows from [RSV20, Lem. 2.2(c)] using that P fixes $1_K \in \mathrm{Irr}_{0,\sigma}(B_0(K))$. □

2.2. **Structure of groups with a cyclic Sylow 3-subgroup.** The following result of Brauer (which relies on a result of Herzog in [Her70]) will be essential in the final step of our reduction theorem for Theorem A. We restate it here for the reader's convenience.

Theorem 2.11. *Let G be a finite group, let $p = 3$ and suppose that G has cyclic Sylow p-subgroups. Then one of the following holds:*

(i) *G is p-solvable and there exists $L \leq G$ with $|G:L| \in \{1,2\}$ such that L has a normal p-complement.*

(ii) *G is not p-solvable and for every $N \lhd G$, either N or G/N is of order not divisible by p.*

Proof. We use the notation of [Bra76, Sec. 2]. Let P be a Sylow 3-subgroup of G and let $r = |\mathbf{N}_G(P) : P\mathbf{C}_G(P)|$. Since $R = \mathbf{N}_G(P)/P\mathbf{C}_G(P) \leq \mathrm{Aut}(P)$, we have that $r = |R|$ is not divisible by p and since P is cyclic, we have that $r \mid p-1 = 2$.

As in [Bra68, p. 562], we say that a group with cyclic Sylow subgroups is of metacyclic type if there exists $K \lhd G$ of index $p^a r$, where $|P| = p^a$.

Now, if G is of metacyclic type, then G is p-solvable by [Bra76, Thm. 2C], and we are in case (i). If $r = 1$, we have that G has a normal p-complement, as wanted. If $r = 2$, the result follows by [Bra76, Prop. 2F].

On the other hand, if G is not of metacyclic type, then it is not p-solvable (again by [Bra76, Thm. 2C]), and we are in case (ii). The result now follows by [Bra76, Thm. 3C]. □

3. Theorem B

The purpose of this section is to prove Theorem B. We start with the following refinement of [Nav18, Lem. 9.3].

Lemma 3.1. *Let $\chi \in \mathrm{Irr}_{p',\tau}(G)$, where $\tau \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$ has p-power order, and let N be a normal subgroup of G. Then χ_N has a $P \times \langle\tau\rangle$ invariant constituent, and any two of them are $\mathbf{N}_G(P)$-conjugate.*

Proof. Let $\mu \in \mathrm{Irr}(N)$ lying under χ. Since $|G : G_\mu|$ is not divisible by p (by the Clifford correspondence and using that p does not divide $\chi(1)$) and $\langle\tau\rangle$ is a p-group, we have that there is at least one $\langle\tau\rangle$-invariant $\theta \in \mathrm{Irr}(N)$ lying under χ. But then for $g \in G$, we have $(\theta^g)^\tau = (\theta^\tau)^g = \theta^g$ and we have that all of the irreducible constituents of χ_N are $\langle\tau\rangle$-invariant. Now, by [Nav18, Lem. 9.3], we conclude that χ_N has a $P \times \langle\tau\rangle$-invariant constituent and any two of them are $\mathbf{N}_G(P)$-conjugate. □

We will also need the following result of Murai. For the convenience of the reader, we restate the theorem in our notation. Recall that if B is a Brauer p-block of G and λ is a linear character of G, then the set $\lambda B = \{\lambda\chi \mid \chi \in \mathrm{Irr}(B)\}$ is a Brauer p-block of G (see [Riz18, Lem. 2.1], for instance).

Lemma 3.2. *Let N be a normal subgroup of G, let B be a p-block of G, let $\xi \in \mathrm{Irr}(G)$ be a linear character and let $\overline{B}$ be a p-block of G/N. Let b and $\overline{b}$ be the Brauer correspondents of B and $\overline{B}$, respectively. Then $\xi\overline{B}$ is dominated by B if, and only if, $\xi_{N_G(P)}\overline{b}$ is dominated by b.*

Proof. This is, essentially, [Mur98, Cor. 2.5]. Write $H = \mathbf{N}_G(P)$. To apply this result, we just need to check that $\xi^{-1}B$ and $\xi_H^{-1}b$ are Brauer correspondents. But this easily follows from the definition of induced blocks (see [Nav98, p. 87]). Let K be a conjugacy class of G; we claim that $\lambda^G_{\xi_H^{-1}b}(\hat{K}) = \lambda_{\xi^{-1}B}(\hat{K})$. Let $L_1, \ldots, L_r$ be the conjugacy classes of H such that $K \cap H = L_1 \cup \ldots \cup L_r$, let $x_i \in L_i$, let $\psi \in \mathrm{Irr}(b)$ and let $\chi \in \mathrm{Irr}(B)$. Then

$$\begin{aligned}\lambda^G_{\xi_H^{-1}b}(\hat{K}) &= \lambda_{\xi_H^{-1}b}\left(\sum_{x\in K\cap H} x\right) = \lambda_{\xi_H^{-1}b}(\hat{L_1} + \ldots + \hat{L_r}) \\ &= \omega_{\xi_H^{-1}b}(\hat{L_1})^* + \ldots + \omega_{\xi_H^{-1}b}(\hat{L_r})^* = \left(\sum_{i=1}^r \frac{|L_i|\overline{\xi_H(x_i)}\psi(x_i)}{\psi(1)}\right)^*\end{aligned}$$

Since ξ is a class funcion of G, we have that $\xi(x_i) = \xi(x_j)$ for all i, j and then

$$\begin{aligned}\lambda^G_{\xi_H^{-1}b}(\hat{K}) &= \overline{\xi(x_1)}^* \left(\sum_{i=1}^r \frac{|L_i|\psi(x_i)}{\psi(1)}\right)^* = \overline{\xi(x_1)}^* \lambda_b(\hat{L_1} + \ldots + \hat{L_r}) \\ &= \overline{\xi(x_1)}^* \lambda_b(K \hat{\cap} H) = \overline{\xi(x_1)}^* \lambda_B(K) = \left(\frac{|K|\xi^{-1}(x_1)\chi(x_1)}{\chi(1)}\right)^* = \lambda_{\xi^{-1}B}(\hat{K}),\end{aligned}$$

proving the claim. □

The following is Theorem B. Recall that $\mathcal{H}$ is the Galois group considered in [Nav04].

Theorem 3.3. *Let $N \triangleleft G$ with N an abelian p-group, and assume G/N has cyclic Sylow p-subgroups. Let $\tau \in \mathcal{H}$ have p-power order. Then $|\mathrm{Irr}_{0,\tau}(B_0(G))| = |\mathrm{Irr}_{0,\tau}(B_0(\mathbf{N}_G(P)))|$, where $P \in \mathrm{Syl}_p(G)$.*

Proof. Let Δ be a $\mathbf{N}_G(P)$-transversal on the set of $P \times \langle\tau\rangle$-invariant elements in $\mathrm{Irr}(N)$. By Lemma 3.1, it follows that

$$\mathrm{Irr}_{0,\tau}(B_0(G)) = \coprod_{\theta\in\Delta} \mathrm{Irr}_{0,\tau}(B_0(G)|\theta),$$

and that

$$\mathrm{Irr}_{0,\tau}(B_0(\mathbf{N}_G(P))) = \coprod_{\theta\in\Delta} \mathrm{Irr}_{0,\tau}(B_0(\mathbf{N}_G(P))|\theta).$$

Hence, it is enough to show that $|\mathrm{Irr}_{0,\tau}(B_0(G)|\theta)| = |\mathrm{Irr}_{0,\tau}(B_0(\mathbf{N}_G(P))|\theta)|$ for every $\theta \in \Delta$. Now, let $\theta \in \Delta$. We claim that the map

$$\iota_G^\theta : \mathrm{Irr}_{0,\tau}(B_0(G_\theta)|\theta) \to \mathrm{Irr}_{0,\tau}(B_0(G)|\theta)$$

defined by $\psi \mapsto \psi^G$ is a bijection. Indeed, since θ is $\langle\tau\rangle$-invariant, it follows that $\langle\tau\rangle$ acts on $\mathrm{Irr}(G_\theta|\theta)$ and on $\mathrm{Irr}(G|\theta)$, and the Clifford correspondence implies that $\psi \mapsto \psi^G$ is a bijection $\mathrm{Irr}(G_\theta|\theta) \to \mathrm{Irr}(G|\theta)$. Since the action of $\langle\tau\rangle$ commutes with character induction, it follows that τ fixes $\psi \in \mathrm{Irr}(G_\theta|\theta)$ if, and only if it fixes $\psi^G \in \mathrm{Irr}(G|\theta)$. Now [Nav98, Cor. 6.2 and Thm. 6.7] imply the claim. Analogously, since $\mathbf{N}_{G_\theta}(P) = \mathbf{N}_G(P)_\theta$, the map

$$\iota_{\mathbf{N}_G(P)}^\theta : \mathrm{Irr}_{0,\tau}(B_0(\mathbf{N}_{G_\theta}(P))|\theta) \to \mathrm{Irr}_{0,\tau}(B_0(\mathbf{N}_G(P))|\theta)$$

defined by $\eta \mapsto \eta^{\mathbf{N}_G(P)}$ is also a bijection.

