Journal club

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Course notes

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Contents

LECTURE 1

Special conjugacy classes and exceptional characters

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This lecture is devoted to the concept of special conjugacy classes and exceptional characters, introduced by M. Suzuki in [**Suz55**]. We will use them to prove the following theorem from [**Suz57**]. We say that G satisfies property (W) if every nonidentity element of G has abelian centralizer in G.

Theorem 1.1 (Suzuki). *Let G be a finite group satisfying property* (W)*. If G has odd order then G is not simple.*

It had been proven previously by Weisner that finite groups with this condition are either solvable or simple. Brauer–Suzuki–Wall proved that nonabelian simple groups of even order satisfying condition (W) had to be isomorphic to $\mathrm{PSL}(2, 2^k)$. These theorems were proved as part of the CFSG, and occured before the odd order theorem.

1.1. Special classes and exceptional characters

Definition 1.2. *Let $H \leqslant G$. We say a conjugacy class $\mathcal{C}$ of H is* ***special*** *in G if $\mathcal{C}$ contains an element x with $\mathbf{C}_G(x) \subseteq H$.*

Lemma 1.3. *Let $\mathcal{C}_1, \ldots, \mathcal{C}_t$ be special classes of $H \leqslant G$ and let $x_i \in \mathcal{C}_i$ be such that $\mathbf{C}_G(x_i) \leqslant H$. Assume that x_i is not G-conjugate to any x_j if $i \neq j$. If ψ is a generalized character of H such that $\psi(g) = 0$ for all $g \in H - \bigcup_i \mathcal{C}_i$ then the following hold:*

(i) $\psi^G(x_i) = \psi(x_i)$,

(ii) $\psi^G(g) = 0$ *if g is not G-conjugate to any x_i,*

(iii) $[\psi^G, \psi^G] = [\psi, \psi]$.

Proof. If $x \in G$ then the induction formula gives

$$\psi^G(x) = \frac{1}{|H|} \sum_{\substack{t \in G \\ x^t \in H}} \psi(x^t)$$

and if $x^t \in H$ but x is not G-conjugate to any x_i, we have $\psi(x^t) = 0$ so $\psi^G(x) = 0$.

Next we set $x = x_i$ and $t \in G$ with $x^t \in H$. We claim that if $\psi(x^t) \neq 0$ then $t \in H$. Indeed, if $\psi(x^t) \neq 0$ then by hypothesis we have x^t is H-conjugate to some x_j. By the hypothesis on the x_k's we have that $x^t = x^h$ for some $h \in H$ and this implies that $th^{-1} \in \mathbf{C}_G(x) \leqslant H$ so $t \in H$, as claimed. Using this, the induction formula gives

$$\psi^G(x) = \frac{1}{|H|} \sum_{\substack{t \in G \\ x^t \in H}} \psi(x^t) = \frac{1}{|H|} \sum_{\substack{t \in H \\ x^t \in H}} \psi(x^t) = \frac{|H|}{|H|}\psi(x) = \psi(x),$$

which proves (i).

Finally let $C_i = \mathbf{C}_G(x_i) \leqslant H$. Set $c_i = |G : C_i|$ and notice that $c_i = |G : H|h_i$ where $h_i = |H : C_i|$. By (ii), we know that ψ^G vanishes on the elements of G not G-conjugate to any x_i. Therefore,

$$[\psi^G, \psi^G] = \frac{1}{|G|} \sum_{i=1}^{t} c_i \psi^G(x)\overline{\psi^G(x)} = \frac{1}{|G|} \sum_{i=1}^{t} |G : H| h_i \psi(x)\overline{\psi(x)} =$$

$$= \frac{1}{|H|} \sum_{i=1}^{t} h_i \psi(x)\overline{\psi(x)} = [\psi, \psi]$$

where we have used (ii) in the second equality. This proves (iii). □

The above result helps us construct a character correspondence in the following special situation.

Proposition 1.4. *Let $\mathcal{C}_1, \dots, \mathcal{C}_t$ be special classes of $H \leqslant G$ and let $x_i \in \mathcal{C}_i$ be such that $\mathbf{C}_G(x_i) \leqslant H$. Assume that x_i is not G-conjugate to any x_j if $i \neq j$. If H has $s \geqslant 2$ distinct irreducible characters $\psi_1, \dots, \psi_s$ such that*

(i) *$\psi_i(x) = \psi_j(x)$ for all $x \in H - \bigcup_i \mathcal{C}_i$ (in particular, $\psi_i(1) = \psi_j(1)$),*

(ii) *$\psi(x) = 0$ if $x \in H$ is G-conjugate but not H-conjugate to an x_i*

then G has s irreducible characters $\chi_1, \dots, \chi_s$ with

$$\psi_i^G = (a + \epsilon)\chi_i + \sum_{j \neq i} a\chi_j + \Delta$$

where $\epsilon = \pm 1$, $a \geqslant 0$, $(a + \epsilon) \geqslant 0$ and Δ does not contain any χ_j's. Further, ϵ, a, Δ are independent of i.

Proof. Consider the $s-1$ generalized characters $\varphi_i = \psi_i - \psi_s$, which satisfy the assumptions of Lemma 1.3. Then $[\varphi_i^G, \varphi_i^G] = 2$ so φ_i^G contains two irreducible constituents with multiplicity ± 1. Arguing analogously for $\varphi_i - \varphi_j = \psi_i - \psi_j$ we get that $[\varphi_i - \varphi_j, \varphi_i - \varphi_j] = 2$ so φ_i^G and φ_j^G have a constituent in common, with the same sign. Let χ_s be this constituent (common to all the φ_i's). We may write

$\varphi_i^G = \epsilon_1\chi_i + \epsilon_2\chi_s$ for $\chi_i \in \mathrm{Irr}(G)$ and ϵ_1, ϵ_2 signs. Now $\varphi_i^G(1) = \psi_i^G(1) - \psi_s^G(1) = 0$ so $\epsilon_1 = -\epsilon_2$, and it follows that

$$\varphi_i^G = \epsilon(\chi_i - \chi_s)$$

for $\epsilon = \pm 1$ for all $1 \leqslant i \leqslant s-1$. Notice that ϵ is the same for all i because the sign of χ_s as a constituent of all φ_i^G is the same.

Therefore $\psi_i^G - \psi_s^G = \epsilon(\chi_i - \chi_s)$ and rearranging we obtain

$$\psi_i^G = \epsilon\chi_i + \sum_{j=1}^{s} a_j\chi_j + \Delta$$

where Δ does not contain any of the χ_i's (we have just written $\varphi_s^G - \epsilon\chi_s = \sum_{j=1}^{s} a_j\chi_j + \Delta$, and notice that Δ does not depend on i either. Since $s \geqslant 2$ we may do the same but fixing a different ψ_j instead of ψ_s and obtain the same formula for ψ_s. We need to show all the a_j's are equal.

First, notice that if x_i^g is H-conjugate to x_i then $g \in H$ (as in the previous result). Using that ψ_i vanishes in the elements of H that are G-conjugate but not H-conjugate to any of the x_k's and that x_k and x_j are never G-conjugate if $k \neq j$ we obtain

$$\psi_i^G(x_j) = \frac{1}{|H|}\sum_{\substack{t\in G\\ x^t\in H}} \psi_i(x_j^t) = \frac{1}{|H|}\sum_{t\in H}\psi_i(x_j^t) = \psi_i(x_j).$$

Secondly, if $x \in G$ is not G-conjugate to any x_i then

$$\psi_i^G(x) = \frac{1}{|H|}\sum_{\substack{t\in G\\ x^t\in H}} \psi_i(x^t) = \frac{1}{|H|}\sum_{\substack{t\in G\\ x^t\in H}} \psi_s(x^t) = \psi_s^G(x)$$

where we have used that ψ_i and ψ_s coincide in the elements of H outside $\bigcup \mathcal{C}_i$.

Setting $c_i = |G : \mathbf{C}_G(x_i)|$ and $h_i = |H : \mathbf{C}_G(x_i)|$ as before, we have

$$\begin{aligned}
[\psi_i^G, \psi_i^G] - [\psi_s^G, \psi_s^G] &= \frac{1}{|G|}\sum_{x\in G}\left(\psi_i^G(x)\overline{\psi_i^G(x)} - \psi_s^G(x)\overline{\psi_s^G(x)}\right) = \\
&= \frac{1}{|G|}\sum_{j=1}^{t} c_i\left(\psi_i(x_j)\overline{\psi_i(x_j)} - \psi_s(x_j)\overline{\psi_s(x_j)}\right) = \\
&= [\psi_i, \psi_i] - [\psi_s, \psi_s] = 0
\end{aligned}$$

Now $[\psi_i, \psi_i] - [\psi_s, \psi_s] = (a_i + \epsilon)^2 + a_s^2 - (a_s + \epsilon)^2 - a_i^2$ which forces $a_i = a_s$, as desired. □

Definition 1.5. *The χ_i's above are called **exceptional** characters associated to the ψ_i's.*

The following might be a more convenient reformulation: in the situation of the Lemma,

$$\psi_i^G = \epsilon\chi_i + \Psi$$

where ϵ, Ψ are independent of i. In particular, $\chi_i \leftrightarrow \psi_i$ is a natural character bijection.

Corollary 1.6. *Assume the hypotheses and notation of Proposition 1.4. Then $\chi_i(x) = \chi_k(x)$ for every element $x \in G$ not G-conjugate to any element in some $\mathcal{C}_i$. In particular, exceptional characters have the same degree.*

Proof. Induction formula and Proposition 1.4(i). □

1.2. Analysis of groups satisfying condition (W)

Notice for example that property (W) guarantees $\mathbf{Z}(G) = 1$ if G is nonabelian. We assume G is simple of odd order satisfying (W) and work to find a contradiction.

Proposition 1.7. *If $A \leqslant G$ is maximal abelian then $A = \mathbf{C}_G(x)$ for all $1 \neq x \in A$, and every nontrivial conjugacy class of A is special in G.*

Proof. We only need to prove the first affirmation, which follows from the fact that $\mathbf{C}_G(x)$ is abelian. □

Proposition 1.8. *Let $N = \mathbf{N}_G(A)$. Two elements of A are G-conjugate if and only if they are N-conjugate. In particular, $w = \frac{|A|-1}{|N:A|}$ is an integer and there are exactly w conjugacy classes of G intersecting $A \backslash \{1\}$ nontrivially.*

Proof. If $A^g \cap A \neq 1$ for $g \in G$ then there is $1 \neq y \in A \cap A^g$ and $A = \mathbf{C}_G(y) = A^g$ by Proposition 1.7. Then $g \in \mathbf{N}_G(A)$, and the first part follows. Since $\mathbf{C}_G(x) = A$ for any $x \in A\backslash\{1\}$ we have that x has exactly $|N : A|$ N-conjugates, and the second part follows. For the final part, w is the number of N-conjugacy classes in $A\backslash\{1\}$ and by the first part, w is the number of different G-conjugacy classes of elements of $A\backslash\{1\}$. □

Proposition 1.9. *Under our hypotheses, N has exactly w irreducible characters $\psi_1, \ldots, \psi_w$ of degree $l = |N : A|$, induced from the nontrivial irreducible characters of A, and every such character induces irreducibly to N.*

Proof. See 6.34 of [**Isa76**] (thanks Juan!). □

Lemma 1.10. *With our hypotheses, if $\chi \in \mathrm{Char}(N)$ satisfies $\chi(x) = 0$ for $x \in N\backslash A$ then $\chi^G(y) = \chi(y)$ for $y \in A$.*

Proof. By Proposition 1.8 we have $y^t \in A$ if, and only if $t \in N$ so

$$\chi^G(y) = \frac{1}{|N|} \sum_{\substack{t \in G \\ y^t \in A}} \chi(y^t) = \frac{1}{|N|} \sum_{t \in N} \chi(y^t) = \frac{|N|}{|N|}\chi(y)$$

as desired. □

Proposition 1.11. *Assume $w \geqslant 2$. The exceptional characters $\chi_1, \ldots, \chi_w$ of G associated to the characters $\psi_1, \ldots, \psi_w$ are linearly independent in $\mathcal{C}_1, \ldots, \mathcal{C}_t$.*