Now, we claim that θ extends to a $\langle\tau\rangle$-invariant character in $\mathrm{Irr}(P)$ if, and only if $\mathrm{Irr}_{0,\tau}(B_0(G)|\theta)$ is nonempty. Indeed, let $\chi \in \mathrm{Irr}_{0,\tau}(B_0(G)|\theta)$ and let $\psi \in \mathrm{Irr}_{0,\tau}(B_0(G_\theta)|\theta)$ be its Clifford correspondent. Then ψ_P is a $\langle\tau\rangle$-invariant character of P, and by [NT19, Lem. 2.1(ii)] we have that ψ_P contains a $\langle\tau\rangle$-invariant linear constituent $\lambda \in \mathrm{Irr}(P)$. Since $\psi_N = \psi(1)\theta$, λ is an extension of θ. Conversely, if θ extends to a $\langle\tau\rangle$-invariant character in $\mathrm{Irr}(P)$ then by Lemma 2.5, θ extends to a character $\psi \in \mathrm{Irr}_{0,\tau}(B_0(G_\theta))$ and $\psi^G \in \mathrm{Irr}_{0,\tau}(B_0(G)|\theta)$. In any case, we conclude that $\mathrm{Irr}_{0,\tau}(B_0(G)|\theta) \neq \emptyset$ if, and only if θ extends to a character $\tilde\theta \in \mathrm{Irr}_{0,\tau}(B_0(G_\theta))$ if, and only if, θ extends to a $\langle\tau\rangle$-invariant character in $\mathrm{Irr}(P)$. Analogously, $\mathrm{Irr}_{0,\tau}(B_0(\mathbf{N}_G(P))|\theta) \neq \emptyset$ if, and only if θ extends to a character $\tilde\theta \in \mathrm{Irr}_{0,\tau}(B_0(\mathbf{N}_{G_\theta}(P)))$ if, and only if, θ extends to a $\langle\tau\rangle$-invariant character in $\mathrm{Irr}(P)$. In particular, $\mathrm{Irr}_{0,\tau}(B_0(\mathbf{N}_G(P))|\theta) \neq \emptyset$ if, and only if, $\mathrm{Irr}_{0,\tau}(B_0(G)|\theta) \neq \emptyset$. Since we want to prove that $|\mathrm{Irr}_{0,\tau}(B_0(G)|\theta)| = |\mathrm{Irr}_{0,\tau}(B_0(\mathbf{N}_G(P))|\theta)|$, we may assume that both are not empty and therefore, we can find $\tilde\theta \in \mathrm{Irr}_{0,\tau}(B_0(G_\theta))$ extending θ. Notice that $\tilde\theta' := \tilde\theta_{\mathbf{N}_{G_\theta}(P)}$ is irreducible and extends θ. Since $|G_\theta : \mathbf{N}_{G_\theta}(P)|$ is not divisible by p, we conclude by [LMNT25, Lem. 2.1] that $\tilde\theta' \in \mathrm{Irr}_{0,\tau}(B_0(\mathbf{N}_{G_\theta}(P)))$ extends θ.

By Gallagher's theorem,

$$\mathrm{Irr}(G_\theta|\theta) = \{\mu\tilde\theta \mid \mu \in \mathrm{Irr}(G_\theta/N)\} \text{ and } \mathrm{Irr}(\mathbf{N}_{G_\theta}(P)|\theta) = \{\beta\tilde\theta' \mid \beta \in \mathrm{Irr}(\mathbf{N}_{G_\theta}(P)/N)\}.$$

Notice that $\mu\tilde\theta$ is $\langle\tau\rangle$-invariant and has degree coprime to p if and only if μ is $\langle\tau\rangle$-invariant and has degree coprime to p. Further, [Riz18, Lem. 2.4] implies that there is a set $\mathcal{B}$ of blocks of G_θ/N of maximal defect such that

$$\mathrm{Irr}_{0,\tau}(B_0(G_\theta)|\theta) = \coprod_{B\in\mathcal{B}} \{\mu\tilde\theta \mid \mu \in \mathrm{Irr}_{0,\tau}(B)\}.$$

If $\mathcal{B}'$ denotes the set of Brauer correspondent blocks in $\mathbf{N}_{G_\theta}(P)/N = \mathbf{N}_{G_\theta/N}(P/N)$ of the blocks in $\mathcal{B}$, then by Lemma 3.2, we have that

$$\mathrm{Irr}_{0,\tau}(B_0(\mathbf{N}_{G_\theta}(P))|\theta) = \coprod_{b\in\mathcal{B}'} \{\beta\tilde\theta' \mid \beta \in \mathrm{Irr}_{0,\tau}(b)\}.$$

Since G/N has cyclic Sylow p-subgroups, [Nav04, Thm. 3.4] implies that for each $B \in \mathcal{B}$, there is a bijection $\Omega_B : \mathrm{Irr}_{0,\tau}(B) \to \mathrm{Irr}_{0,\tau}(b)$, where $b \in \mathcal{B}'$ is the Brauer correspondent block of B. It follows that

$$\Omega_\theta : \mathrm{Irr}_{0,\tau}(B_0(G_\theta)|\theta) \to \mathrm{Irr}_{0,\tau}(B_0(\mathbf{N}_{G_\theta}(P)|\theta)$$
$$\mu\tilde\theta \mapsto \Omega_{\mathrm{bl}(\mu)}(\mu)\tilde\theta'$$

is a bijection, where $\mathrm{bl}(\mu)$ denotes the block of G_θ/N that contains μ (which belongs to $\mathcal{B}$).

Therefore, the map

$$\Psi_\theta := \iota_{\mathbf{N}_G(P)}^\theta \circ \Omega_\theta \circ (\iota_G^\theta)^{-1} : \mathrm{Irr}_{0,\tau}(B_0(G)|\theta) \to \mathrm{Irr}_{0,\tau}(\mathbf{N}_G(P)|\theta)$$

is a bijection, and we can construct a bijection $\Psi : \mathrm{Irr}_{0,\tau}(B_0(G)) \to \mathrm{Irr}_{0,\tau}(B_0(\mathbf{N}_G(P)))$ by setting $\Psi(\chi) = \Psi_\theta(\chi)$ if χ lies over $\theta \in \mathrm{Irr}_P(N)$. (Notice that the map is surjective because $\mathrm{Irr}_{0,\tau}(B_0(G)|\theta) \neq \emptyset$ if, and only if $\mathrm{Irr}_{0,\tau}(B_0(\mathbf{N}_G(P)|\theta) \neq \emptyset$, as proved above.) The result follows. □

Remark 3.4. *The hypothesis on the Sylow subgroups of G/N being cyclic is only used in the second-to-last paragraph of the proof of Theorem 3.3, to apply* [Nav04, Thm. 3.4]. *In general, we could assume only that, for every block B of maximal defect of G/N with Brauer correspondent b, there is a bijection $\Omega_B : \mathrm{Irr}_{0,\tau}(B) \to \mathrm{Irr}_{0,\tau}(b)$. Moreover, if one assumes $\Omega_B(\psi)(1) \leq \psi(1)$ for all $\psi \in \mathrm{Irr}_{0,\tau}(B)$ and all blocks B of maximal defect, then the proof of Theorem 3.3 gives a bijection $\Omega : \mathrm{Irr}_{0,\tau}(B_0(G)) \to \mathrm{Irr}_{0,\tau}(B_0(\mathbf{N}_G(P)))$ with $\Omega(\chi)(1) \leq \chi(1)$. The existence of such bijections has been proposed recently in* [Gia25].

A similar proof gives the following result.

Proposition 3.5. *Let $N \lhd G$ be perfect and let $P \in \mathrm{Syl}_p(G)$. Assume every P-invariant character $\theta \in \mathrm{Irr}_{0,\sigma}(B_0(N))$ that extends to a σ-invariant character of PN also extends to a σ-invariant character of $B_0(G_\theta)$ and assume G/N has cyclic Sylow p-subgroups. Then $|\mathrm{Irr}_{0,\sigma}(B_0(G))| = |\mathrm{Irr}_{0,\sigma}(B_0(N\mathbf{N}_G(P)))|$.*

Proof. We may argue as in the proof of Theorem 3.3 that every $\chi \in \mathrm{Irr}_{0,\sigma}(B_0(G))$ lies over a P-invariant $\theta \in \mathrm{Irr}(B_0(N))$. Since θ has a number of G-conjugates not divisible by p and $\langle\sigma\rangle$ acts on the set of G-conjugates of θ, it follows that θ is also σ-invariant. Arguing again as in Theorem 3.3 we conclude that $\chi \in \mathrm{Irr}_0(B_0(G))$ is σ-invariant if and only if its Clifford correspondent $\psi \in \mathrm{Irr}_0(B_0(G_\theta))$ is σ-invariant.

Now each $\theta \in \mathrm{Irr}_{0,\sigma}(B_0(N))$ extends to some $\hat\theta \in \mathrm{Irr}_{0,\sigma}(B_0(PN))$ by [Nav18, Cors. 6.2, 6.4] (where we are using that N is perfect so $o(\theta) = 1$). By hypothesis, θ extends to a character in $\mathrm{Irr}_{0,\sigma}(B_0(G_\theta))$ and we may follow the proof of Theorem 3.3 starting at the fourth paragraph to construct a bijection $\mathrm{Irr}_{0,\sigma}(B_0(G)) \to \mathrm{Irr}_{0,\sigma}(B_0(N\mathbf{N}_G(P)))$. □

As before, we could obtain more general versions of Proposition 3.5, but given its technicality we have chosen to state it in the simplest way possible.

4. Reduction of Theorem A

In this section, we prove Theorem A assuming the results in Section 5 below.

Theorem 4.1 (Navarro–Rizo–Schaeffer Fry–Vallejo)**.** *Suppose that $G = NP$, where N is a nonabelian nonsimple minimal normal subgroup of G, $\mathbf{C}_G(N) = 1$, G/N is cyclic and $P \in \mathrm{Syl}_3(G)$. Let $S \lhd N$ be simple and suppose that the Sylow 3-subgroups of S are not cyclic, $H = \mathbf{N}_G(S)$, $C = \mathbf{C}_G(S)$ and let $V \in \mathrm{Syl}_3(H/C)$. Then the following hold.*

(a) $|\mathrm{Irr}_{3',\sigma}(B_0(G))| \in \{6, 9\}$ *if and only if* $H > SC$ *and* $|\mathrm{Irr}_{3',\sigma}(B_0(H/C))| \in \{6, 9\}$.
(b) $|P : \Phi(P)| = 9$ *if and only if* $H > SC$ *and* $|V : \Phi(V)| = 9$.

Proof. Mimic the proof of [NRSV21, Thm. 3.4]. Notice that [NRSV21, Thm. 3.4] did not require the hypothesis on non-cyclic Sylow subgroups of S, because nonabelian simple groups never have cyclic Sylow 2-subgroups. □

Next is the proof of Theorem A, assuming the results of Section 5. For the remainder of the section, we let $p = 3$.

Theorem 4.2. *Let G be a finite group, let $B_0(G)$ be the principal 3-block of G, and let $P \in \mathrm{Syl}_3(G)$. Suppose that $k_{0,\sigma}(B_0(G)) \in \{6, 9\}$. Then $|P : \Phi(P)| = 9$.*

Proof. We argue by induction on $|G|$. Since $\mathrm{Irr}_{0,\sigma}(B_0(G)) = \mathrm{Irr}_{0,\sigma}(B_0(G/\mathbf{O}_{p'}(G)))$ by [Nav98, Thm. 9.9], we may assume $\mathbf{O}_{p'}(G) = 1$. We can also assume that P is not cyclic by [RSV20, Thm. A].

Step 1: We may assume P is not normal in G.

If $P \lhd G$, since $\mathbf{O}_{p'}(G) = 1$, we have that $|\mathrm{Irr}(G/\Phi(P))| \in \{6, 9\}$ (see [RSV20, Lem. 2.2], for instance). Now we conclude by examining the structure of the Sylow 3-subgroups of groups having 6 or 9 conjugacy classes and normal Sylow 3-subgroups (see [VLVL85]) (notice that $|P/\Phi(P)| > p$, since otherwise P is cyclic).

Let N be a minimal normal subgroup of G.

Step 2: We may assume N is not p-elementary abelian.
Assume N is p-elementary abelian. We have the following cases.

(i) $k_{0,\sigma}(B_0(G/N)) = 1$. In this case $|G/N|$ is not divisible by p by Lemma 2.10, so $P = N \lhd G$ and we are done by Step 1.
(ii) $k_{0,\sigma}(B_0(G/N)) = k_{0,\sigma}(B_0(G))$. In this case $N \subseteq \ker(\chi)$ for all $\chi \in \mathrm{Irr}_{0,\sigma}(B_0(G))$. By Lemma 2.6, $N \subseteq \Phi(P)$. By induction, $|P : \Phi(P)| = |P/N : \Phi(P)/N| = 9$ and we are done.
(iii) $k_{0,\sigma}(B_0(G/N)) = 6$. By the previous case, $k_{0,\sigma}(B_0(G)) = 9$. In this case, by induction we have $|P/N : \Phi(P)N/N| = 9$ and it suffices to show that $N \subseteq \Phi(P)$. Assume otherwise, and let $\lambda \in \mathrm{Irr}(P/\Phi(P))$, and suppose that $1_N \neq \theta = \lambda_N \in \mathrm{Irr}(N)$ (here we are using that $N \not\subseteq \Phi(P)$). Now, θ extends to $\lambda \in \mathrm{Irr}_{0,\sigma}(P)$ and we can apply Lemma 2.5, then θ extends to $\hat{\theta} \in \mathrm{Irr}_{0,\sigma}(B_0(G_\theta))$. By Proposition 2.8 and the hypothesis, we have that $k_{0,\sigma}(B_0(G_\theta)|\theta) = 3$. Since $\hat{\theta}$ is σ-invariant, by Gallagher's theorem and [Riz18, Lem. 2.4], $k_{0,\sigma}(B_0(G/N)) \leq k_{0,\sigma}(B_0(G_\theta)|\theta) = 3 < 6 = k_{0,\sigma}(B_0(G/N))$, a contradiction.
(iv) $k_{0,\sigma}(B_0(G/N)) = 3$. In this case, G/N has cyclic Sylow 3-subgroups by the main result of [RSV20], so by Theorem 3.3 we have $k_{0,\sigma}(B_0(G)) = k_{0,\sigma}(B_0(\mathbf{N}_G(P)))$ and we are done by Step 1.

Therefore we may assume $N = S_1 \times \cdots \times S_t$ where the S_i's are nonabelian simple groups of order divisible by p.

Step 3: If $k_{0,\sigma}(B_0(G/N)) = 1$ then we are done.
In this case G/N is not divisible by p and hence $P \subseteq N$. Since $\mathbf{O}_{p'}(G) = 1$, we also have that $\mathbf{C}_G(N) = 1$. If $t = 1$, we conclude by applying the main result of [Ket25]. Hence, we may assume that $t > 1$. By the Frattini argument, we have that $G = N\mathbf{N}_G(P)$, so $M = N\mathbf{C}_G(P)$ is normal in G. Then $\mathrm{Irr}(G/M) \subseteq \mathrm{Irr}_{0,\sigma}(B_0(G))$ (see [GRSV25, Lem. 2.1]). Then, by Theorem 5.1, there are at least 5 character degrees in $\mathrm{Irr}_{0,\sigma}(B_0(N))$, and so the same holds in $\mathrm{Irr}_{0,\sigma}(B_0(M))$ by Theorem 2.3. For every $\psi \in \mathrm{Irr}_{0,\sigma}(B_0(M))$ there is at least one $\chi \in \mathrm{Irr}_{p',\sigma}(B_0(G))$ over it, so the set of character degrees of $\mathrm{Irr}_{0,\sigma}(B_0(M)) \setminus \{1_M\}$ has size at most $9 - k(G/M)$. This, together with the fact that $|G/M|$ is not divisible by 3, forces $k(G/M) \in \{1, 2, 4, 5\}$. Now, we may argue as in [GRSV25, Thm. 6.3], Step 2, cases (2.a.i)–(2.a.iv) using Theorem 5.2(iii) instead of Theorem 3.1(c) in loc. cit. to finish:

(i) Case $k(G/M) = 1$. In this case $G = M$, and by Theorem 2.3, we obtain
$$k_{0,\sigma}(B_0(S))^t = k_{0,\sigma}(B_0(N)) = k_{0,\sigma}(B_0(M)) = 9.$$
This forces $t = 2$ and $k_{0,\sigma}(B_0(S)) = 3$. By [RSV20, Thm. A], this means that S has cyclic Sylow subgroups and hence $P \subseteq N \cong S \times S$ is 2-generated, as wanted.
(ii) Case $k(G/M) = 2$. In this case, arguing as in (2.a.ii) of [GRSV25, Thm. 6.3], we obtain again that $k_{0,\sigma}(B_0(S)) = 3$ and $t = 2$. We remark that one needs to apply [NT21, Lem. 4.2] to show that there is a height-zero σ-invariant $\chi \in \mathrm{Irr}(B_0(G))$ lying above every character in $\mathrm{Irr}_{0,\sigma}(B_0(M))$.
(iii) Case $k(G/M) \in \{4, 5\}$. Arguing as in cases (2.a.iii) and (2.a.iv) of [GRSV25, Thm. 6.3] (and applying again [NT21, Lem. 4.2]), we obtain that either $t = 2$ and $k_{0,\sigma}(B_0(S)) = 3$ (and we conclude as above), or $t = 2$ and $k_{0,\sigma}(B_0(S)) = 6$. In this case, S is not cyclic (by the main result of [RSV20], and hence $9 \mid |S|$. We conclude as in cases (2.a.iii) and (2.a.iv) by using Theorem 5.2(iii).

Step 4: If $k_{0,\sigma}(B_0(G/N)) = k_{0,\sigma}(B_0(G))$ then we are done.