Proof. Assume that $\sum_{i=1}^{w} a_i \chi_i(x) = 0$ for all $x \in \mathcal{C}_j$ $(j = 1, \ldots, t)$. Then by Corollary 1.6 we have that

$$\left(\sum_{i=1}^{w} a_i \chi_i\right)(\overline{\chi_k} - \overline{\chi_l}) = 0$$

for any pair $k, l \in \{1, \ldots, t\}$. Now

$$0 = [\left(\sum_i a_i \chi_i\right)(\overline{\chi_k} - \overline{\chi_l}), 1_G] = \sum_i a_i[\chi_i, \chi_k] - \sum_i a_i[\chi_i, \chi_l] = a_k - a_l$$

for any pair k, l. Therefore

$$\sum_i \chi_i(x) = 0$$

for all $x \in \mathcal{C}_j$, $j = 1, \ldots, t$. Now recall that we may write

$$\psi_i^G = \epsilon \chi_i + \Psi$$

where ϵ and Ψ are independent of i. Therefore $\chi_i(x) = \epsilon \psi_i^G(x) - \epsilon \Psi(x)$ so

$$0 = \sum_i \chi_i(x) = \sum_i \epsilon \psi_i^G(x) - \epsilon w \Psi(x). \tag{1.2.1}$$

Now Lemma 1.3, $\psi_i^G(x) = \psi_i(x) = \psi_i(x_j)$ for $x_j \in A \cap \mathcal{C}_j$ and $(\psi_i)_A = \mu_1 + \cdots + \mu_k$ for certain linear characters μ, where $\{\mu_1, \ldots, \mu_k\}$ is the set of N-conjugates of μ. By Proposition 1.9, every $1_A \neq \mu \in \mathrm{Lin}(A)$ appears as a constituent of exactly one of the $(\psi_i)_A$'s. Therefore

$$\sum_i \psi_i^G(x) = \sum_{1_A \neq \mu \in \mathrm{Irr}(A)} \mu(x_j) = -1$$

(because column sums of character tables of abelian groups are 0). Then Equation 1.2.1 implies $w\Psi(x) = -1$ but $w \geqslant 2$ contradicts the fact that $\Psi(x)$ is an algebraic integer. □

Corollary 1.12. *If B is a maximal abelian subgroup of G not G-conjugate to A then the exceptional characters of A are not exceptional for B.*

Proof. We have that χ_i and χ_j agree on B but by Proposition 1.11 they should be linearly independent in every nontrivial conjugacy class of B. □

If H is a finite group, ρ_H denotes its regular character.

PROPOSITION 1.13. *The induced character 1_A^G contains every exceptional character χ_i of A with the same multiplicity.*

PROOF. Notice that $\chi_i - \chi_j = \epsilon(\psi_i - \psi_j)$, and that this (generalized) character vanishes in the elements that are not G-conjugate to some element of A by Corollary 1.6. Write A^G for the set of G-conjugates of elements in A. We have that

$$\sum_{x \in G} 1_A^G(x)\left(\overline{\chi_i}(x) - \overline{\chi_j}(x)\right) = \epsilon \sum_{x \in A^G} 1_A^G(x)\left(\overline{\psi_i^G}(x) - \overline{\psi_j^G}(x)\right) \tag{1.2.2}$$

and notice that if $x \in A$ we have $1_A^G(x) = 1_A(x)$ and $\overline{\psi_k^G}(x) = \overline{\psi}_k(x)$ by Lemma 1.10. Now Proposition 1.8 implies that every element $x \in A$ has exactly $|G : N||N : \mathbf{C}_N(x)|$ conjugates. Using this we can rewrite equation 1.2.2 as

$$\epsilon|G : N| \sum_{x \in A} 1_A(x)\left(\overline{\psi_i}(x) - \overline{\psi_j}(x)\right) = \epsilon|G : A| \sum_{x \in A}\left(\overline{\psi_i}(x) - \overline{\psi_j}(x)\right) = 0$$

(where we have used that $\sum_{x \in A} \psi_k(x) = |N|[\psi_k, 1_N] = 0$). From equation 1.2.2 we have

$$[1_A^G, \chi_i] = [1_A^G, \chi_j]$$

as desired. $\square$

PROBLEM 1.14. *Show that*

$$[1_A^G - \psi_i^G, 1_A^G - \psi_i^G] = \frac{1}{|G|}\sum_{x \in G} |1_A^G(x) - \psi_i^G(x)|^2 = 1 + |N : A|.$$

REMARK 1.15. *Write $\rho = 1_A^G$. By Propositions 1.13 and 1.4 we may write*

$$\rho - \psi_i^G = 1_G - \epsilon\chi_i + b\sum_{k=1}^{w} \chi_k + \sum a_\gamma \gamma$$

where the last sum runs over the nontrivial and nonexceptional characters of G. Using Problem 1.14 we have that

$$1 + |N : A| = [\rho - \psi_i^G, \rho - \psi_i^G] = 1 + (b - \epsilon)^2 + (w - 1)b^2 + \sum a_\gamma^2$$

so

$$|N : A| = (b - \epsilon)^2 + (w - 1)b^2 + \sum a_\gamma^2.$$

We will use this formula in the future.

PROPOSITION 1.16. *Assume $w \geqslant 2$. If χ is a nonexceptional character of A then χ takes integer values on A. More precisely, if $x \in A$ we have $\chi(x) = a_\chi$ in the notation of Remark 1.15, and $\chi(x)$ is the unique integer satisfying $\chi(x) \equiv \chi(1) \bmod |A|$ and $|\chi(x)| \leqslant \frac{|A|-1}{2}$. In particular, χ vanishes in $A\backslash\{1\}$ if and only if $|A|$ divides $\chi(1)$.*

PROOF. Let $1_A \neq \lambda \in \mathrm{Lin}(A)$. Then $\lambda^G = \psi_i^G$ for some i (because these ψ_i are induced from the nontrivial linear characters of A, see Lemma 1.3. If χ is a nonexceptional character then by Proposition 1.4, χ appears in every ψ_i^G with the same multiplicity m. If $l = [1_A^G, \chi]$ then the $a_\chi = l - m$ where a_χ is from Remark 1.15. By Frobenius reciprocity we have

$$\chi_A = l 1_A + m \sum_{1_A \neq \xi \in \mathrm{Lin}(A)} \xi$$

and therefore we have

$$\chi(1) = l - m(|A| - 1) \equiv l - m \bmod |A|.$$

Now for $x \in A \backslash \{1\}$ we have

$$\sum_{\gamma \in \mathrm{Lin}(A)} \gamma(x) = 0$$

so

$$\chi(x) = (l - m)1_A(x) + m \sum_{\gamma \in \mathrm{Lin}(A)} \gamma(x) = l - m$$

and the first part of the result follows. From the formula in Remark 1.15 we have $|\chi(x)| \leqslant \chi(x)^2 \leqslant |N : A| \leqslant \frac{|A|-1}{2}$. □

COROLLARY 1.17. *If χ, ψ are nonexceptional characters of the same degree, then* $[(1_A)^G - \psi_i^G, \chi] = [(1_A)^G - \psi_i^G, \psi]$.

PROOF. Notice that $a_\chi = \chi(x)$ which is uniquely determined by $\chi(1)$ by the previous result. □

PROPOSITION 1.18. *We have*

$$\sum_{x \in A \backslash \{1\}} |\chi_i(x)|^2 \geqslant |N : A|(|A| - |N : A|).$$

PROOF. . We have that $(\psi_i)_A = \sum \lambda$ for exactly $|N : A|$ different N-conjugate $\lambda \in \mathrm{Lin}(A)$. If $\xi \in \mathrm{Lin}(A)$ is not contained in $(\psi_i)_A$ then ξ^G is one of the ψ_k's for $k \neq i$. Let m be the multiplicity of χ_i in ψ_k^G. Then m is independent of j and $[\chi_i, \psi_i^G] = m + \epsilon$. If $l = [\chi_i, 1_A^G]$ then $[\chi_i, 1_A^G - \psi_i^G] = l - m - \epsilon$ and by Frobneius reciprocity

$$(\chi_i)_A = l 1_A + m \sum_{1_A \neq \xi \in \mathrm{Lin}(A)} \xi + \epsilon \sum \lambda$$

where the last sum is over the $|N : A|$ constituents of $(\psi_i)_A$. Arguing as before, if $1 \neq x \in A$ we have

$$\chi_i(x) = l - m + \epsilon \psi_i(x).$$

Writing $n = l - m \in \mathbb{Z}$ we have

$$\sum_{1\neq x\in A} |\chi_i(x)|^2 = \sum_{1\neq x\in A} (\epsilon\psi_i(x) + n)(\epsilon\psi_i(x^{-1} + n) = \tag{1.2.3}$$

$$= \sum_{1\neq x\in A} |\psi_i(x)|^2 + \epsilon \sum_{1\neq x\in A} (\psi_i(x) - \psi_i(x^{-1}) + a^2(|A| - 1). \tag{1.2.4}$$

Now if $t = |N : A|$ and $(\psi_i)_A = \sum_{j=1}^{t} \xi_i$ we have

$$\sum_{1\neq x\in A} |\chi_i(x)|^2 = \sum_{1\neq x\in A} \left(\sum_{i,j=1}^{t} \xi_i(x)\xi_j(x^{-1}) \right) =$$

$$= |A| \sum_{i=1}^{t} [\xi_i, \xi_i] - |N : A|^2 = |A||N : A| - |N : A|^2.$$

Furthermore, since $\sum_{1\neq x\in A} \lambda(x) = -1$ for $\lambda \in \mathrm{Lin}(A)$ we have

$$\sum_{1\neq x\in A} \psi_i(x) = -|N : A|$$

because $(\psi_i)_A$ has $|N : A|$ constituents. This shows that equation 1.2.3 equals

$$|A||N : A| - |N : A|^2 - 2\epsilon n|N : A| + n^2(|A| - 1) \geqslant |N : A|(|A| - |N : A|)$$

using that $2 \leqslant w = \frac{|A|-1}{|N:A|}$ so $a^2w - 2\epsilon n \geqslant 0$ for $w \geqslant 2$. □

1.3. Simple groups satisfying (W)

We now assume G is nonabelian simple of odd order and satisfies (W) and work to find a contradiction. We note the following.

Proposition 1.19. *The maximal abelian subgroups of G form a partition of $G\backslash\{1\}$.*

We now set some notation. Let $A_1, \dots, A_s$ be representatives of the G-conjugacy classes of maximal abelian subgroups of G. Let $N_i = \mathbf{N}_G(A_i)$, $n_i = |A_i|$ and $l_i = |N_i : A_i|$. Recall that $w_i := (n_i - 1)/l_i$ is an integer by Proposition 1.8, and that there are exactly w_i G-conjugacy classes intersecting $A_i\backslash\{1\}$. Since n_i and l_i are odd, w_i is even and in particular $w_i \geqslant 2$. Therefore all results from the previous section apply, and each A_i contributes exactly w_i exceptional characters of$G$ which are not exceptional for the other A_j's by Corollary 1.12. In fact, we have the following:

Proposition 1.20. $|\,\mathrm{Irr}(G)| = 1+\sum w_i$. *In particular, every nontrivial character of G is exceptional for exactly one A_i.*

Proof. Every $g \in G\backslash\{1\}$ is G-conjugate to an element of some A_i. Since each A_i contributes w_i conjugacy classes of G we have that G has $1 + \sum w_i$ conjugacy classes and the first part follows. The sdecond part follows from Corollary 1.12. □

More notation! We let $\{\chi_1^k, \ldots, \chi_{w_k}^k\}$ be the exceptional characters of A_k coming from characters $\{\psi_1^k, \ldots, \psi_{w_k}^k\}$ of N_k (these were induced from the linear nontrivial characters of A_k). We assume now that s is such that $|A_s| = n_s$ is the smallest order among the A_i's. Recall that by Corollary 1.6, exceptional characters of A_s have the same degree d_s. We reorder the A_i's such that there is some t with n_i dividing d_s for $i \leqslant t$ and n_j not dividing d_s for $i > t$. For the A_s and all its related invariants and characters we omit the super or sub-index s.

Proposition 1.21. *We have that every $1_A^G - \psi_m^G$ contains A_i-characters if $i \leqslant t$.*

Proof. Write $\Gamma = 1_A^G - \psi_m^G$. Using the notation from Remark 1.15, and using that exceptional characters of the same subgroup have the same degree and Corollary 1.17 we may write

$$1_G - \chi_m + b \sum_{j \neq m} \chi_j + \sum_{k=1}^{s-1} a_k \sum_{r=1}^{w_k} \chi_r^k.$$

Now 1_A^N and ψ_m coincide in $N\backslash A$ so using the induction formula we have that Γ vanishes in the elements of G not G-conjugate to any element of A, so Γ vanishes in $A_1, \ldots, A_{s-1}$.

Assume by way of contradiction that $a_i = 0$. Then from Proposition 1.16 we have that the χ_j's vanish in $x \in A_i\backslash\{1\}$ and that $\chi_r^k(x) = y_k$ is an integer independent of r for $k \neq i$ and $k < s$. Therefore $\Gamma(x) = 0$ implies

$$0 = 1 + \sum_{\substack{k<s \\ k \neq i}} a_k y_k w_k$$

which is a contradiction because w_k is even. □

Now we find the final contradiction by doing a series of computations. Recall that, from Remark 1.15 applied to $A = A_s$ and using the formula from the previous proof, we have

$$l = a^2(w-1) + (a-\epsilon)^2 + \sum_{k \neq s} a_k^2 w_k \geqslant 1 + \sum_{k \neq s} a_k^2 w_k$$

using that $w - 1 > 0$ and at least one of a or $a - \epsilon$ is nonzero. By Proposition 1.21 we have that each $a_k \neq 0$ for $k = 1, \ldots, t$ so we conclude

$$l - 1 \geqslant \sum_{k=1}^{t} w_k. \tag{1.3.1}$$

Now fixing an A-exceptional character χ with degree d we have

$$|G| = \sum_{g \in G} |\chi(g)|^2 = d^2 + \sum_{k=1}^{s} \frac{|G|}{n_k l_k} \sum_{g \in A_i \backslash \{1\}} |\chi(g)|^2$$

where we have used that if $g \in A_k \backslash \{1\}$ then $|G : \mathbf{C}_G(g)| = |G : A_k| = |G|/n_k$ (in the left sum we coubnt the G-conjugates of any g inside A_k exactly l_k times!).

If $k \leqslant t$ then $\chi(g) = 0$ for any $g \in A_k \backslash \{1\}$ by Proposition 1.16 which implies that

$$\sum_{g \in A_k \backslash \{1\}} |\chi(g)|^2 = 0.$$

Further, if $k > t$ we know from the same result that $\chi(g)$ is a nonzero integer. Therefore

$$\sum_{g \in A_k \backslash \{1\}} |\chi(g)|^2 \geqslant n_i - 1.$$

Finally, if $k = s$ then we have from Proposition 1.18 that

$$\sum_{g \in A \backslash \{1\}} |\chi(g)|^2 \geqslant l(n - l).$$

We put this all together and obtain that

$$|G| \geqslant d^2 + \sum_{k=t+1}^{s-1} \frac{|G|}{n_k l_k}(n_k - 1) + \frac{|G|}{n}(n - l).$$

Now every element of $G \backslash \{1\}$ is G-conjugate to some element of an A_i. This implies that

$$|G| = 1 + \sum_{k=1}^{s} \frac{|G|}{n_k l_k}(n_k - 1)$$

(we are doing the same computation as before more or less). This implies that

$$1 + \sum_{k=1}^{s} \frac{|G|}{n_k l_k}(n_k - 1) \geqslant d^2 + \sum_{k=t+1}^{s-1} \frac{|G|}{n_k l_k}(n_k - 1) + \frac{|G|}{n}(n - l)$$

and therefore

$$1 + \sum_{k=1}^{t} \frac{|G|}{n_k l_k}(n_k - 1) + \frac{|G|}{nl}(n - 1) \geqslant d^2 + \frac{|G|}{n}(n - l)$$

and rearranging and dividing by $|G|$ we obtain

$$\sum_{k=1}^{t} \frac{(n_k - 1)}{n_k l_k} + \frac{(n - 1)}{nl} \geqslant (d^2 - 1)/|G| + 1 - \frac{l}{n}$$

which gives

$$\sum_{k=1}^{t} \frac{(n_k-1)}{n_k l_k} + \frac{(n-1+l^2)}{nl} \geqslant (d^2-1)/|G| + 1. \tag{1.3.2}$$

Now recall that by definition, $n = |A_s| \leqslant |A_k| = n_k$ if $k < s$, and that $w_k = (n_k-1)/l_k$. This implies

$$\sum_{k=1}^{t} (n_i-1)/n_i l_i \leqslant \left(\sum_{k=1}^{t} w_i\right)/n.$$

Using this and inequality 1.3.1 we have

$$(l-1)/n \geqslant \sum_{k=1}^{t} (n_i-1)/n_i l_i$$

and using inequlaity 1.3.2 we have

$$(l-1)/n + (n-1+l^2)/nl \geqslant 1 + (d^2-1)/|G|.$$

Since G is a nonabelian simple group, $d > 1$**.**. This implies that $(d^2-1)/|G| > 0$ so we obtain the strict inequality

$$(l-1)/n + (n-1+l^2)/nl > 1.$$

Now using $wl = n-1$ and multiplying by n on both sides we can rewrite the above as

$$l - 1 + (wl + l^2)/l > n$$

so

$$l - 1 + w + l > wl + 1$$

which implies $2(l-1) - w(l-1) > 0$ and therefore $(2-w)(l-1) > 0$ but this contradicts $w \geqslant 2$ and $l \geqslant 1$. We have finally proved

THEOREM 1.22 (Suzuki). *If a nonsolvable group G satisfies (W) then $|G|$ must be even.*

Using Brauer–Suzuki–Wall and Weisner this implies

THEOREM 1.23. *If G satisfies (W) then either G is solvable or G is* $\mathrm{PSL}(2, 2^k)$ *for some integer k.*

LECTURE 2

Generalized blocks and isometries

Gabriel A. L. Souza

This lecture is devoted to studying some of the fundamental ideas introduced in [**KOR03**] and relating them to the picky conjectures formulated by A. Moretó and N. Rizo. These are mostly generalizations of concepts from modular representation theory that can be found in [**Nav98**] - we will draw the parallels as we go, though none of that theory is needed here.

For a bit of background, Donkin used an isomorphism between the group algebra kS_n and one particular Hecke algebra to define "l-modular representations" of S_n, where l is a possibly composite number, as representations of other Hecke algebras (see [**Don03**]). Here, we will drop the assumption that $G = S_n$ and see some things we can still retain.

Remark 2.1. *Before starting, a note on notation. G will always be a finite group and $\mathcal{C}$ will always be a union of conjugacy classes of G (we will restrict this a bit later on). The contentious bit is that* ***the standard inner product of two complex class functions*** α, β ***of*** G ***will be denoted*** $\langle \alpha, \beta \rangle$*. *The reason for this will become clearer as we go (in part, it is because it follows* [**KOR03**]*).*

2.1. $\mathcal{C}$-Blocks

Definition 2.2. *Let $\alpha, \beta \in \mathrm{cf}(G)$. We define:*

$$\langle \alpha, \beta \rangle_{\mathcal{C}} = \frac{1}{|G|} \sum_{g \in \mathcal{C}} \alpha(g) \overline{\beta(g)}.$$

If $\langle \alpha, \beta \rangle_{\mathcal{C}} \neq 0$, we say they are ***directly $\mathcal{C}$-linked****; otherwise, we say they are* ***orthogonal across $\mathcal{C}$****.*

In principle, there is no way to guarantee that $g^{-1} \in \mathcal{C}$ when $g \in \mathcal{C}$. So, given two irreducible characters χ, ψ, $\langle \chi, \psi \rangle_{\mathcal{C}}$ has no reason to be real, much less an integer. Also, since we don't have $1 \in \mathcal{C}$, it can even happen that $\langle \chi, \chi \rangle_{\mathcal{C}} = 0$!

*Miquel, si prefieres cambiarlo a $[\alpha, \beta]$, hazlo (basta cambiar la definición de "inn") - es que, con el número de matrices que aparecen (+ el hecho que el proprio paper lo hace), creo que igual queda más claro así. Tambien siéntete libre para quitar los "left - right" en las definiciones de inn, innC, innD; de verdad que no sé cómo lo prefiero...

Using the relation of $\mathcal{C}$-linking, we can construct a graph with vertices $\mathrm{Irr}(G)$ where two irreducible characters are connected by an edge if and only if they are directly $\mathcal{C}$-linked.

Definition 2.3. *The $\mathcal{C}$-**blocks** of G are the connected components of the graph defined by $\mathcal{C}$-linking. If two irreducible characters χ, ψ are in the same $\mathcal{C}$-block, we say they are $\mathcal{C}$-**linked**.*

From the definition, if $\chi, \psi \in B$, where B is a $\mathcal{C}$-block of G, then there exist $\mu_i \in \mathrm{Irr}(G)$, $1 \leqslant i \leqslant n$, such that $\chi = \mu_0$, $\psi = \mu_n$ and $\langle \mu_i, \mu_{i+1} \rangle_{\mathcal{C}} \neq 0$ for all $0 \leqslant i \leqslant n-1$.

Remark 2.4. *When $\mathcal{C}$ is the union of the classes of the elements of G of order not divisible by a given prime p, the above definition is one of the characterizations of the Brauer p-blocks of G.*

Whenever $\alpha \in \mathrm{cf}(G)$, we write $\alpha^{\mathcal{C}}$ to denote the class function of G which coincides with α on the classes in $\mathcal{C}$ and is 0 for all other classes. We will also write $\mathcal{C}'$ to denote the complement of $\mathcal{C}$ in G. Note that this is also a union of conjugacy classes (namely, those not in $\mathcal{C}$).

Proposition 2.5. *Let $\alpha, \beta \in \mathrm{cf}(G)$. Then, $\left\langle \alpha^{\mathcal{C}}, \beta \right\rangle = \langle \alpha, \beta \rangle_{\mathcal{C}}$. In particular, if $\chi \in \mathrm{Irr}(G)$ is in a $\mathcal{C}$-block B, then both $\chi^{\mathcal{C}}$ and $\chi^{\mathcal{C}'}$ are $\mathbb{C}$-linear combinations of the characters in B.*

Proof. By definition, we have

$$\begin{aligned} \left\langle \alpha^{\mathcal{C}}, \beta \right\rangle &= \frac{1}{|G|} \sum_{g \in G} \alpha^{\mathcal{C}}(g) \overline{\beta(g)} \\ &= \frac{1}{|G|} \sum_{g \in \mathcal{C}} \alpha(g) \overline{\beta(g)} = \langle \alpha, \beta \rangle_{\mathcal{C}} . \end{aligned}$$

Thus, if $\psi \in \mathrm{Irr}(G)$, $\left\langle \chi^{\mathcal{C}}, \psi \right\rangle \neq 0$ if and only if $\langle \chi, \psi \rangle_{\mathcal{C}} \neq 0$, and this can only happen for irreducible characters in the block B.

Also, it is immediate from the definition that $\langle \chi, \psi \rangle_{\mathcal{C}} + \langle \chi, \psi \rangle_{\mathcal{C}'} = \langle \chi, \psi \rangle$. But then, if $\psi \neq \chi$ is directly $\mathcal{C}'$-linked to χ - equivalently, if it is one of the elements of $\mathrm{Irr}(G)$ in the decomposition of $\chi^{\mathcal{C}'}$ - it is also directly $\mathcal{C}$-linked to χ, meaning it is in B by the above. □

Of course, the properties we can hope to get depend highly on $\mathcal{C}$. To illustrate this, we have a couple of examples.

Example 2.6. *Suppose $\mathcal{C} = G$. Then, $\langle \chi, \psi \rangle_{\mathcal{C}} = \langle \chi, \psi \rangle$ for all $\chi, \psi \in \mathrm{Irr}(G)$. By the First Orthogonality Relation, this means χ is not directly $\mathcal{C}$-linked to any* *other ψ. Thus, the $\mathcal{C}$-linking graph is totally disconnected and each irreducible character is a $\mathcal{C}$-block.*

Example 2.7. *Suppose $\mathcal{C} = \{1\}$. Then, $\langle \chi, \psi \rangle_{\mathcal{C}} = \frac{\chi(1)\psi(1)}{|G|}$, which is always nonzero for all $\chi, \psi \in \mathrm{Irr}(G)$. Thus, every irreducible character is directly $\mathcal{C}$-linked to every other irreducible character, and the $\mathcal{C}$-linking graph is complete. In particular, there is a unique $\mathcal{C}$-block.*

Remark 2.8. *Notice that the two cases above are exactly what happen in modular representation theory when $p \nmid |G|$ and when G is a p-group, respectively!*

However, no matter which $\mathcal{C}$ we choose, we have a sufficient condition for a character to constitute a block. This is an indicator of some examples to come later.