In this case, every character in $\mathrm{Irr}_{0,\sigma}(B_0(G))$ contains N in its kernel. Since P acts on $\mathrm{Irr}_{0,\sigma}(B_0(N))$ (a set of size divisible by 3 by Lemma 2.10) and fixes at least one character, then it fixes at least 3. Let $\eta \in \mathrm{Irr}_{0,\sigma}(B_0(N))$ be nontrivial and P-invariant. By [Nav18, Cors. 6.2, 6.4] there is a $\psi \in \mathrm{Irr}_{0,\sigma}(PN|\eta)$ extending η. Then ψ extends to $\hat{\psi} \in \mathrm{Irr}_{0,\sigma}(PN\mathbf{C}_G(P)|\eta)$ by Theorem 2.3, and $\hat{\psi}^G$ contains some $\chi \in \mathrm{Irr}_{0,\sigma}(B_0(G))$ by [RSV20, Lem. 2.4]. Since χ_N contains η then $\chi \in \mathrm{Irr}_{0,\sigma}(B_0(G)) - \mathrm{Irr}_{0,\sigma}(B_0(G/N))$, a contradiction.

Step 5: If $k_{0,\sigma}(B_0(G/N)) = 6$ then we are done.

In this case, we may assume $k_{0,\sigma}(B_0(G)) = 9$ by Step 4. We first prove that P acts transitively on the simple direct factors of N. Suppose the contrary and let $N = R \times T$, where R is the direct product of the elements in a P-orbit of $\{S_1, \ldots, S_t\}$. Since p divides $|R|$ and $|T|$, we know that p divides $k_{0,\sigma}(B_0(R))$ and $k_{0,\sigma}(B_0(T))$, so there are at least two nontrivial P-invariant characters $\theta_R, \xi_R \in \mathrm{Irr}_{0,\sigma}(B_0(R))$ and two nontrivial P-invariant characters $\theta_T, \xi_T \in \mathrm{Irr}_{0,\sigma}(B_0(T))$. Now, $1_N, 1_R \times \theta_T$ and $\theta_R \times \theta_T$ are $P \times \langle\sigma\rangle$-invariant and not G-conjugate. But this is a contradiction, since there are at least 6 irreducible characters in $\mathrm{Irr}_{0,\sigma}(B_0(G))$ lying over 1_N and there are at least 3 distinct characters in $\mathrm{Irr}_{0,\sigma}(B_0(G))$ lying over each $1_R \times \theta_T$ and $\theta_R \times \theta_T$ by Proposition 2.8. Hence P acts transitively on $\{S_1, \ldots, S_t\}$.

Now, by induction we have that $|PN : \Phi(P)N| = 9$. Suppose that Theorem 5.1(i)(a) holds for S and let θ_1, θ_2 be as therein. By Lemma 2.1(v), let

$$\tilde{\theta}_i = \theta_i \times \cdots \times \theta_i \in \mathrm{Irr}_{p',\sigma}(B_0(N)),$$

so $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are P-invariant. Note that $\tilde{\theta}_1$ and $\tilde{\theta}_2$ extend to characters in $\mathrm{Irr}_{0,\sigma}(B_0(PN))$ by [Nav18, Cors. 6.2, 6.4], and that $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are not G-conjugate. By Proposition 2.8 each of these characters has at least 3 irreducible characters in $\mathrm{Irr}_{0,\sigma}(B_0(G))$ lying above them, a contradiction. Finally suppose that Theorem 5.1(i)(b) holds for S and let $\theta \in \mathrm{Irr}_{p'}(B_0(S))$ be the character given by the statement of Theorem 5.1(i)(b). Let $\tilde{\theta} \in \mathrm{Irr}_{p'}(B_0(N))$ be the product of copies of θ, so that $\tilde{\theta}$ is P-invariant. Notice then that $G_{\tilde{\theta}}$ acts transitively on the simple direct factors of N (because P does). By Lemma 2.7 we also know that $\tilde{\theta}$ extends to some $\phi \in \mathrm{Irr}_{0,\sigma}(B_0(G_{\tilde{\theta}}))$. Since ϕ is σ-invariant, by Gallagher's theorem and [Riz18, Lem. 2.4], we have that $k_{0,\sigma}(B_0(G_{\tilde{\theta}})|\tilde{\theta}) \geq k_{0,\sigma}(B_0(G_{\tilde{\theta}}/N))$. By Clifford's theorem, $k_{0,\sigma}(B_0(G)|\tilde{\theta}) \geq k_{0,\sigma}(B_0(G_{\tilde{\theta}}/N))$. Since $k_{0,\sigma}(B_0(G/N)) = 6$ we have that PN/N is not cyclic, then $k_{0,\sigma}(B_0(G_{\tilde{\theta}}/N)) > 3$ by applying [RSV20, Thm. A] twice, and this gives a contradiction as $k_{0,\sigma}(B_0(G)) - k_{0,\sigma}(B_0(G/N)) = 3$.

Therefore we have that $k_{0,\sigma}(B_0(G/N)) = 3$ and in particular G/N has cyclic Sylow 3-subgroups by [RSV20, Thm. A]. Recall that $N = S_1 \times \cdots \times S_t$, where $S_i \cong S$, nonabelian simple group of order divisible by p.

Step 6: N is the unique minimal normal subgroup of G and $t \geq 2$.

If M is another minimal normal subgroup, then by Steps 3, 4 and 5 we may assume that $k_{0,\sigma}(B_0(G/M)) = 3$. By [RSV20, Thm. A] we have that $N \cong NM/M$ and G/N have cyclic Sylow 3-subgroups, then P is metacyclic but not cyclic, and thus it is 2-generated, as wanted. If N is simple then G is almost simple and we are done by Theorem 5.2, so we assume $t \geq 2$.

Our next goal is to show that PN acts transitively on the set $\{S_1, \ldots, S_t\}$. We write $N = M_1 \times \cdots \times M_s$ where each M_i is the direct product of a single P-orbit on $\{S_1, \ldots, S_t\}$.

Step 7: We may assume $s \leq 2$.

Since $M_i \triangleleft PN$, for each M_i we can find a nontrivial P-invariant $\theta_i \in \mathrm{Irr}_{0,\sigma}(B_0(M_i))$, using Lemma 2.10. Then, by Lemma 2.1(v), we can produce $s+1$ characters

$$1_N, (\theta_1 \times 1_{M_2} \times \cdots \times 1_{M_s}), (\theta_1 \times \theta_2 \times 1_{M_3} \times \cdots \times 1_{M_s}), \ldots, (\theta_1 \times \cdots \times \theta_s)$$

that lie in $\mathrm{Irr}_{0,\sigma}(B_0(N))$ and are P-invariant and not G-conjugate. Since N is perfect, all of these extend to characters in $\mathrm{Irr}_{0,\sigma}(B_0(PN))$ by [Nav18, Cors. 6.2, 6.4] and Lemma 2.1(iv). Proposition 2.8 produces 3 distinct characters in $\mathrm{Irr}_{0,\sigma}(B_0(G))$ above each of these, so $k_{0,\sigma}(B_0(G)) \geq 3(s+1)$. If $s > 2$ we arrive at a contradiction.

Step 8: We may assume $s \neq 2$.

If $s = 2$ then $N = M_1 \times M_2$ and M_1 and M_2 are G-conjugate but not P-conjugate. Suppose that Theorem 5.1(i)(a) holds for S_i and let θ_1 and θ_2 be the characters as therein. Let $\tilde{\theta}_i = \theta_i \times \cdots \times \theta_i \in \mathrm{Irr}(B_0(M_1))$, using Lemma 2.1(v). Then the characters

$$1_N, \tilde{\theta}_1 \times 1_{M_2}, \tilde{\theta}_2 \times 1_{M_2}$$

are not G-conjugate. Further, they are not conjugate to the character $\eta \in \mathrm{Irr}_{0,\sigma}(B_0(N))$ constructed as the direct product of copies of θ_1. Now all the characters $1_N, \tilde{\theta}_1 \times 1_{M_2}, \tilde{\theta}_1 \times 1_{M_2}$ and η are P-invariant so by [Nav18, Cors. 6.2, 6.4] and Lemma 2.1(iv), they extend to characters in $\mathrm{Irr}_{0,\sigma}(B_0(PN))$. By Proposition 2.8, we have that $k_{0,\sigma}(B_0(G)) \geq 12$, which is absurd.

Therefore we assume that the S_i are as in Theorem 5.1(i)(b). Write $S = S_1$ and let $\theta \in \mathrm{Irr}(S)$ be the character of S given in that statement. Let $\eta_i = \theta \times \theta \times \cdots \times \theta \in \mathrm{Irr}_{0,\sigma}(B_0(M_i))$ (using Lemma 2.1(v)). We claim that every P-invariant character in $\mathrm{Irr}_{0,\sigma}(B_0(N))$ is G-conjugate to 1_N, $\eta_1 \times 1_{M_2}$ and $\eta_1 \times \eta_2$. Indeed, if there is $\xi \in \mathrm{Irr}_{0,\sigma}(B_0(N))$ not G-conjugate to any of these, then ξ extends to $\mathrm{Irr}_{0,\sigma}(B_0(PN))$ by [Nav18, Cors. 6.2, 6.4] and Lemma 2.1(iv). By Proposition 2.8 we have that $k_{0,\sigma}(B_0(G)) \geq 12$, a contradiction that proves the claim. Moreover, notice that $\mathrm{Irr}_{0,\sigma}(B_0(G)|1_N) = 3$.