Proposition 2.9. *Write $\mathrm{Irr}^{\mathcal{C}}(G) = \{\chi \in \mathrm{Irr}(G) \mid \exists x \in \mathcal{C}$ s.t. $\chi(x) \neq 0\}$. Then, if $\chi \notin \mathrm{Irr}^{\mathcal{C}}(G)$, $\{\chi\}$ is a $\mathcal{C}$-block of G.*

Proof. χ is not in $\mathrm{Irr}^{\mathcal{C}}(G)$ if and only if $\chi(x) = 0$ for all $x \in \mathcal{C}$. Then, it follows from the definition that χ cannot be directly $\mathcal{C}$-linked to any other irreducible character. $\square$

The converse is clearly false, as the first example of $\mathcal{C} = G$ shows, for instance.

Proposition 2.10. *Let B be a block of G and let $\theta = \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi$, $a_\chi \in \mathbb{C}$, be such that $\theta(x) = 0$ for all x not in $\mathcal{C}$. Then, the same is true of $\theta_B = \sum_{\chi \in B} a_\chi \chi$. The result also holds replacing $\mathcal{C}$ for $\mathcal{C}'$.*

Proof. For the sake of simplicity, we will prove only the stated implication, as the other one is nearly identical, using the symmetry in the preceding proposition.

Let $\mu \in \mathrm{Irr}(G)$. If $\mu \notin B$, then $\left\langle \chi^{\mathcal{C}'}, \mu \right\rangle = 0$ for all $\chi \in B$, using Proposition 2.5. Then,

$$\left\langle \sum_{\chi \in B} a_\chi \chi^{\mathcal{C}'}, \mu \right\rangle = 0. \tag{2.1.1}$$

Also, by hypothesis, $\sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi^{\mathcal{C}'} = 0$, meaning

$$\left\langle \sum_{\chi \in \mathrm{Irr}(G)} a_\chi \chi^{\mathcal{C}'}, \mu \right\rangle = 0,$$

whether μ is in B or not. This equation can be rewritten as

$$\left\langle \sum_{\chi\in B} a_\chi \chi^{\mathcal{C}'}, \mu \right\rangle + \left\langle \sum_{\chi\notin B} a_\chi \chi^{\mathcal{C}'}, \mu \right\rangle = 0. \tag{2.1.2}$$

Now suppose μ is in B. By Proposition 2.5, $\left\langle \sum_{\chi\notin B} a_\chi \chi^{\mathcal{C}'}, \mu \right\rangle = 0$, since $\left\langle \chi^{\mathcal{C}'}, \mu \right\rangle = 0$ for all $\chi \notin B$. Plugging this into Equation 2.1.2 and combining it with Equation 2.1.1, we get

$$\left\langle \sum_{\chi\in B} a_\chi \chi^{\mathcal{C}'}, \mu \right\rangle = 0, \forall \mu \in \mathrm{Irr}(G),$$

meaning $\sum_{\chi\in B} a_\chi \chi^{\mathcal{C}'} = 0$. But this just says that $\theta_B(x) = 0$ if x is not in $\mathcal{C}$, as we wanted. □

Corollary 2.11. *If B is a $\mathcal{C}$-block, $x \in \mathcal{C}$ and $y \in \mathcal{C}'$, then*

$$\sum_{\chi\in B} \chi(x)\overline{\chi(y)} = 0.$$

Proof. Let $\theta = \sum_{\chi\in \mathrm{Irr}(G)} \overline{\chi(y)}\chi$. Since $x \in \mathcal{C}$, it is not conjugate to y and thus, by the Second Orthogonality Relation, $\theta(x) = 0$. Since x is arbitrary then, by the preceding proposition, $\theta_B(x) = 0$ for all $x \in \mathcal{C}$. □

Remark 2.12. *In modular representation theory, this result is known as "Weak Block Orthogonality".*

In modular representation theory, there is a theorem due to Osima that says that the Brauer p-blocks are the minimal subsets of $\mathrm{Irr}(G)$ with the preceding property. The same result is actually true in our context, as we now show.

Proposition 2.13. *Let $S \subseteq \mathrm{Irr}(G)$ be such that, for all $x \in \mathcal{C}$, $y \in \mathcal{C}'$, we have*

$$\sum_{\chi\in S} \chi(x)\overline{\chi(y)} = 0.$$

Then, there exist $\mathcal{C}$-blocks $B_1, ..., B_k$ such that $S = B_1 \sqcup ... \sqcup B_k$.

Proof. Let $\theta = \sum_{\chi\in S} \overline{\chi(x)}\chi$. By hypothesis, θ is identically 0 on $\mathcal{C}'$. If $\mu \in \mathrm{Irr}(G)\backslash S$, then

$$\sum_{\chi\in S} \left\langle \mu^{\mathcal{C}}, \chi \right\rangle \chi(x) = \left\langle \mu^{\mathcal{C}}, \theta \right\rangle = \langle \mu, \theta \rangle = 0.$$

Then, it follows that $\sum_{\chi\in S}\left\langle \mu^{\mathcal{C}},\chi\right\rangle \chi$ is identically 0 on $\mathcal{C}$, since x was arbitrary. Write $\mu^{\mathcal{C}} = \alpha+\beta$, where all the irreducible summands of α are in S and those of β are not. Then, given $\chi \in S$, $\langle \beta,\chi\rangle = 0$, meaning $\left\langle \mu^{\mathcal{C}},\chi\right\rangle = \langle\alpha,\chi\rangle$. But

$$\alpha = \sum_{\chi\in S}\langle\alpha,\chi\rangle\chi = \sum_{\chi\in S}\left\langle\mu^{\mathcal{C}},\chi\right\rangle\chi$$

and thus $\alpha(x) = 0$ for all x in $\mathcal{C}$. Then, by definition, $\left\langle\alpha,\mu^{\mathcal{C}}\right\rangle = 0$, using that $\mathcal{C}\sqcup\mathcal{C}' = G$. Consequently, $\langle\alpha,\alpha\rangle = 0$, which is equivalent to saying $\alpha = 0$. In particular, $\langle\mu,\chi\rangle_{\mathcal{C}} = \left\langle\mu^{\mathcal{C}},\chi\right\rangle = 0$ for all χ in S. Thus, no element of S is directly $\mathcal{C}$-linked to an element outside of it, which is to say that S is a union of connected components of the $\mathcal{C}$-linking graph of G (which are exactly the $\mathcal{C}$-blocks). □

The class functions $\{\chi^{\mathcal{C}} \mid \chi\in\mathrm{Irr}(G)\}$ are almost never linearly independent (as a multiset), since there are $k(G)$ of them, and yet there can be fewer than $k(G)$ classes constituting $\mathcal{C}$. What we will now do is decompose them in a certain way and construct a couple of important matrices along the way.

Let $k = k(G)$ and write $\mathcal{C} = \bigsqcup_{i=1}^{t} C_i$. Consider the following ordering of the character table of G:

	C_1	C_2	$\cdots$	C_t	$\cdots$	C_k
χ_1	$*$	$*$	$\cdots$	$*$	$\cdots$	$*$
χ_2	$*$	$*$	$\cdots$	$*$	$\cdots$	$*$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
χ_t	$*$	$*$	$\cdots$	$*$	$\cdots$	$*$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$
χ_k	$*$	$*$	$\cdots$	$*$	$\cdots$	$*$

Let $X(\mathcal{C})$ denote the submatrix of the cells shaded in red; $X(\mathcal{C}) = [\chi_i(C_j)]_{i,j}$. Let $\Delta(\mathcal{C}) = \overline{X(\mathcal{C})^t}X(\mathcal{C})$. Then, $\Delta(\mathcal{C}) = \left[\sum_{r=1}^{k}\overline{\chi_k(C_i)}\chi_k(C_j)\right]_{i,j} = \mathrm{diag}\left(\frac{|G|}{|C_1|},\ldots,\frac{|G|}{|C_t|}\right)$ by the Second Orthogonality Relation.

Definition 2.14. *The matrix of $\mathcal{C}$-contributions of G is defined as* $\Gamma(\mathcal{C}) = [\langle\chi_i,\chi_j\rangle_{\mathcal{C}}]_{i,j}$.

Notice how

$$X(\mathcal{C})\Delta(\mathcal{C})^{-1} = \left[\chi_i(C_j)\frac{|G|}{|C_j|}\right]_{i,j},$$

from which

$$X(\mathcal{C})\Delta(\mathcal{C})^{-1}\overline{X(\mathcal{C})^t} = \left[\sum_{r=1}^{t}\chi_i(C_r)\frac{|G|}{|C_r|}\overline{\chi_j(C_r)}\right]_{i,j} = \Gamma(\mathcal{C}),$$

by definition.

Also, we note that, ordering the characters by $\mathcal{C}$-blocks, $\Gamma(\mathcal{C})$ admits a block-diagonal form, each block corresponding to one of the $\mathcal{C}$-blocks of G. We denote these matrices $\Gamma(\mathcal{C}, B)$.

Proposition 2.15. *$\Gamma(\mathcal{C})$ is an idempotent matrix of rank and trace t.*

Proof. By the preceding equations,

$$\begin{aligned}\Gamma(\mathcal{C})^2 &= X(\mathcal{C})\Delta(\mathcal{C})^{-1}\left(\overline{X(\mathcal{C})^t}X(\mathcal{C})\right)\Delta(\mathcal{C})^{-1}\overline{X(\mathcal{C})^t}\\ &= X(\mathcal{C})\Delta(\mathcal{C})^{-1}\overline{X(\mathcal{C})^t} = \Gamma(\mathcal{C}).\end{aligned}$$

Also, $X(\mathcal{C})$ clearly has rank t, since the character table has full rank. But $\Gamma(\mathcal{C})X(\mathcal{C}) = X(\mathcal{C})$, meaning $\Gamma(\mathcal{C})$ also has rank at least t, and, from $\Gamma(\mathcal{C}) = X(\mathcal{C})\Delta(\mathcal{C})^{-1}\overline{X(\mathcal{C})^t}$, its rank is also at most t. Finally, by the properties of the trace, $\operatorname{tr}(\Gamma(\mathcal{C})) = \operatorname{tr}\left(\Delta(\mathcal{C})^{-1}\Delta(\mathcal{C})\right) = t$. □

Now let $\{\Phi_i \mid 1 \leqslant i \leqslant t\}$ be a basis for the complex vector space of class functions which are identically 0 outside $\mathcal{C}$ and let $\mathbf{\Phi}(\mathcal{C}) = [\Phi_i(C_j)]_{i,j}$. By their linear independence, $\mathbf{\Phi}(\mathcal{C})$ is an invertible $(t \times t)$-matrix. Now, for each $i \in \{1, ..., k\}$, there exist uniquely determined $d_{ij} \in \mathbb{C}$ such that $\chi_i^{\mathcal{C}} = \sum_{j=1} d_{ij}\Phi_j$. As such, defining $D(\mathcal{C}) = [d_{ij}]_{i,j}$, we have $X(\mathcal{C}) = D(\mathcal{C})\mathbf{\Phi}(\mathcal{C})$.

Definition 2.16. *The $\mathcal{C}$-**decomposition matrix** of G regarding the basis $\{\Phi_i \mid 1 \leqslant i \leqslant t\}$ is the matrix $D(\mathcal{C})$ defined above.*

Remark 2.17. *We may regard the Φ_i as analogues to the Brauer characters (extended to the whole of G) from modular representation theory.*

We then have

$$\begin{aligned}\Gamma(\mathcal{C}) &= X(\mathcal{C})\Delta(\mathcal{C})^{-1}\overline{X(\mathcal{C})^t} = D(\mathcal{C})\mathbf{\Phi}(\mathcal{C})\Delta(\mathcal{C})^{-1}\overline{\mathbf{\Phi}(\mathcal{C})^t D(\mathcal{C})^t}\\ &= D(\mathcal{C})\mathbf{\Phi}(\mathcal{C})\left(\overline{\mathbf{\Phi}(\mathcal{C})^t D(\mathcal{C})^t}D(\mathcal{C})\mathbf{\Phi}(\mathcal{C})\right)^{-1}\overline{\mathbf{\Phi}(\mathcal{C})^t D(\mathcal{C})^t}\\ &= D(\mathcal{C})\left(\overline{D(\mathcal{C})^t}D(\mathcal{C})\right)^{-1}\overline{D(\mathcal{C})^t}.\end{aligned}$$

Definition 2.18. *The $\mathcal{C}$-**Cartan matrix** of G with respect to the basis regarding the basis $\{\Phi_i \mid 1 \leqslant i \leqslant t\}$ is the matrix $C(\mathcal{C}) = \overline{D(\mathcal{C})^t}D(\mathcal{C})$.*

Notice how

$$\begin{aligned}C(\mathcal{C})^{-1} &= \left(\overline{D(\mathcal{C})^t}D(\mathcal{C})\right)^{-1} = \mathbf{\Phi}(\mathcal{C})\left(\overline{\mathbf{\Phi}(\mathcal{C})^t D(\mathcal{C})^t}D(\mathcal{C})\mathbf{\Phi}(\mathcal{C})\right)^{-1}\overline{\mathbf{\Phi}(\mathcal{C})^t}\\ &= \mathbf{\Phi}(\mathcal{C})\Delta(\mathcal{C})^{-1}\overline{\mathbf{\Phi}(\mathcal{C})^t} = [\langle\Phi_i, \Phi_j\rangle]_{i,j}\end{aligned}$$

2.2. Closed unions

Thus far, we have considered an arbitrary union of conjugacy classes $\mathcal{C}$. We will now only consider a special kind of union, where we will have more to say about the structure of $\mathcal{C}$-blocks and associated concepts.

Definition 2.19. *A union $\mathcal{C}$ of conjugacy classes of G is called* ***closed*** *if, when $x \in \mathcal{C}$, then every generator of $\langle x \rangle$ is also in $\mathcal{C}$.*

An immediate consequence of this restriction is that the inner product $\langle \chi, \psi \rangle_{\mathcal{C}}$ of two irreducible characters is more controlled, as we can see below.

Proposition 2.20. *If $\mathcal{C}$ is closed, then $|G| \langle \chi, \psi \rangle_{\mathcal{C}} \in \mathbb{Z}$ for all $\chi, \psi \in \mathrm{Irr}(G)$.*

Proof. Let $n = \exp(G)$ and let $\sigma \in \mathrm{Gal}(\mathbb{Q}_n : \mathbb{Q})$. Then, we have

$$\begin{aligned}\sigma\left(|G| \langle \chi, \psi \rangle_{\mathcal{C}}\right) &= \sigma\left(\sum_{g \in \mathcal{C}} \chi(g)\overline{\psi(g)}\right) = \sum_{g \in \mathcal{C}} \sigma(\chi(g))\sigma\left(\overline{\psi(g)}\right) \\ &= \sum_{g \in \mathcal{C}} \chi(g^k)\overline{\psi(g^k)} = \sum_{g \in \mathcal{C}} \chi(g)\overline{\psi(g)} = |G| \langle \chi, \psi \rangle_{\mathcal{C}} ,\end{aligned}$$

where k is the positive integer coprime with n such that $\sigma(\zeta_n) = \zeta_n^k$ (where ζ_n is a primitive n-root of unity) and the fourth equality uses that $g \mapsto g^k$ is a bijection from $\mathcal{C}$ to $\mathcal{C}$, as this union is closed. Since σ is arbitrary, $|G| \langle \chi, \psi \rangle_{\mathcal{C}} \in \mathbb{Q}$. But this is an algebraic integer, so $|G| \langle \chi, \psi \rangle_{\mathcal{C}} \in \mathbb{Z}$. $\square$

Most cases we are interested in actually satisfy this condition.

Example 2.21. *If g is p-regular, then so is every generator of $\langle g \rangle$. Thus, the set of p-regular elements of G is a closed union of conjugacy classes. Recall that this is the case studied in modular representation theory.*

Example 2.22. *Let $\mathcal{P} = \{x \in G \mid x \text{ is } p\text{-picky}\}$, where p is a fixed prime. $\mathcal{P}$ is clearly a union of conjugacy classes of G and it is closed, since, if $(p, k) = 1$ and x^k is contained in more than one Sylow p-subgroup, so is $x^{kl} = x$ (l is the multiplicative inverse of k mod p).*

More on Example 2.22

Let $P \in \mathrm{Syl}_p(G)$ and let $x \in P$ be picky. If x is G-conjugate to y and $y \in \mathbf{N}_G(P)$, then there exists $g \in G$ such that $y = x^g \in P^g$. But $y \in P$ (since y is a p-element in $\mathbf{N}_G(P)$). Thus, $g \in \mathbf{N}_G(P)$, by pickyness, and we get $x^G \cap \mathbf{N}_G(P) = x^{\mathbf{N}_G(P)}$. Also, if $c \in \mathbf{C}_G(x)$, then $x \in P^c$, meaning $c \in \mathbf{N}_G(P)$; i.e., $\mathbf{C}_G(x) \subseteq \mathbf{N}_G(P)$ and $x^{\mathbf{N}_G(P)}$ is a conjugacy class of $\mathbf{N}_G(x)$ satisfying Definition 1.2!

Example 2.23. *Let $\mathcal{P}$ be as in Example 2.22. Let $x \in \mathcal{P}$ and define $\mathcal{S}(x) = \{y \in G \mid y_p$ is G-conjugate to $x\}$ and $\mathcal{S} = \{x \in G \mid x_p \in (P)\}$ ($\mathcal{S}(x)$ is called the p-**section** of x). Let $\{x_1, ..., x_t\}$ be a complete set of representatives of the picky classes of G, so that $\mathcal{P} = \bigsqcup_{i=1}^t x_i^G$. Then, we have $\mathcal{S} = \bigsqcup_{i=1}^t \mathcal{S}(x_i)$.*

Now take $x \in \mathcal{P}$ and let $\{y_1, ..., y_{s_x}\}$ be a complete set of representatives of the conjugacy classes of p-regular elements in $\mathbf{C}_G(x)$. If $z \in \mathcal{S}(x)$, then there exists $u \in G$ such that $z_p^u = x$. In particular, $z_{p'}^u \in \mathbf{C}_G(x)$ and thus, there exist some $v \in \mathbf{C}_G(x)$ and a uniquely determined j such that $z_{p'}^{uv} = y_j$. Thus, $z^{uv} = xy_j$. Also, if $(xy_i)^g = xy_j$, then $g \in \mathbf{C}_G(x)$, by the uniqueness of the decomposition into p and p'-parts, and $y_i^g = y_j$, implying $i = j$. In summary, $\mathcal{S}(x) = \bigsqcup_{j=1}^{s_x} (xy_j)^G$. Combining this with what we had before shows that $\mathcal{S}$ is a union of conjugacy classes of G. If $x \in \mathcal{S}$, then x_p is picky, and thus so is $x_p^k = (x^k)_p$, meaning this union is closed.

More on Example 2.23

Let $P \in \mathrm{Syl}_p(G)$ and let $\{x_1, ..., x_t\}$ be a complete set of representatives of the p-picky G-conjugacy classes, such that $x_i \in P$ for all i. Then, $\mathcal{P}_{\mathbf{N}_G(P)} := \mathcal{P} \cap \mathbf{N}_G(P) = \bigsqcup_{i=1}^t x_i^{\mathbf{N}_G(P)}$ is a disjoint union of conjugacy classes of $\mathbf{N}_G(P)$. Also, $\mathbf{C}_G(x_i) \subseteq \mathbf{N}_G(P)$, as we had seen, so they all satisfy Definition 1.2.

Just like what was done in Example 2.23, we can show that $\mathcal{S}_{\mathbf{N}_G(P)} := \mathcal{S} \cap \mathbf{N}_G(P)$ is a disjoint union of conjugacy classes of $\mathbf{N}_G(P)$, whose representatives are $x_i y_{ij}$, where $\{y_{i1}, ..., y_{is_i}\}$ is a complete set of representatives of the classes of p-regular elements of $\mathbf{C}_G(x_i)$. Since $\mathbf{C}_G(x_i y_{ij}) \subseteq \mathbf{C}_G(x_i) \subseteq \mathbf{N}_G(P)$ (again using the uniqueness of the decomposition into p and p'-parts), these are all classes satisfying Definition 1.2!

If $z \in \mathcal{S}_{\mathbf{N}_G(P)}$, there exists $g \in \mathbf{N}_G(P)$ such that $z_p^g = x_i$ for some (uniquely determined) i. Then, $(z_p^k)^g = x_i^k$, which is also a picky element in $\mathbf{N}_G(P)$, and so there exists $h \in \mathbf{N}_G(P)$ such that $(x_i^k)^h = x_j$ for some (uniquely determined) j. Thus, $(z^k)_p^{gh} = x_j$ and $z^k \in \mathcal{S}_{\mathbf{N}_G(P)}$. Thus, the union is closed.

From now on, **we will assume that $\mathcal{C}$ is closed**. We may define a couple of special $\mathbb{Z}$-modules associated to $\mathcal{C}$.

First, we denote $\mathfrak{R}(\mathcal{C}) = \mathbb{Z}\left[\chi^{\mathcal{C}} \mid \chi \in \mathrm{Irr}(G)\right]$, the $\mathbb{Z}$-module generated by the $\chi^{\mathcal{C}}$, which is contained in $\mathrm{cf}(G)$. We also denote $\mathfrak{P}(\mathcal{C}) = \mathfrak{R}(\mathcal{C}) \cap \mathbb{Z}[\mathrm{Irr}(G)]$, which is the $\mathbb{Z}$-module of generalized characters in $\mathfrak{R}(\mathcal{C})$. Finally, we write $\mathrm{Cart}(\mathcal{C}) = \mathfrak{R}(\mathcal{C})/\mathfrak{P}(\mathcal{C})$.

Proposition 2.24. *$|G|\mathfrak{R}(\mathcal{C}) \subseteq \mathfrak{P}(\mathcal{C}) \subseteq \mathfrak{R}(\mathcal{C})$; in particular, both are free $\mathbb{Z}$-modules of the same rank.*

Proof. Since $\mathrm{cf}(G)$ is torsion-free as a $\mathbb{Z}$-module, the same is true of $\mathfrak{R}(\mathcal{C})$ and $\mathfrak{P}(\mathcal{C})$. Also, they are both finitely generated. Thus, by a standard result on modules over PIDs, all we have to do is prove the inclusions to show that they have the same rank.

Let $\chi \in \mathrm{Irr}(G)$ and let $\psi \in \mathrm{Irr}(G)$. Since $|G| \langle \chi, \psi \rangle_{\mathcal{C}} \in \mathbb{Z}$ and $\langle \chi, \psi \rangle_{\mathcal{C}} = \left\langle \chi^{\mathcal{C}}, \psi \right\rangle$, we have $\left\langle |G|\chi^{\mathcal{C}}, \psi \right\rangle \in \mathbb{Z}$. Since ψ is arbitrary, $|G|\chi^{\mathcal{C}}$ is a generalized character, which finishes the proof. □

Corollary 2.25. $\mathrm{Cart}(\mathcal{C})$ *is a finite abelian group whose exponent divides* $|G|$.

Proof. By the preceding proposition, $\mathrm{Cart}(\mathcal{C})$ is a $\mathbb{Z}$-module of rank 0. The fact on the exponent follows from $|G|\mathfrak{R}(\mathcal{C}) \subseteq \mathfrak{P}(\mathcal{C})$. □

Definition 2.26. *A $\mathcal{C}$-**basic set** is a $\mathbb{Z}$-basis for $\mathfrak{R}(\mathcal{C})$.*

We will denote $k(\mathcal{C})$ the number of conjugacy classes of G whose union composes $\mathcal{C}$.