Let $L = \mathbf{N}_G(M_1) = \mathbf{N}_G(M_2)$ and notice that $|G : L| = 2$, $G_{\eta_1 \times 1_{M_2}} = L_{\eta_1 \times 1_{M_2}} = L_{\eta_1}$ and $M_2 \subseteq L_{\eta_1}$. Since PN acts transitively on the copies of S inside M_1 and $PN \subseteq L_{\eta_1}$, by Lemma 2.7 we have that η_1 extends to $\hat{\eta}_1 \in \mathrm{Irr}_{0,\sigma}(B_0(L_{\eta_1}))$. Now since M_2 is perfect $(\hat{\eta}_1)_N = \eta_1 \times 1_{M_2}$. Similarly, $1_{M_1} \times \eta_2$ extends to $\hat{\eta}_2 \in \mathrm{Irr}_{0,\sigma}(B_0(L_{\eta_2}))$.

We claim that $\eta = \eta_1 \times \eta_2$ extends to a character in $\mathrm{Irr}_{0,\sigma}(B_0(G_\eta))$. If $G_\eta L = G$ then G_η acts transitively on the set $\{S_1, \ldots, S_t\}$ and we are done applying Lemma 2.7. Otherwise, $G_\eta \subseteq L$ and therefore $G_\eta = L_{\eta_1} \cap L_{\eta_2}$. Now, since $PN \leq G_\eta$, G_η also acts transitively on the copies of S contained in M_1 and in M_2. Repeating the previous argument, $\eta_1 \times 1_{M_2}$ and $1_{M_1} \times \eta_2$ extend to $\mu_1, \mu_2 \in \mathrm{Irr}_{0,\sigma}(B_0(G_\eta))$ respectively. Now, by [GRSS20, Lem. 2.3], $\mu_1\mu_2 \in \mathrm{Irr}_{0,\sigma}(B_0(G_\eta))$ and

$$(\mu_1\mu_2)_N = (\eta_1 \times 1_{M_2})(1_{M_1} \times \eta_2) = \eta$$

and we are done.

Therefore, we are in the case of the hypotheses of Proposition 3.5, so

$$k_{0,\sigma}(B_0(G)) = k_{0,\sigma}(B_0(N\mathbf{N}_G(P)))$$

so if $N\mathbf{N}_G(P) \leq G$ we are done by induction. Otherwise $PN \lhd G$.

Let $X = PN\mathbf{C}_G(P)$. Then $\mathrm{Irr}(G/X) \subseteq \mathrm{Irr}_{0,\sigma}(B_0(G)|1_N)$. By the previous paragraphs and Proposition 2.8, we have $k_{0,\sigma}(B_0(G)|\eta_1 \times 1_{M_2}) = 3 = k_{0,\sigma}(B_0(G)|\eta_1 \times \eta_2)$, and this implies that $k_{0,\sigma}(B_0(G)|1_N) = 3$. Now p does not divide $|G : X|$, so we have $|G : X| \leq 2$. By Theorem 2.3, $k_{0,\sigma}(B_0(X)) = k_{0,\sigma}(B_0(PN))$. If $X = G$, by induction we may assume $PN = G$ but then N is minimal normal in PN and P acts transitively on $\{S_1, \ldots, S_t\}$ so we are done. Otherwise, $|G : X| = 2$, so $\mathrm{Irr}_{0,\sigma}(B_0(X)) \leq 15$. By Theorem 2.3 and [NRSV21, Lem. 2.5] we have that

$$3|\mathrm{Irr}_{0,P,\sigma}(B_0(N))| = |\mathrm{Irr}_{0,\sigma}(B_0(PN))| = |\mathrm{Irr}_{0,\sigma}(X)| \leq 15.$$

Since 3 divides $|\mathrm{Irr}_{0,P,\sigma}(B_0(N))|$, this forces $|\mathrm{Irr}_{0,P,\sigma}(B_0(N))| = 3$, so $|\mathrm{Irr}_{0,\sigma}(X)| = 9$ and we are done by induction.

Final step.

We conclude that $s = 1$ so PN acts transitively on the set $\{S_1, \ldots, S_t\}$. Recall that PN/N is cyclic.

Assume first S_i has noncyclic Sylow p-subgroups. Consider $M = \bigcap \mathbf{N}_G(S_i) \lhd G$ and notice that $M \subseteq \mathbf{N}_G(S)$. Since p divides $|G : \mathbf{N}_G(S)|$ then p divides $|G : M|$. By Theorem 2.11, either G/N is p-solvable or $|M/N|$ is not divisible by p.

Suppose first that $|M/N|$ is not divisible by p. Notice that, since PN/N is abelian and PN acts transitively on $\{S_1, \ldots, S_t\}$, then $\mathbf{N}_{PN}(S_i) = \mathbf{N}_{PN}(S_j)$ for every i, j. Then, $N \subseteq \mathbf{N}_{PN}(S) \subseteq M$, and therefore $N = \mathbf{N}_{PN}(S)$. Now, we can apply Lemma 2.2 and Theorem 5.1(iii) to obtain three P-invariant characters in $\mathrm{Irr}_{0,\sigma}(B_0(N))$ that are not G-conjugate. Now, Proposition 2.8 gives 9 characters in $\mathrm{Irr}_{0,\sigma}(B_0(G))$ not in $\mathrm{Irr}_{0,\sigma}(B_0(G/N))$, a contradiction.

Hence, we may assume that G/N is p-solvable. Theorem 2.11 implies that there exists $L \leq G$ with $|G : L| \in \{1, 2\}$ such that L/N has a normal p-complement K/N. If $|G : L| = 2$, then $k_{0,\sigma}(B_0(L)) \leq 16$ and since $k_{0,\sigma}(B_0(L))$ is divisible by 3, we obtain that $k_{0,\sigma}(B_0(L)) \in \{3, 6, 9, 12, 15\}$. Notice that $K = \mathbf{O}^p(L)$, and hence, by [NRSV21, Lem. 2.5], we obtain that $k_{0,\sigma,P}(B_0(K)) \in \{1, 2, 3, 4, 5\}$. Since 3 divides $|K|$, $k_{0,\sigma,P}(B_0(K)) \equiv k_{0,\sigma}(B_0(K)) \equiv 0 \bmod 3$, and then the only possibility is $k_{0,\sigma,P}(B_0(K)) = 3$. But in this case $k_{0,\sigma}(B_0(L)) = 9$ and we are done by induction.

Hence we may assume that $G = L = KP$, with $K \triangleleft G$. Now by [NRSV21, Thm. 2.8], we have that $k_{0,\sigma}(B_0(G)) = k_{0,\sigma}(B_0(N\mathbf{N}_G(P))$ and therefore, by induction, we can assume that $PN \triangleleft G$. Now consider $X = PN\mathbf{C}_G(P)$ and notice that $\mathrm{Irr}(G/X) \subseteq \mathrm{Irr}_{0,\sigma}(B_0(G)|1_N)$ by [GRSV25, Lem. 2.1]. Since p divides $|N|$, we have that $k_{0,\sigma,P}(B_0(N)) \equiv k_{0,\sigma}(B_0(N)) \equiv 0 \bmod p$. By Proposition 2.8, we obtain that $|\mathrm{Irr}_{0,\sigma}(B_0(G)|1_N)| = 3$. Since $|G : X|$ is not divisible by 3, we obtain that $|G : X| \in \{1, 2\}$. If $|G : X| = 2$, then

$$p \cdot k_{0,\sigma,P}(B_0(N)) = k_{0,\sigma}(B_0(PN)) = k_{0,\sigma}(B_0(X)) \leq 15,$$

where we have used [NRSV21, Lem. 2.5] in the first equality and Theorem 2.3 in the second equality. This forces $|\mathrm{Irr}_{0,P,\sigma}(B_0(N))| = 3$ and then $k_{0,\sigma}(B_0(PN)) = 9$. By induction we may assume $PN = G$. If $G = X$, by Theorem 2.3 we may assume $PN = G$ aswell, and the result follows from Theorem 4.1 and the main result of [Ket25].