Proposition 2.27. *Let $t = k(\mathcal{C})$. Then,* $\mathrm{rank}(\mathfrak{P}(\mathcal{C})) = \mathrm{rank}(\mathfrak{R}(\mathcal{C})) = t$ *and every $\mathcal{C}$-basic set is $\mathbb{C}$-linearly independent.*

Proof. Since $X(\mathcal{C})$ is a matrix with full rank equal to t, $\mathrm{rank}_{\mathbb{Z}}(\mathfrak{R}(\mathcal{C})) \geqslant t$, as $\mathbb{C}$-linear independence implies $\mathbb{Z}$-independence. Analogously, $\mathrm{rank}_{\mathbb{Z}}(\mathfrak{R}(\mathcal{C}')) \geqslant k(G) - t$. If $\chi \in \mathrm{Irr}(G)$, then, by definition, $\chi = \chi^{\mathcal{C}} + \chi^{\mathcal{C}'}$. Thus, $|G|\chi \in \mathfrak{P}(\mathcal{C}) \oplus \mathfrak{P}(\mathcal{C}')$. It easily follows that $|G|\mathbb{Z}[\mathrm{Irr}(G)] \subseteq \mathfrak{P}(\mathcal{C}) \oplus \mathfrak{P}(\mathcal{C}') \subseteq \mathbb{Z}[\mathrm{Irr}(G)]$ and thus $\mathrm{rank}_{\mathbb{Z}}(\mathfrak{R}(\mathcal{C})) + \mathrm{rank}_{\mathbb{Z}}(\mathfrak{R}(\mathcal{C})) = k(G) = t + (k(G) - t)$. From the previous inequalities, we get $\mathrm{rank}_{\mathbb{Z}}(\mathfrak{R}(\mathcal{C})) = t$.

Now, suppose $\mathscr{B} = \{\theta_i \mid 1 \leqslant i \leqslant t\}$ is a $\mathcal{C}$-basic set. Then $\chi^{\mathcal{C}} \in \mathrm{span}_{\mathbb{Z}} \mathscr{B} \subseteq \mathrm{span}_{\mathbb{C}} \mathscr{B}$ for all $\chi \in \mathrm{Irr}(G)$. In particular, $\mathrm{span}_{\mathbb{C}}\{\chi^{\mathcal{C}} \mid \chi \in \mathrm{Irr}(G)\} \subseteq \mathrm{span}_{\mathbb{C}} \mathscr{B}$. But the former has dimension t, since $X(\mathcal{C})$ has full rank. Thus, $\mathscr{B}$ is $\mathbb{C}$-linearly independent. □

Let B be a $\mathcal{C}$-block. Then, since $\chi^{\mathcal{C}}$ has all of its components in the block B, if we define $\mathfrak{R}(\mathcal{C}, B) = \mathbb{Z}\left[\chi^{\mathcal{C}} \mid \chi \in B\right]$ and $\mathfrak{P}(\mathcal{C}, B) = \mathfrak{R}(\mathcal{C}, B) \cap \mathbb{Z}[\mathrm{Irr}(G)]$, then we obtain two direct sum decompositions:

$$\mathfrak{R}(\mathcal{C}) = \bigoplus_{B \in \mathrm{Bl}_{\mathcal{C}}(G)} \mathfrak{R}(\mathcal{C}, B)$$

$$\mathfrak{P}(\mathcal{C}) = \bigoplus_{B \in \mathrm{Bl}_{\mathcal{C}}(G)} \mathfrak{P}(\mathcal{C}, B),$$

from which we also have a decomposition

$$\mathrm{Cart}(\mathcal{C}) = \bigoplus_{B \in \mathrm{Bl}_{\mathcal{C}}(G)} \frac{\mathfrak{R}(\mathcal{C}, B)}{\mathfrak{P}(\mathcal{C}, B)}.$$

We write $\mathrm{Cart}(\mathcal{C}, B) = \mathfrak{R}(\mathcal{C}, B)/\mathfrak{P}(\mathcal{C}, B)$.

Definition 2.28. *Let B be a $\mathcal{C}$-block of G and let $\{\theta_i \mid 1 \leqslant i \leqslant s\}$ be a $\mathbb{Z}$-basis for $\mathfrak{P}(\mathcal{C}, B)$. The* ***Cartan matrix*** *associated to B in this basis is the matrix $[\langle \theta_i, \theta_j \rangle_{\mathcal{C}}]_{i,j}$*[†]

Proposition 2.29. *Let B be a $\mathcal{C}$-block, let $\{\theta_i \mid 1 \leqslant i \leqslant s\}$ and $\{\eta_i \mid 1 \leqslant i \leqslant s\}$ be two $\mathbb{Z}$-bases for $\mathfrak{P}(\mathcal{C}, B)$ and let C, C' be the corresponding Cartan matrices. Then, C, C' are integer matrices and there exists a matrix $A \in \mathrm{Mat}_s(\mathbb{Z})$ such that $|\det(A)| = 1$ and $C' = A^t C A$.*

Proof. Since the θ_i are generalized characters which are identically 0 outside of $\mathcal{C}$, we have $\langle \theta_i, \theta_j \rangle_{\mathcal{C}} = \langle \theta_i, \theta_j \rangle \in \mathbb{Z}$, for all i, j. Let $a_{ij} \in \mathbb{Z}$ be such that $\eta_i = \sum_{k=1}^{s} a_{ik} \theta_k$. Then, we get:

$$\begin{aligned}
\left[\langle \eta_i, \eta_j \rangle_{\mathcal{C}}\right]_{i,j} &= \left[\sum_{k=1}^{s} a_{ik} \langle \theta_k, \eta_j \rangle_{\mathcal{C}}\right]_{i,j} = [a_{ij}] \left[\langle \theta_i, \eta_j \rangle_{\mathcal{C}}\right] \\
&= [a_{ij}] \left[\sum_{k=1}^{s} a_{jk} \langle \theta_i, \theta_k \rangle_{\mathcal{C}}\right] = [a_{ij}] \left[\langle \theta_i, \theta_j \rangle_{\mathcal{C}}\right] [a_{ij}]^t \\
&= A^t \left[\langle \theta_i, \theta_j \rangle_{\mathcal{C}}\right]_{i,j} A,
\end{aligned}$$

where $A = [a_{ij}]^t$. Reversing the roles of η and θ, we get the desired result. The part on the determinant of A follows from a well-known result on modules over integral domains (notice that A^t is the matrix of the η_j written on the basis $\{\theta_i \mid 1 \leqslant i \leqslant s\}$). □

Problem 2.30. *Let $d_1 \mid \cdots \mid d_s$ be the invariant factors of the finite $\mathbb{Z}$-module $\mathrm{Cart}(\mathcal{C}, B)$. Show that $d_s = \min\{d \in \mathbb{N}^* \mid d\Gamma(\mathcal{C}, B)$ is an integer matrix$\}$ and that the invariant factors of the (integer) matrix $d_s\Gamma(\mathcal{C}, B)$ are*

$$1, \frac{d_s}{d_{s-1}}, \frac{d_s}{d_{s-2}}, \ldots, \frac{d_s}{d_1}$$

[†] I do not like this name, as it is analogous to the **inverse** to our previous "Cartan matrix". But it is how they define it in [**KOR03**]...

2.3. Isometries

In this final short section, we define the concept of a **generalized perfect isometry** and apply it to our case of picky elements; this is related to some work in progress by A. Moretó and N. Rizo. For reference, in [**Bro90**], M. Broué gives a definition which can be stated as follows:

DEFINITION 2.31 (Broué). *Let G, H be finite groups and let B, B' be Brauer p-blocks of G, H respectively. Then, an isometry $I : \mathbb{Z}[\mathrm{Irr}(B)] \to \mathbb{Z}[\mathrm{Irr}(B')]$ (i.e., a bijection preserving the ordinary inner product of $\mathrm{cf}(G)$) is called **perfect** if the function*

$$\begin{aligned}\mu_I : G \times H &\to \mathbb{C}\\ (g, h) &\mapsto \sum_{\chi \in \mathrm{Irr}(B)} \chi(g) I(\chi)(h)\end{aligned}$$

satisfies the following two conditions:

(i) *if exactly one of g, h is p-regular, then $\mu_I(g, h) = 0$;*

(ii) $\frac{\mu_I(g,h)}{|\mathbf{C}_G(g)|_p}$ *and* $\frac{\mu_I(g,h)}{|\mathbf{C}_H(h)|_p}$ *are algebraic integers.*

*If such a function exists, B and B' are called **perfectly isometric**.*

There is a lot of literature on (perfect) isometries and their relations to other kinds of equivalences (such as Morita equivalences); see [**Sam20**]. For our context, [**KOR03**] gives the following definition. It looks much weaker in principle, but according to [**Sam20**], it actually preserves many important theorems on perfect isometries.

DEFINITION 2.32. *As before, let G, H be finite groups and $\mathcal{C}$, $\mathcal{D}$ closed unions of conjugacy classes of G, H (respectively). Let B be a $\mathcal{C}$-block of G and B', a $\mathcal{D}$-block of H. Write $B = \{\chi_i \mid 1 \leqslant i \leqslant n\}$. A **generalized perfect isometry** from B to B' is a bijection $I : B \to B'$ such that there exist $\epsilon_i \in \{1, -1\}$, $1 \leqslant i \leqslant n$ satisfying $\langle \chi_i, \chi_j \rangle_{\mathcal{C}} = \langle \epsilon_i I(\chi_i), \epsilon_j I(\chi_j) \rangle_{\mathcal{D}}$, for all i, j. In this case, we say B, B' are **perfectly isometric**.*

Equivalently, the blocks B, B' are perfectly isometric in this sense if there exists a matrix $S = \mathrm{diag}(\epsilon_1, ..., \epsilon_n)$, with $\epsilon_i = \pm 1$ for all i, such that $\Gamma(\mathcal{C}, B) = S\Gamma(\mathcal{D}, B')S$.

We remark that essentially the same notion works for *unions of* blocks, rather then blocks individually. So the question of finding generalized perfect isometries, as well as most of what we have been doing, admits both the local version at a specific block and the global version, taking unions of the blocks.

Proposition 2.33. *Let B, B' be perfectly isometric (unions of) blocks and write $B = \{\chi_i \mid 1 \leqslant i \leqslant s\}$, $B' = \{\mu_i \mid 1 \leqslant i \leqslant s\}$ such that $\langle \chi_i, \chi_j \rangle_{\mathcal{C}} = \langle \epsilon_i \mu_i, \epsilon_j \mu_j \rangle_{\mathcal{D}}$. Then:*

- *there exists a $\mathbb{Z}$-module isomorphism $\varphi : \mathfrak{R}(\mathcal{C}, B) \to \mathfrak{R}(\mathcal{D}, B')$ which restricts to an isomorphism from $\mathfrak{P}(\mathcal{C}, B)$ to $\mathfrak{P}(\mathcal{D}, B')$;*
- *there exist bases of $\mathfrak{P}(\mathcal{C}, B)$ and $\mathfrak{P}(\mathcal{D}, B')$ such that the Cartan matrices of B, B' regarding those bases are equal.*

Proof. Let $\mathscr{X} = \sum_{i=1}^{s} a_i \chi_i^{\mathcal{C}}$ be an arbitrary element of $\mathfrak{R}(\mathcal{C}, B)$. Notice that

$$\left\langle \sum_{i=1}^{s} a_i \epsilon_i \mu_i^{\mathcal{D}}, \mu_j \right\rangle = \left\langle \sum_{i=1}^{s} a_i \epsilon_i \mu_i^{\mathcal{D}}, \mu_j \right\rangle_{\mathcal{D}} = \sum_{i=1}^{s} a_i \langle \epsilon_i \mu_i, \mu_j \rangle_{\mathcal{D}}$$
$$= \sum_{i=1}^{s} a_i \langle \chi_i, \epsilon_j \chi_j \rangle_{\mathcal{C}} = \epsilon_j \left\langle \sum_{i=1}^{s} a_i \chi_i^{\mathcal{C}}, \chi_j \right\rangle = \epsilon_j \langle \mathscr{X}, \chi_j \rangle.$$

If $\mu \in \mathrm{Irr}(H) \backslash B'$, then, since the $\mu_i^{\mathcal{D}}$ only have components in the block B', the inner product above is 0.