We are left with the case that the S_i's have cyclic Sylow p-subgroups, and let $C_1 \in \mathrm{Syl}_p(S_1)$. Let $C_i = C_1^{y^i} \in \mathrm{Syl}_p(S_1^{y^i})$, $Q = C_1 \times \cdots \times C_t \in \mathrm{Syl}_3(N)$ where each $C_i \in \mathrm{Syl}_3(S_i)$ and where $Q \subseteq P$, with $P/Q = \langle yQ \rangle$ cyclic. Then $\langle yQ \rangle$ permutes $\{C_1, \ldots, C_t\}$ transitively, and since C_i is normal in Q for all i, we conclude that $\langle y \rangle$ permutes $\{C_1, \ldots, C_t\}$ transitively. This implies that there are generators $x_1, \ldots, x_t$ of $C_1, \ldots, C_t$ respectively such that $\langle y \rangle$ permutes the set $\{x_1, \ldots, x_t\}$ transitively. Write $z_i = (1, \ldots, 1, x_i, 1, \ldots, 1)$. Notice that $Q = \langle z_1, \ldots, z_t \rangle$. We claim that $P = \langle y, z_1 \rangle$. Since $\langle y \rangle$ acts transitively on $\{x_1, \ldots, x_t\}$ then it acts transitively on $\{z_1, \ldots, z_t\}$. It follows that every $z_j \in \langle y, z_1 \rangle$, so $Q \subseteq \langle y, z_1 \rangle$ and $P = \langle y, Q \rangle \subseteq \langle y, z_1 \rangle$. Since P is not cyclic, it is 2-generated and we are done. □

5. Simple groups

The goal of this section is to prove the following "σ-version" of [GRSV25, Thm. 3.1].

Theorem 5.1. *Let S be a nonabelian simple group of order divisible by 3 and let $X \in \mathrm{Syl}_3(\mathrm{Aut}(S))$.*

(i) *One of the following holds.*
  - (a) *There exist $1_S \neq \theta_1, \theta_2 \in \mathrm{Irr}_{0,\sigma}(B_0(S))$ nonconjugate in $\mathrm{Aut}(S)$ and invariant under X, or*
  - (b) *There is an X-invariant $1_S \neq \theta \in \mathrm{Irr}_{0,\sigma}(B_0(S))$ that extends to some σ-invariant character in $B_0(T)$ for all $S \leq T \leq \mathrm{Aut}(S)_\theta$.*

(ii) *The set of degrees of characters in $\mathrm{Irr}_{0,\sigma}(B_0(S))$ has size at least 3.*

(iii) *If S has noncyclic Sylow 3-subgroups, then there exist $1_S \neq \theta_1, \theta_2, \theta_3 \in \mathrm{Irr}_{0,\sigma}(B_0(S))$ nonconjugate in $\mathrm{Aut}(S)$.*

We remark that part (iii) of Theorem 5.1 is already found as [RSV20, Thm. C(b)]. In addition to Theorem 5.1, we will need the main result from [Ket25], restated here:

Theorem 5.2 (E. Ketchum)**.** *Let A be an almost simple group, and let $P \in \mathrm{Syl}_3(A)$. Then $k_{0,\sigma}(B_0(A)) \in \{6, 9\}$ if and only if $|P : \Phi(P)| = 9$.*

5.1. **Previous Results.** We begin by recording some previous results that will be useful for proving Theorem 5.1.

Lemma 5.3. *Let $N \triangleleft G$ with $3 \nmid [G : N]$.*

- *If $\theta \in \mathrm{Irr}(N)$ and $\chi \in \mathrm{Irr}(G \mid N)$, then χ is σ-fixed if and only if θ is σ-fixed.*
- *If $\theta \in \mathrm{Irr}_\sigma(B_0(N))$, then there is some $\chi \in \mathrm{Irr}_\sigma(B_0(G)|\theta)$.*

Proof. This is from [MMSV26, Lem. 3.5, Cor. 3.6]. □

Assume $S = G/\mathbf{Z}(G)$ is a simple group of Lie type, with $G = \mathbf{G}^F$, where $\mathbf{G}$ is a simple, simply connected algebraic group defined over a field of characteristic q_0 and $F\colon \mathbf{G} \to \mathbf{G}$ is a Steinberg endomorphism. Let $(\mathbf{G}^*, F)$ be dual to $(\mathbf{G}, F)$ and write $G^* := (\mathbf{G}^*)^F$.

Lemma 5.4. *Let S be as above with $q_0 = p \geq 3$. Then every $\chi \in \mathrm{Irr}(S)$ is σ-invariant.*

Proof. This follows from [TZ04, Thm. 1.3 and Prop. 10.12]. □

Lemma 5.5. *Assume S is as above with $q_0 \neq p$ and $p \geq 3$. Then any unipotent character of S is σ-invariant.*

Proof. This is [MMSV26, Lem. 4.7]. □

Lemma 5.6. *Assume S and G are as above with $q_0 \neq p$ and p good for G. Assume $s \in G^*$ has order p and $p \nmid |\mathbf{Z}(G)|$. Then the semisimple character χ_s deflates to a character in $\mathrm{Irr}_\sigma(B_0(S))$.*

Proof. This is exactly as in [MMSV26, Lem. 4.6] and [HS23, Thm. 5.1], which uses [His90, Cor. 3.4] and [CE04, Thm. 21.13]. Note that χ_s is trivial on $\mathbf{Z}(G)$ since $s \in [G^*, G^*]$ (see [NT13, Lem. 4.4]), and $\mathbf{C}_{\mathbf{G}^*}(s)$ is connected, using [MT11, Ex. 20.16]. □

5.2. **Proof of Theorem 5.1.**

Lemma 5.7. *Theorem 5.1 holds when S is an alternating or sporadic group, a group of Lie type defined in characteristic 3, or ${}^2\mathrm{F}_4(2)'$. Further, (i)(a) holds unless $S \in \{\mathrm{PSL}_2(3^a), \mathrm{PSL}_3^\epsilon(3^a)\}$.*

Proof. Let S be as in the statement. If $S = J_3$, we see the statement from computation in [GAP24], and we see that (i)(a) holds in this case. So, we now assume that $S \neq J_3$.

Lemma 5.4 and [MMSV26, Lem. 4.1] give that Theorem 5.1 is equivalent to [GRSV25, Thm. 3.1] in these cases whenever condition (a1) of the latter holds. Thus, by [GRSV25, Rem. 3.2], we need only consider groups of the form $\mathrm{PSL}_2(3^a)$ and $\mathrm{PSL}_3^\epsilon(3^a)$. Further, we need only verify that condition (i)(b) of Theorem 5.1 holds in these cases.

Let $S = \mathrm{PSL}_2(3^a)$ or $S = \mathrm{PSL}_3^\epsilon(3^a)$. From [GRSV25, Thm. 3.1 and Rem. 3.2] we obtain some $1 \neq \theta \in \mathrm{Irr}_0(B_0(S))$, which is X-invariant, such that for every $S \leq T \leq \mathrm{Aut}(S)_\theta$ we have that θ extends to some character in $B_0(T)$ (in both cases $\mathrm{Aut}(S)_\theta/S$ is abelian, so in fact all characters lying above θ in $\mathrm{Aut}(S)_\theta$ are extensions). Further, we have that θ is σ-fixed by Lemma 5.4. Let $S \leq T \leq \mathrm{Aut}(S)_\theta$. In both cases we observe that T/S has a normal Sylow 3-subgroup. Let M be the preimage under the natural projection $\pi : T \to T/S$ of this Sylow 3-subgroup. Since S is perfect, we have the determinantal order $o(\theta)$ satisfies $o(\theta) = 1$ and we can use [Nav18, Cor. 6.4] to obtain a σ-fixed extension of θ in the principal block of M. (Recall that since M/S has 3-power order, $B_0(M)$ is the only block above $B_0(S)$ by Lemma 2.1.) Then Lemma 5.3 gives a σ-fixed extension in the principal block of T. □

Lemma 5.8. *Theorem 5.1 holds when S is a group of Lie type defined in characteristic $p \neq 3$, with (i)(a) holding unless $S = \mathrm{PSL}_2(q)$ or $3|(q+\epsilon)$ and $S = \mathrm{PSL}_3^\epsilon(q)$.*

Proof. Assume S is a simple group of Lie type defined in characteristic $p \neq 3$.

First we consider (i). Assume $S = \mathrm{PSL}_2(q)$ or that $3|(q+\epsilon)$ and $S = \mathrm{PSL}_3^\epsilon(q)$. We then claim that (*b*) holds. Take $\theta = \hat{\theta}_S$, with $\hat{\theta}$ being the Steinberg character in $\widetilde{S}$, where $\widetilde{S}$ is $\mathrm{PGL}_2(q)$ and $\mathrm{PGL}_3^\epsilon(q)$ in the respective cases. Then θ is $\mathrm{Aut}(S)$-invariant by [Mal08, Thm. 2.5] and extends to $\mathrm{Aut}(S)$ by [Mal08, Thm. 2.4] and we see that $\hat{\theta} \in \mathrm{Irr}_\sigma(B_0(\widetilde{S}))$ by arguing as in the second and third paragraphs of the proof of [RSV20, Prop. 3.9]. Let $S \leq T \leq \mathrm{Aut}(S)_\theta = \mathrm{Aut}(S)$. We clearly have that θ extends to $\hat{\theta}_{T\cap\widetilde{S}} \in \mathrm{Irr}(B_0(T \cap \widetilde{S}))$. Since the Steinberg character is rational-valued, so is $\hat{\theta}_{T\cap\widetilde{S}}$. We have that $T/(T \cap \widetilde{S})$ is an abelian group and therefore has a normal Sylow 3-subgroup. Let M be the preimage under $\pi : T \to T/(T \cap \widetilde{S})$ of this Sylow 3-subgroup. Then $B_0(M)$ is the unique block above $B_0(T \cap \widetilde{S})$ by Lemma 2.1 and we use [Nav18, Cor. 6.6(a)] to obtain a rational extension χ of $\hat{\theta}_{T\cap\widetilde{S}}$ in $\mathrm{Irr}(B_0(M))$. Finally, using Lemma 5.3, we obtain a character ψ in $\mathrm{Irr}_\sigma(B_0(T)|\chi)$. Since ψ lies above $\hat{\theta}_{T\cap\widetilde{S}}$ and $T/(T \cap \widetilde{S})$ is abelian, it follows that ψ must be an extension of $\hat{\theta}_{T\cap\widetilde{S}}$ (and hence of θ), as desired.

Now Assume S is not one of the groups discussed above. Then [GRSV25, Prop. 5.8] gives 3 $\mathrm{Aut}(S)$-invariant unipotent characters. Further, Lemma 5.5 gives that these characters must be σ-fixed. Thus, (i)(a) holds for S.

We next show (ii). Note that 3 does not divide the size of ${}^2\mathrm{B}_2(q)$, so we may assume $S \neq {}^2\mathrm{B}_2(q)$. If $S \neq \mathrm{PSL}_2(q)$ and $S \neq \mathrm{PSL}_3^\epsilon(q)$ with $3 \mid (q+\epsilon)$, then [GRSV25, Prop. 5.8] gives 3 unipotent characters with distinct $3'$ degrees lying in the principal block, and these characters are σ-fixed by Lemma 5.5.

So, now assume $S = \mathrm{PSL}_2(q)$ or that $3 \mid (q+\epsilon)$ and $S = \mathrm{PSL}_3^\epsilon(q)$. In these cases let $G = \mathrm{SL}_2(q)$ and $G = \mathrm{SL}_3^\epsilon(q)$ respectively. We also let $\widetilde{G} = \mathrm{GL}_2(q)$, resp. $\widetilde{G} = \mathrm{GL}_3^\epsilon(q)$ and $G^* = \mathrm{PGL}_2(q)$, resp. $\mathrm{PGL}_3^\epsilon(q)$. Let $s \in G^*$ be a semisimple element whose preimage under the canonical projection map $\pi : \tilde{G} \to G^*$ has nontrivial eigenvalues (ζ_3, ζ_3^2) for a primitive third root of unity ζ_3. Consider a semisimple character $\chi_s \in \mathcal{E}(G, s)$. Lemma 5.6 gives that χ_s deflates to a character in $\mathrm{Irr}_\sigma(B_0(S))$. Further, this character is height-zero by the degree properties of semisimple characters, which we see by computing the centralizer explicitly in $\widetilde{G}$. Then the deflation of χ_s alongside the 2 unipotent characters given in [GRSV25, Prop. 5.8] yield three σ-fixed characters in $B_0(S)$ with distinct $3'$-degrees, as desired. □

Theorem 5.1 now follows from Lemmas 5.7 and 5.8, recalling that part (iii) is [RSV20, Thm. C(b)].

5.3. **Related Results.** We end by making some observations to give evidence of the analogue of Theorem A for arbitrary blocks. We first obtain the statement for groups of Lie type defined in characteristic 3 whose ambient algebraic group has connected center.

Corollary 5.9. *Let $G = \mathbf{G}^F$ be a group of Lie type, where $\mathbf{G}$ is a connected reductive group over $\overline{\mathbb{F}}_3$ such that $\mathbf{Z}(\mathbf{G})$ is connected. Let B be a 3-block of G with defect group D. Then $[D : \Phi(D)] = 9$ if $|\mathrm{Irr}_{0,\sigma}(B)| \in \{6, 9\}$.*

Proof. By a result of Dagger and Humphreys, the only blocks with positive defect have maximal defect, and these are indexed by the characters of $\mathbf{Z}(G)$ (see [Hum71]). First assume $\mathbf{G}$ has no component of type G_2. Then any block B with positive defect satisfies $\mathrm{Irr}_0(B) = \mathrm{Irr}_{3'}(B)$ is comprised of semisimple characters in B, using the degree properties of Jordan decomposition and that nontrivial unipotent characters are divisible by 3 by [Mal07, Thm. 6.8].

Since $\mathbf{Z}(\mathbf{G})$ is connected and the semisimple elements of G^* have order prime to 3 and σ fixes $3'$-roots of unity, we then have $\mathrm{Irr}_0(B) = \mathrm{Irr}_{0,\sigma}(B)$ using the main result of [SV20]. Assume that $|\mathrm{Irr}_{0,\sigma}(B)| \in \{6, 9\}$ and let B be the block indexed by the character $\theta \in \mathrm{Irr}(\mathbf{Z}(G))$. Now, the number of semisimple characters in $\mathrm{Irr}(B)$ is the number of G^*-classes of semisimple elements $s \in G^*$ such that the characters in $\mathcal{E}(G, s)$ lie above θ. But as noted in [ST23, Prop. 2.7(iii)] (see also the proofs of [NT13, Lem. 4.4], [MNST24, Prop. 3.7]) this is the number of semisimple classes in the corresponding coset in $G^*/\mathbf{O}^{3'}(G^*)$. Now, for each semisimple $s \in G^*$, there is some $T^* := (\mathbf{T}^*)^F$ containing s, where $\mathbf{T}^* \leq \mathbf{G}^*$ is some F-stable maximal torus. By [GM20, Rem. 1.5.13(b, c)], T^* induces all cosets of $G^*/\mathbf{O}^{3'}(G^*)$. Then by [FG12, Cor. 2.4], we see each such coset contains the same number of semisimple classes. This means that $|\mathrm{Irr}_0(B)| = |\mathrm{Irr}_0(B_0(G))|$. Recalling that $\mathrm{Irr}_0(B_0(G)) = \mathrm{Irr}_{0,\sigma}(B_0(G))$ as before, we therefore have $|\mathrm{Irr}_{0,\sigma}(B_0(G))| \in \{6, 9\}$. Then $[P : \Phi(P)] = 9$ by Theorem A.

Finally, suppose that $\mathbf{G}$ has components of type G_2. Here by [Mal07, Thm. 6.8], the group $\mathrm{G}_2(q)$ (where q is a power of 3) has 7 unipotent characters in $\mathrm{Irr}_{3'}(\mathrm{G}_2(q))$. The unipotent characters of $\mathrm{G}_2(q)$ are all σ-stable, using [Gec03, Table 1 and Prop. 5.5]. Note that these 7 unipotent characters then lie in $\mathrm{Irr}_{0,\sigma}(B_0(\mathrm{G}_2(q)))$ since $\mathbf{Z}(\mathrm{G}_2)$ is connected, and hence there is a unique block of positive defect from the first paragraph. Hence the statement is trivially satisfied if $\mathbf{G}$ is simple.

So, assume that $\mathbf{G}$ is not simple. Since we have assumed $\mathbf{Z}(\mathbf{G})$ is connected, we have by [GM20, Lem. 1.7.7] that $\mathbf{G}^F = [\mathbf{G}, \mathbf{G}]^F \mathbf{T}^F$ for any F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$. Then G is a quotient of the direct product of $[\mathbf{G}, \mathbf{G}]^F$ with an abelian group $\mathbf{T}^F$ of order prime to 3, and hence we may without loss replace $\mathbf{G}$ with $[\mathbf{G}, \mathbf{G}]$, and therefore assume that $\mathbf{G}$ is semisimple.

(Indeed, recall that σ will fix any character in $\mathrm{Irr}(\mathbf{T}^F)$, each of which lie in their own block.) Then $\mathbf{G} = \prod_{i=1}^n \mathbf{G}_i$ for some simple reductive groups $\mathbf{G}_i$ which commute pairwise. We may assume that $\mathbf{G}_1, \dots, \mathbf{G}_k$ are the components of type G_2 and write $\mathbf{H}$ for the product of these. Since each $\mathbf{G}_i$ for $1 \leq i \leq k$ has trivial center (as the only simple group of type G_2 is the adjoint group), it follows that $\mathbf{H}$ is a direct product and $\mathbf{G} = \mathbf{H} \times \mathbf{H}'$ where $\mathbf{H}'$ is the product of simple components not of type G_2. In particular, by [GM20, Cor. 1.5.16], G has a direct factor of the form $\mathrm{G}_2(q)$ with q some power of 3, and the result follows from the previous paragraph. □

The next observation follows from Theorem A and the fact that blocks of general unitary and linear groups behave similarly to certain principal blocks.

Corollary 5.10. *Let $\widetilde{G} = \mathrm{GL}_n^\epsilon(q)$ and let B be a 3-block of $\widetilde{G}$ with defect group D. Then $[D : \Phi(D)] = 9$ if $|\mathrm{Irr}_{0,\sigma}(B)| \in \{6, 9\}$.*

Proof. If q is a power of 3, this is from Corollary 5.9. So now assume that q is a power of some prime $p \neq 3$. Let B be a block in $\mathcal{E}(\widetilde{G}, s)$ for some semisimple $3'$-element $s \in \widetilde{G}^*$. Now, by [FS82, Thm. (7A)], Jordan decomposition yields a bijection between $\mathrm{Irr}_0(B)$ and $\mathrm{Irr}_0(b)$, where b is some unipotent block of $C := \mathbf{C}_{\widetilde{G}^*}(s)$. Here B and b share a defect group D. This bijection is further σ-equivariant by [SV20].