Define $\varphi : \mathfrak{R}(\mathcal{C}, B) \to \mathfrak{R}(\mathcal{D}, B')$ by sending $\mathscr{X}$ to $\sum_{i=1}^{s} a_i \epsilon_i \mu_i^{\mathcal{D}}$. This is clearly $\mathbb{Z}$-linear. To show it is well defined and injective, note that $\varphi(\mathscr{X}) = 0$ if and only if $\langle \varphi(\mathscr{X}), \mu \rangle = 0$ for all $\mu \in \mathrm{Irr}(H)$. By the observation in the previous paragraph, this is equivalent to $\langle \varphi(\mathscr{X}), \mu_j \rangle$ for all $1 \leqslant j \leqslant s$. But, using the equation above, $\langle \varphi(\mathscr{X}), \mu_j \rangle = 0$ if and only if $\langle \mathscr{X}, \chi_j \rangle = 0$. Thus, $\varphi(\mathscr{X}) = 0$ if and only if $\mathscr{X} = 0$.

Finally, we can define $\psi : \mathfrak{R}(\mathcal{D}, B') \to \mathfrak{R}(\mathcal{C}, B)$ by sending an element $\sum_{i=1}^{s} b_i \mu_i^{\mathcal{D}}$ to $\sum_{i=1}^{s} b_i \epsilon_i \chi_i^{\mathcal{C}}$. It follows by arguing as above that ψ is well-defined, and a simple computation gives $\psi = \varphi^{-1}$.

From our previous computations, $\langle \mathscr{X}, \chi_j \rangle \in \mathbb{Z}$ if and only if $\langle \varphi(\mathscr{X}), \mu_j \rangle \in \mathbb{Z}$. Thus, φ restricts to an isomorphism from $\mathfrak{P}(\mathcal{C}, B)$ to $\mathfrak{P}(\mathcal{D}, B')$. Notice as well, writing $\mathscr{Y} = \sum_{i=1}^{s} b_i \chi_i^{\mathcal{C}}$, that

$$\langle \varphi(\mathscr{X}), \varphi(\mathscr{Y}) \rangle = \sum_{j=1}^{s} b_j \epsilon_j \langle \varphi(\mathscr{X}), \mu_j \rangle = \sum_{j=1}^{s} b_j \epsilon_j^2 \langle \mathscr{X}, \chi_j \rangle$$
$$= \left\langle \mathscr{X}, \sum_{j=1}^{s} b_j \chi_j \right\rangle = \langle \mathscr{X}, \mathscr{Y} \rangle.$$

Thus, if $\mathscr{B}$ is a $\mathbb{Z}$-basis for $\mathfrak{P}(\mathcal{C}, B)$, the Cartan matrices in the bases $\mathscr{B}$ and $\varphi(\mathscr{B})$ are equal. □

Finally, we end-off by obtaining what is almost a generalized perfect isometry in a particular case we are interested in.

Proposition 2.34. *Let $P \in \mathrm{Syl}_p(G)$, let $N = \mathbf{N}_G(P)$ and take the notation from Example 2.23. Then, the induction map $\uparrow_N^G\colon \mathbb{Z}[\mathrm{Irr}(N)] \to \mathbb{Z}[\mathrm{Irr}(G)]$ defines a not-necessarily bijective generalized perfect isometry from $\mathfrak{P}(\mathcal{S}_N)$ into $\mathfrak{P}(\mathcal{S})$.*

Proof. If χ is any generalized character of N which is 0 outside of the picky sections, then the same is trivially true for χ^G, by the induction formula (and since $P \subseteq N$). Thus, the map $\chi \mapsto \chi^G$ is well-defined from $\mathfrak{P}(\mathcal{S}_N)$ into $\mathfrak{P}(\mathcal{S})$, and is known to be $\mathbb{Z}$-linear. All we have to do, then, is show that $\left\langle \alpha^G, \beta^G \right\rangle_{\mathcal{S}} = \langle \alpha, \beta \rangle_{\mathcal{S}_N}$ for $\alpha, \beta \in \mathfrak{P}(\mathcal{S}_N)$.

For simplicity, let $\{x_1, ..., x_t\}$ be a complete set of representatives of the picky conjugacy classes of G such that $x_i \in P$ for all i. If $g \in S$, then g is G-conjugate to $x_i y_{ij}$, where the $\{y_{i1}, ..., y_{is_i}\}$ is a complete set of representatives of the p-regular classes in $\mathbf{C}_G(x_i)$. Thus, we only need to calculate $(x_i y_{ij})^G$.

Fix a right transversal $\mathcal{T}$ for N in G. Then, for each coset $N\mathfrak{t}$, $|(x_i y_{ij})^{N\mathfrak{t}}| = |(x_i y_{ij})^N|$. If $(x_i y_{ij})^{n\mathfrak{t}} = (x_i y_{ij})^{n'\mathfrak{t}'}$ for some $n, n' \in N$, $\mathfrak{t}, \mathfrak{t}' \in \mathcal{T}$, then $x_i^{n\mathfrak{t}\mathfrak{t}'^{-1}n'^{-1}} = x_i \in P$. Thus, $x_i^n \in P^{\mathfrak{t}'\mathfrak{t}^{-1}}$. But, since $n \in N$, we also have $x_i^n \in P$. As x_i is picky, $\mathfrak{t}'\mathfrak{t}^{-1} \in N$, which means $\mathfrak{t} = \mathfrak{t}'$.

So there are $[G:N]$ G-conjugates of $x_i y_{ij}$ for each N-conjugate. I.e., $|(x_i y_{ij})^G| = [G:N]|(x_i y_{ij})^N|$. Thus, we have

$$\left\langle \alpha^G, \beta^G \right\rangle_{\mathcal{S}} = \frac{1}{|G|} \sum_{g \in S} \alpha^G(g)\overline{\beta^G(g)} = \frac{1}{|G|} \sum_{i=1}^{t} \sum_{j=1}^{s_i} |(x_i y_{ij})^G| \alpha^G(x_i y_{ij})\overline{\beta^G(x_i y_{ij})}$$

$$= \frac{1}{|N|} \sum_{i=1}^{t} \sum_{j=1}^{s_i} |(x_i y_{ij})^N| \alpha^G(x_i y_{ij})\overline{\beta^G(x_i y_{ij})}.$$

Now, by the induction formula, we have

$$\alpha^G(x_i y_{ij}) = \frac{1}{|N|} \sum_{h \in G} \alpha^\circ((x_i y_{ij})^h) = \frac{1}{|N|} \sum_{h \in N} \alpha((x_i y_{ij})^h) = \alpha(x_i y_{ij}),$$

where the second equality comes since x_i is picky. Combining this equation with the previous one, we obtain

$$\left\langle \alpha^G, \beta^G \right\rangle_{\mathcal{S}} = \frac{1}{|N|} \sum_{i=1}^{t} \sum_{j=1}^{s_i} |(x_i y_{ij})^N| \alpha(x_i y_{ij})\overline{\beta(x_i y_{ij})} = \langle \alpha, \beta \rangle_{\mathcal{S}_N},$$

as wanted. □

LECTURE 3

Requirement for existence of perfect isometries

David Cabrera Berenguer

This lecture is devoted to study a conditions required to ensure the possible existence of a perfect isometry between the principal block of $\mathbf{N}_G(P)$ and G, when G is simple and $\mathbf{N}_G(P)/\mathbf{O}_{p'}(\mathbf{N}_G(P))$ is a Frobenius group. We shall follow the structure of [**Rob00**]

3.1. Preliminaries

Let $\mathbf{R}$ be the ring of algebraic integers in $\mathbb{C}$ and let p be a fixed prime. Let M be a maximal ideal of $\mathbf{R}$ with $p\mathbf{R} \subseteq M$. Thus $F = \mathbf{R}/M$ is a field a characteristic p and the projection $^* : \mathbf{R} \to F$ is a ring homomorphism.

Let $\mathbf{U} = \{\varepsilon \in \mathbb{C} : \varepsilon^m = 1 \text{ for some integer } m \text{ with } (m, p) = 1\}$.

Lemma 3.1. *The restriction of * to $\mathbf{U}$ defines an isomorphism of groups $\mathbf{U} \to F^\times$. Also, F is algebraically closed.*

Proof. See Lemma (2.1) of [**Nav98**]. □

Definition 3.2. *Let G^0 be the set of p-regular elements of G and let $\mathcal{X} : G \to \mathrm{GL}_n(F)$ be a representation of G. Let $g \in G^0$. Since F is algebraically closed, by the previous lemma we have that the eigenvalues of $\mathcal{X}(g)$ are of the form $\varepsilon_1^*, \ldots, \varepsilon_n^*$ for uniquely determined $\varepsilon_1, \ldots, \varepsilon_n \in \mathbf{U}$. We say that*

$$\begin{array}{rccc} \varphi : & G^0 & \longrightarrow & \mathbb{C} \\ & g & \longmapsto & \varepsilon_1 + \ldots + \varepsilon_n \end{array}$$

*is the **Brauer character** afforded by the representation $\mathcal{X}$. We say that φ is **irreducible** if $\mathcal{X}$ is irreducible, and we denote the set of irreducible Brauer characters by $\mathrm{IBr}(G)$.*

Theorem 3.3. *$\mathrm{IBr}(G)$ is a basis of $\mathrm{cf}(G^0)$, the class functions of G^0. Thus, $|\mathrm{IBr}(G)|$ is the number of conjugacy classes of p-regular elements of G.*

Proof. See Corollary (2.10) of [**Nav98**]. □

If $\chi \in \mathrm{Irr}(G)$ and $\mathcal{X}$ is a representation affording χ then $\mathcal{X}(\hat{K}) = \omega_\chi(\hat{K})I$ for every class sum $\hat{K}$ of G. Thus $\omega_\chi : \mathbf{Z}(\mathbb{C}G) \to \mathbf{R}$ defines an algebra homomorphism. We may construct an F-linear map

$$\begin{array}{rccc} \lambda_\chi : & \mathbf{Z}(FG) & \longrightarrow & F \\ & \hat{K} & \longmapsto & \omega_\chi(\hat{K})^* \end{array}$$

by linear extension. It can be seen that λ_χ is an algebra homomorphism.

Now, let $\varphi \in \mathrm{IBr}(G)$ be afforded by an F-representation $\mathcal{X}$. Again, $\mathcal{X}(\hat{K}) = \lambda_\varphi(\hat{K})I$ is a scalar matrix, which only depends on φ, and then $\lambda_\varphi : \mathbf{Z}(FG) \to F$ defines an algebra homomorphism.

Definition 3.4. *The p-blocks of G are the equivalence classes in* $\mathrm{Irr}(G) \cup \mathrm{IBr}(G)$ *via $\chi \sim \varphi$ if $\lambda_\chi = \lambda_\varphi$ for $\chi, \varphi \in \mathrm{Irr}(G) \cup \mathrm{IBr}(G)$.*

Hence, $\alpha, \beta \in \mathrm{Irr}(G)$ lie in the same p-block if and only if

$$\left(\frac{|K|\alpha(x_K)}{\alpha(1)}\right)^* = \left(\frac{|K|\beta(x_K)}{\beta(1)}\right)^*$$

for every $K \in \mathrm{Cl}(G)$, where $x_K \in K$.

We denote the set of p-blocks of G by $\mathrm{Bl}(G)$. The unique p-block which contains the trivial character 1_G is the **principal block**, and it shall be denoted by $B_0(G)$.

There is another way of characterize the complex irreducible characters of a p-block. Let $B \in \mathrm{Bl}(G)$. We denote $\mathrm{Irr}(B) = \mathrm{Irr}(G) \cap B$, $\mathrm{IBr}(B) = \mathrm{IBr}(G) \cap B$, and we define $k(B) = |\mathrm{Irr}(B)|$, $l(b) = |\mathrm{IBr}(B)|$.

Definition 3.5. *Let G, H be groups and let $B \in \mathrm{Bl}(G)$, $B' \in \mathrm{Bl}(H)$. A perfect isometry between B and B' is a $\mathbb{Z}$-linear bijective map $I : \mathbb{Z}[\mathrm{Irr}(B)] \to \mathbb{Z}[\mathrm{Irr}(B')]$ that preserves the inner product and such that the map*

$$\begin{array}{rccc} \mu_I : & G \times H & \longrightarrow & \mathbb{C} \\ & (g,h) & \longmapsto & \sum_{\chi \in \mathrm{Irr}(B)} \chi(g)I(\chi)(h) \end{array}$$

verifies the following.