Now, let e be the order of ϵq modulo 3, so that $e \in \{1, 2\}$. If $e = 1$, then in fact the only unipotent block of C is the principal block, so the result now follows from Theorem A.

So, suppose $e = 2$. We may write $C := \prod C_i$ as the product of lower-rank general linear or unitary groups C_i, depending on the invariant factors of s. Say $C_i := \mathrm{GL}_{m_i}(\eta_i q_i)$ with q_i some power of q, $\eta_i \in \{\pm 1\}$, and m_i a positive integer. Let e_i be the order of $\eta_i q_i$ modulo 3. Then the results of [MO83] yield that D is isomorphic to a Sylow 3-subgroup of $C' := \prod \mathrm{GL}_{e_i w_i}(\eta_i q_i)$ for some positive integer w_i and further there is a bijection between $\mathrm{Irr}_0(b)$ and $\mathrm{Irr}_0(B_0(C'))$. This is further σ-equivariant, which can be seen from the construction in [MO83, P.209-210] and using [SV20] to see the action of σ on characters in $\mathcal{E}(C_i, t)$ depend only on s_i and t, and these will have the same eigenvalues as the corresponding embeddings in C_i'. Hence, the result again follows from Theorem A. □

References

[Alp76] J. L. Alperin. Isomorphic blocks. *J. Algebra*, 43(2):694–698, 1976.

[BP80] M. Broué and L. Puig. A Frobenius theorem for blocks. *Invent. Math.*, 56(2):117–128, 1980.

[Bra68] R. Brauer. On blocks and sections in finite groups. II. *Amer. J. Math.*, 90:895–925, 1968.

[Bra76] R. Brauer. On finite groups with cyclic Sylow subgroups. I. *J. Algebra*, 40(2):556–584, 1976.

[CE04] M. Cabanes and M. Enguehard. *Representation theory of finite reductive groups*, volume 1 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2004.

[Dad77] E. C. Dade. Remarks on isomorphic blocks. *J. Algebra*, 45(1):254–258, 1977.

[FS82] P. Fong and B. Srinivasan. The blocks of finite general linear and unitary groups. *Invent. Math.*, 69(1):109–153, 1982.

[FG12] J. Fulman and R. Guralnick. Bounds on the number and sizes of conjugacy classes in finite Chevalley groups with applications to derangements. *Trans. Amer. Math. Soc.*, 364(6):3023–3070, 2012.

[GAP24] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.13.0*, 2024.

[Gec03] M. Geck. Character values, Schur indices and character sheaves. *Represent. Theory*, 7:19–55, February 2003.

[GM20] M. Geck and G. Malle. *The character theory of finite groups of Lie type*, volume 187 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2020. A guided tour.

[Gia25] E. Giannelli. McKay bijections and character degrees. *arXiv:2507.01730*, 2025.

[GRSS20] E. Giannelli, N. Rizo, B. Sambale, and A. A. Schaeffer Fry. Groups with few p'-character degrees in the principal block. *Proc. Amer. Math. Soc.*, 148(11):4597–4614, 2020.

[GRSV25] E. Giannelli, N. Rizo, A. A. Schaeffer Fry, and C. Vallejo. Characters and Sylow subgroup abelianization. *J. Algebra*, 667:824–864, 2025.

[GI83] D. Gluck and I. M. Isaacs. Tensor induction of generalized characters and permutation characters. *Illinois J. Math.*, 27(3):514–518, 1983.

[Her70] M. Herzog. On a problem of E. Artin. *J. Algebra*, 15:408–416, 1970.

[His90] G. Hiss. Regular and semisimple blocks of finite reductive groups. *J. London Math. Soc. (2)*, 41(1):63–68, 1990.

[Hum71] J. E. Humphreys. Defect groups for finite groups of Lie type. *Math. Z.*, 119:149–152, 1971.

[HS23] N. N. Hung and A. A. Schaeffer Fry. On Héthelyi-Külshammer's conjecture for principal blocks. *Algebra Number Theory*, 17(6):1127–1151, 2023.

[IN02] I. M. Isaacs and G. Navarro. New refinements of the McKay conjecture for arbitrary finite groups. *Ann. of Math. (2)*, 156(1):333–344, 2002.

[Ket25] E. Ketchum. Characters and the generation of Sylow 3-subgroups for almost simple groups. *arXiv:2509.02854*, 2025.

[LMNT25] R. Lyons, J. M. Martínez, G. Navarro, and P. H. Tiep. Principal blocks, irreducible restriction, fields and degrees. *arXiv:2507.22503,* 2025.

[Mal07] G. Malle. Height 0 characters of finite groups of Lie type. *Represent. Theory*, 11:192–220, 2007.

[Mal08] G. Malle. Extensions of unipotent characters and the inductive McKay condition. *J. Algebra*, 320(7):2963–2980, 2008.

[MNST24] G. Malle, G. Navarro, A. A. Schaeffer Fry, and P. H. Tiep. Brauer's height zero conjecture. *Ann. of Math. (2)*, 200(2):557–608, 2024.

[MT11] G. Malle and D. Testerman. *Linear Algebraic Groups and Finite Groups of Lie Type*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2011.

[MMSV26] A. Maróti, J. M. Martínez, A. A. Schaeffer Fry, and C. Vallejo. On almost p-rational characters in principal blocks. *Publ. Mat.*, 70(1):55–78, 2026.

[MO83] G. O. Michler and J. B. Olsson. Character correspondences in finite general linear, unitary and symmetric groups. *Math. Z.*, 184(2):203–233, 1983.

[MS23] A. Moretó and B. Sambale. Groups with 2-generated Sylow subgroups and their character tables. *Pacific J. Math.*, 323(2):337–358, 2023.

[Mur98] M. Murai. Blocks of factor groups and heights of characters. *Osaka J. Math.*, 35(4):835–854, 1998.

[Nav98] G. Navarro. *Characters and blocks of finite groups*, volume 250 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.

[Nav04] G. Navarro. The McKay conjecture and Galois automorphisms. *Ann. of Math. (2)*, 160(3):1129–1140, 2004.

[Nav18] G. Navarro. *Character theory and the McKay conjecture*, volume 175 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2018.

[NRSV21] G. Navarro, N. Rizo, A. A. Schaeffer Fry, and C. Vallejo. Characters and generation of Sylow 2-subgroups. *Represent. Theory*, 25:142–165, 2021.

[NT13] G. Navarro and P. H. Tiep. Characters of relative p'-degree over normal subgroups. *Ann. of Math. (2)*, 178(3):1135–1171, 2013.

[NT16] G. Navarro and P. H. Tiep. Real groups and Sylow 2-subgroups. *Adv. Math.*, 299:331–360, 2016.

[NT19] G. Navarro and P. H. Tiep. Sylow subgroups, exponents, and character values. *Trans. Amer. Math. Soc.*, 372(6):4263–4291, 2019.

[NT21] G. Navarro and P. H. Tiep. The fields of values of characters of degree not divisible by p. *Forum Math. Pi*, 9:Paper No. e2, 28, 2021.

[Riz18] N. Rizo. p-blocks relative to a character of a normal subgroup. *J. Algebra*, 514:254–272, 2018.

[RSV20] N. Rizo, A. A. Schaeffer Fry, and C. Vallejo. Galois action on the principal block and cyclic Sylow subgroups. *Algebra Number Theory*, 14(7):1953–1979, 2020.

[RS25] L. Ruhstorfer and A. A. Schaeffer Fry. The Isaacs–Navarro Galois conjecture. *arXiv:2509.02300*, 2025.

[ST23] A. A. Schaeffer Fry and J. Taylor. Galois automorphisms and classical groups. *Transform. Groups*, 28(1):439–486, 2023.

[SV20] B. Srinivasan and C. R. Vinroot. Galois group action and Jordan decomposition of characters of finite reductive groups with connected center. *J. Algebra*, 558:708–727, 2020.

[TZ04] P. H. Tiep and A. E. Zalesskii. Unipotent elements of finite groups of Lie type and realization fields of their complex representations. *J. Algebra*, 271:327–390, 2004.

[Va23] C. Vallejo Rodríguez. A lower bound on the number of generators of a defect group. *Vietnam J. Math.*, 51(3):571–576, 2023.

[VLVL85] A. Vera López and J. Vera López. Classification of finite groups according to the number of conjugacy classes. *Israel J. Math.*, 51(4):305–338, 1985.

(Ketchum, Schaeffer Fry) Department of Mathematics, University of Denver, Denver, CO 80210, USA

(Martínez, Rizo) Departament de Matemàtiques, Universitat de València, 46100 Burjassot, València, Spain

Email address, Ketchum: eden.ketchum@du.edu
Email address, Martínez: josep.m.martinez@uv.es
Email address, Rizo: noelia.rizo@uv.es
Email address, Schaeffer Fry: mandi.schaefferfry@du.edu