(i) *If exactly one of $g \in G$, $h \in H$ is p-regular then $\mu_I(g,h) = 0$.*

(ii) *The numbers $\frac{\mu_I(g,h)}{|\mathbf{C}_G(g)|_p}, \frac{\mu_I(g,h)}{|\mathbf{C}_H(h)|_p}$ are algebraic integers for $(g,h) \in G \times H$.*

Definition 3.6. *G is a **Frobenius group** if there is a proper $1 \neq N \trianglelefteq G$ and a complement A of N in G such that $\mathbf{C}_G(n) \subseteq N$ for every $1 \neq n \in N$. We say that N is the **Frobenius kernel** of G and A is a **Frobenius complement** of G.*

LEMMA 3.7. *If G is a Frobenius group with respectively Frobenius kernel and complement N and A, then* $(|N|, |A|) = 1$.

PROOF. Consider the action of A on N by conjugation and let $1 \neq n \in N$. As $\mathbf{C}_G(n) \subseteq N$ then $\mathbf{C}_A(n) \subseteq N \cap A = 1$. Therefore,

$$|N| = 1 + \sum_i |A : \mathbf{C}_A(n_i)| = 1 + n|A|.$$

□

LEMMA 3.8. *Let G be a Frobenius group with Frobenius kernel N. Then the following hold.*

(i) *If $\theta \in \mathrm{Irr}(N)$ is nontrivial then $\theta^G \in \mathrm{Irr}(G)$ and hence $G_\theta = N$.*

(ii) *If $\chi \in \mathrm{Irr}(G)$ satisfies $[\chi_N, 1_N] = 0$ then $\chi = \theta^G$ for some $\theta \in \mathrm{Irr}(N)$.*

PROOF. See Theorem (6.34) of [**Isa76**]. □

3.2. The requirement

Suppose that G is a finite simple group. Let $P \in \mathrm{Syl}_p(G)$ and let $H = \mathbf{N}_G(P)$. Let b_1 be the principal p-block of H, and let B_1 be the principal p-block of G. We further assume that there is a perect isometry $\tau : \mathbb{Z}[\mathrm{Irr}(b_1)] \to \mathbb{Z}[\mathrm{Irr}(B_1)]$ and that $H/\mathbf{O}_{p'}(H)$ is a Frobenius group with Frobenius kernel $P\mathbf{O}_{p'}(H)/\mathbf{O}_{p'}(H)$ and Frobenius complement E, where E has e conjugacy classes.

DEFINITION 3.9. *The **non-exceptional** characters of b_1 are the irreducible charactes of $\mu \in \mathrm{Irr}(b_1)$ such that $P \subseteq \ker(\mu)$.*

CLAIM 3.10. *There are e non-exceptional characters.*

First we need a few lemmas.

LEMMA 3.11. *Let B be a p-block of G and let $N \trianglelefteq G$ be a p'-group. Then,* $\mathrm{Irr}(B_0(G/N)) = \mathrm{Irr}(B_0(G))$ *and* $\mathrm{IBr}(B_0(G/N)) = \mathrm{IBr}(B_0(G))$

PROOF. See Theorem (9.9) of [**Nav98**]. □

LEMMA 3.12. *Let G be finite and p-solvable with $\boldsymbol{O}_{p'}(G) = 1$. Then G has a unique p-block.*

PROOF. See Problem (4.9) of [**Nav98**]. □

Now we can prove Claim (3.10)

Proof. We first prove that $\mathrm{Irr}(b_1) = \mathrm{Irr}(H/\mathbf{O}_{p'}(H))$. Since H is p-solvable ($\mathbf{N}_G(P)/P$ is a p'-group) then so is $H/\mathbf{O}_{p'}(H)$. Also,

$$\mathbf{O}_{p'}(H/\mathbf{O}_{p'}(H)) = \mathbf{O}_{p'}(H)/\mathbf{O}_{p'}(H) = 1.$$

Thus, by lemma (3.12) we have that the unique p-block of $H/\mathbf{O}_{p'}(H)$ is the principal one and hence by Lemma (3.11) we have that

$$\mathrm{Irr}(H/\mathbf{O}_{p'}(H)) = \mathrm{Irr}(B_0(H/\mathbf{O}_{p'}(H))) = \mathrm{Irr}(b_1).$$

Thus, by definition we have that the number of non-exceptional characters of b_1 is

$$\begin{aligned}&|\{\alpha \in \mathrm{Irr}(b_1) : P \subseteq \ker(\alpha)\}| =\\ &|\{\alpha \in \mathrm{Irr}(H/\mathbf{O}_{p'}(H)) : P \subseteq \ker(\alpha)\}| =\\ &|\,\mathrm{Irr}(H/P\mathbf{O}_{p'}(H))|.\end{aligned}$$

As $H/P\mathbf{O}_{p'}(H) \cong (H/\mathbf{O}_{p'}(H))/(P\mathbf{O}_{p'}(H)/\mathbf{O}_{p'}(H)) \cong E$, then $|\mathrm{Irr}(H/P\mathbf{O}_{p'}(H))| = e$, since e is the number of conjugacy classes of E. □

Claim 3.13. $e = l(b_1)$

Again, we first need a few lemmas.

Lemma 3.14. *Let p be a prime such that p does not divide $|G|$. Then* $\mathrm{Irr}(G) = \mathrm{IBr}(G)$.

Proof. See Theorem (2.12) of [**Nav98**]. □

Lemma 3.15. *Let $N \trianglelefteq G$ a p-group. Then there is a bijection between the p-regular classes of G onto the regular classes of G/N.*

Proof. See Lemma (3.9) of [**Isa18**]. □

Now we prove Claim (3.13).

Proof. Since the Frobenius action is coprime and $P\mathbf{O}_{p'}(H)/\mathbf{O}_{p'}(H)$ is a p-number, then p does not divide $|E|$. By Theorem (3.14) we have that $|\,\mathrm{Irr}(E)| = |\mathrm{IBr}(E)|$. Therefore, $e = |\mathrm{IBr}(E)| = |\mathrm{IBr}(H/P\mathbf{O}_{p'}(H))|$ and thus e is the number of p-regular classes of $H/P\mathbf{O}_{p'}(H)$. Since $P\mathbf{O}_{p'}(H)/\mathbf{O}_{p'}(H)$ is a normal p-subgroup of $H/\mathbf{O}_{p'}(H)$, by Lemma (3.15) we have that there is a bijection between the p-regular classes of $H/\mathbf{O}_{p'}(H)$ onto the set of p-regular classes of $(H/\mathbf{O}_{p'}(H))/(P\mathbf{O}_{p'}(H)/\mathbf{O}_{p'}(H)) \cong H/P\mathbf{O}_{p'}(H)$. As a consequence, $|\mathrm{IBr}(H/P\mathbf{O}_{p'}(H))| = |\mathrm{IBr}(H/\mathbf{O}_{p'}(H))|$. But, by Lemma (3.11) this is equal to $|\mathrm{IBr}(b_1)| = l(b_1)$. □

Now, let $\{\mu_1, \ldots, \mu_e\}$ be the non-exceptional characters of b_1. As τ is an isometry then $\mu_i^\tau = \epsilon_i \chi_i$ for a sign ϵ_i and some $\chi_i \in \mathrm{Irr}(G)$. We refer to $\{\chi_1, \ldots, \chi_e\}$ as the **non-exceptional** characters of B_1.

Obviously, if $i \neq j$ then $\chi_i \neq \chi_j$, since in other case $0 \neq [\epsilon_i\chi_i, \epsilon_j\chi_j] = [\mu_i^\tau, \mu_j^\tau] = [\mu_i, \mu_j]$, and this is not possible.

We want to prove the following.

THEOREM 3.16. *All non-exceptional characters in B_1 are constant on p-singular elements. In particular, there are at least $l(b_1)$ irreducible characters of B_1 which take constant (nonzero) values on p-singular elements.*

Furthermore, it is always possible to modify the perfect isometry τ so that it sends 1_H to 1_G.

We will distinguish two cases: when H acts transitively on $P\backslash\{1\}$ and when it does not. For the first case, we need some claims to establish the result.

LEMMA 3.17. *G has a normal p-complement if and only if* $\mathrm{IBr}(B_0(G)) = \{1_{G^0}\}$.

PROOF. See Corollary (6.13) of [**Nav98**]. □

LEMMA 3.18. *Let $x_1, \ldots, x_k$ be representatives of the G-conjugacy classes of p-elements of G. Then,*

$$k(B_0(G)) = \sum_{i=1}^{k} l(B_0(\mathbf{C}_G(x_i))).$$

PROOF. See Theorem (5.12) of [**Nav98**] and apply the third main theorem. □

CLAIM 3.19. *Suppose that H acts transitively on $P\backslash\{1\}$. Then $k(b_1) = 1 + l(b_1)$.*

PROOF. Let $1 \neq x \in P$. As $H/\mathbf{O}_{p'}(H)$ is a Frobenius group with kernel $P\mathbf{O}_{p'}(H)/\mathbf{O}_{p'}(H)$ and $1 \neq x \in P$ then it follows that

$$\mathbf{C}_H(x)\mathbf{O}_{p'}(H)/\mathbf{O}_{p'}(H) \subseteq \mathbf{C}_{H/\mathbf{O}_{p'}(H)}(x\mathbf{O}_{p'}(H)) \subseteq P\mathbf{O}_{p'}(H)/\mathbf{O}_{p'}(H),$$

and therefore $\mathbf{C}_H(x) \leqslant P \times \mathbf{O}_{p'}(H)$. As $P \times \mathbf{O}_{p'}(H)$ is p-nilpotent then so is $\mathbf{C}_H(x)$ and by Lemma (3.17) we have that $l(B_0(\mathbf{C}_H(x))) = 1$. As P consists on 2 H-conjugacy classes, then $1, x$ are the representatives of the p-classes, and applying now Lemma (3.18) we have that

$$k(b_1) = l(B_0(\mathbf{C}_H(1))) + l(B_0(\mathbf{C}_H(x))) = l(b_1) + l(B_0(\mathbf{C}_H(x))) = l(b_1) + 1.$$

□

LEMMA 3.20. *Let $x \in G$ be a p-element and let $y \in \mathbf{O}_{p'}(\mathbf{C}_G(x))$. If $\chi \in \mathrm{Irr}(G)$ is in the principal block then $\chi(xy) = \chi(x)$.*

Proof. See Theorem (7.7) of [**Nav98**]. □

Now, we prove the Theorem in the case where H acts transitively on $P\backslash\{1\}$.

Proof. Now, let $\chi \in \mathrm{Irr}(B_1)$. We check that χ is constant and nonzero on the p-singular elements. As H acts transitively on $P\backslash\{1\}$, then the nontrivial p-elements are G-conjugate, and therefore χ is constant on the nontrivial p-elements.

By our previous claims we have that $k(b_1) = 1 + l(b_1)$. Now, since τ is a perfect isometry, by Proposition (2.33) we have that $k(B_1) = k(b_1)$ and $l(B_1) = l(b_1)$ and thus, $k(B_1) = 1 + l(B_1)$. Let $1 \neq x \in P$. Using again Lemma (3.18) we have that $l(B_0(\mathbf{C}_G(x))) = 1$, and Lemma (3.17) yields that $\mathbf{C}_G(x)$ has a normal p-complement. As a consequence, if $y \in \mathbf{C}_G(x)^0$ then necessarily $y \in \mathbf{O}_{p'}(\mathbf{C}_G(x))$, and by Lemma (3.20) we deduce that $\chi(x) = \chi(xy)$, and hence χ is constant on the p-singular elements. Let $K \in \mathrm{Cl(G)}$ be a class of p-singular elements and let $x \in K$. If $\chi(x) = 0$ then

$$0 = \left(\frac{|K|\chi(x)}{\chi(1)}\right)^* = |K|^*,$$

since χ is in the principal block. Then p divides $|K|$, since in other case $a|K| + bp = 1$ for some $a, b \in \mathbb{Z}$ and hence $1 \in M$, which is not possible as M is maximal. However, as P consists on 2 H-conjugacy classes then it is a minimal normal subgroup of $\mathbf{N}_G(P)$, which is necessarily p-elementary abelian. Thus $P \subseteq \mathbf{C}_G(x)$ and hence p does not divide $|K|$. Therefore χ is nonzero on the p-singular elements. □

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