0. Introduction

DEF A p-element of a group G is picky if it belongs to a unique Sylow p-subgroup.

Conjecture (Moretó - Rizo 2024) Let $x \in P \in Syl_p(G)$ be picky. Then $\exists$ bijection

$$f_x : \mathrm{Irr}^x(G) \longrightarrow \mathrm{Irr}^x(N_G(P)) \qquad \left(\mathrm{Irr}^x(G) := \{\chi \in \mathrm{Irr}(G) : \chi(x) \neq 0\} \right)$$

such that $\begin{cases} \chi(1)_p = f_x(\chi)(1)_p \\ \mathbb{Q}(\chi(x)) = \mathbb{Q}(f_x(\chi)(x)) \end{cases}$

$N_G(P)$ appears but the right way to think about it is as the subnormalizer of x.

DEF $H \leq G$, $S_G(H) := \{ g \in G : H \lhd\lhd \langle g, H \rangle \}$; the subnormalizer of H.
Let $x \in G$, $\mathrm{Sub}_G(x) := \langle S_G(\langle x \rangle) \rangle$; the subnormalizer of x.

Lemma Let $x \in P \in Syl_p(G)$ be a p-element then $N_G(P) \subseteq \mathrm{Sub}_G(x)$ and equality holds iff x is picky.

The subnormalizer conjecture takes any p-elmt x in Moretó - Rizo and substitutes $N_G(P)$ by $\mathrm{Sub}_G(x)$. Finite groups of Lie type always play a fundamental role in group theory and Malle has recently studied picky elements and subnormalizers in these groups "Picky elements, subnormalizers and character correspondences".

The goal of this session(s) is to present without any prerequisites some of the results in Malle's paper. Therefore we need to introduce some theory of Algebraic groups.

1. ROOT SYSTEMS

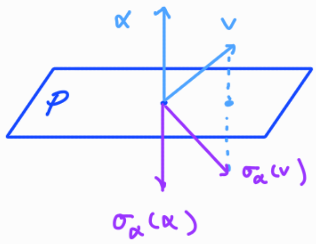
Root systems are of key importance in Lie theory. They arise when studying Lie Algebras / Lie Groups / LAGs.

Let E be a f.dim $\mathbb{R}$-vs endowed with $(,): E \times E \to \mathbb{R}$ positive definite symmetric bilinear form. (Then $\exists$ orthonormal basis so one can really think of $\mathbb{R}^n$ and usual product)

Let $\alpha \in E$, the reflection through α is $s_\alpha : E \to E$

$$v \longmapsto v - \underbrace{\frac{2(v,\alpha)}{(\alpha,\alpha)}}_{\langle v,\alpha\rangle} \alpha$$

Note it fixes a hyperplane (subspace of codim 1) P.



DEF Let E be a euclidean space as above, a subset $\Phi \subseteq E$ is a root system if

(R1) $|\Phi| < \infty$, $0 \notin \Phi$, $\langle \Phi \rangle_{\mathbb{R}} = E$

(R2) If $\alpha \in \Phi$ and $c\alpha \in \Phi$ for $c \in \mathbb{R}$ then $c = \pm 1$

(R3) $\forall \alpha \in \Phi$, $s_\alpha(\Phi) = \Phi$

(R4) $\forall \alpha, \beta \in \Phi$, $\langle \beta, \alpha \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$. These integers are called Cartan integers. The definition is variable, sometimes R4 is omitted.

The elements of Φ are called roots. $W := W(\Phi) := \langle s_\alpha : \alpha \in \Phi \rangle \leq GL(E)$ is called the Weyl group of Φ.

↪ It is a finite Coxeter group!

DEF Let E, E' be euclidean spaces, Φ, Φ' root systems in E, E' respectively. We say that the root systems are isomorphic if $\exists\ \Psi : E \to E'$ isom st

- $\Psi(\Phi) = \Phi'$
- $\langle \Psi(\alpha), \Psi(\beta) \rangle = \langle \alpha, \beta \rangle$.

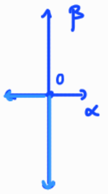
(preserves angles and relative root lengths of roots belonging to same irred component)

Example Any root system of rank 2 (dim of the v.s. E) is isom to one of the following (which have to be thought in $\mathbb{R}^2$ with usual product)

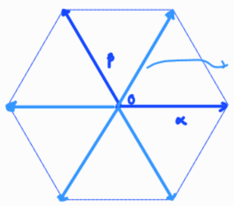
We say that Φ is reducible if $\Phi = \Phi_1 \sqcup \Phi_2$, $\Phi_1 \perp \Phi_2$. (irred otherwise)

$\nearrow SL_3, PGL_3$

- $2A_1$ or $A_1 \times A_1$, A_2



Relative root lengths can be whatever

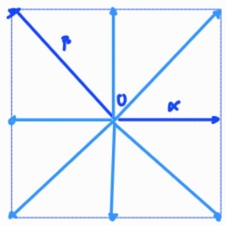


drawn by taking regular hexagon

of course the picture tells us that this is $\alpha+\beta$

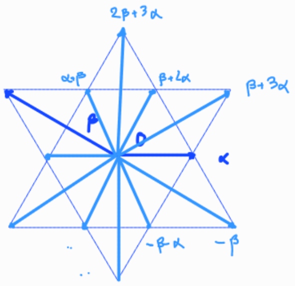
Here $|\alpha| = |\beta|$ and the angle is $2\pi/3$ ($|\alpha|$ whatever)

- B_2



Here the square condition forces $\|\alpha\|$ arbitrary but $2\|\alpha\|^2 = \|\beta\|^2$ and the angle is $3\pi/4$

G_2



Now here the triangle conditions force $\|\alpha\|$ arbitrary but $\frac{\|\beta\|^2}{\|\alpha\|^2} = 3$ with angle $5\pi/6$.

<u>DEF</u> Let $\Phi \subseteq E$ be a root system with Weyl Group W. A subset $\Delta \subseteq \Phi$ is called a **base** if it satisfies : B1) Δ basis of E

B2) $\forall \alpha \in \Phi$, $\alpha = \sum_{\delta \in \Delta} x_\delta \delta$ with either all $x_\delta \in \mathbb{Z}_{\geq 0}$ or all $x_\delta \in \mathbb{Z}_{\leq 0}$

$\alpha \in \Delta$ is called a **simple root** (s_α a **simple reflection**). Let $\beta \in \Phi$, $\beta = \sum_{\delta \in \Delta} x_\delta \delta$; we set $ht(\beta) = \sum_{\delta \in \Delta} x_\delta$; the **height of β**. We let

$$\Phi^+ = \{ \beta \in \Phi : ht(\beta) > 0 \} , \quad \Phi^- = \{ \beta \in \Phi : ht(\beta) < 0 \} = -\Phi^+ .$$

positive roots negative roots

<u>Theorem (nontrivial)</u> If (Φ, E) root system , bases exist . Also

i) W acts transitively on bases. Moreover if $\sigma(\Delta) = \Delta \to \sigma = 1$ ($\sigma \in W$)

ii) $W = \langle s_\alpha : \alpha \in \Delta \rangle$

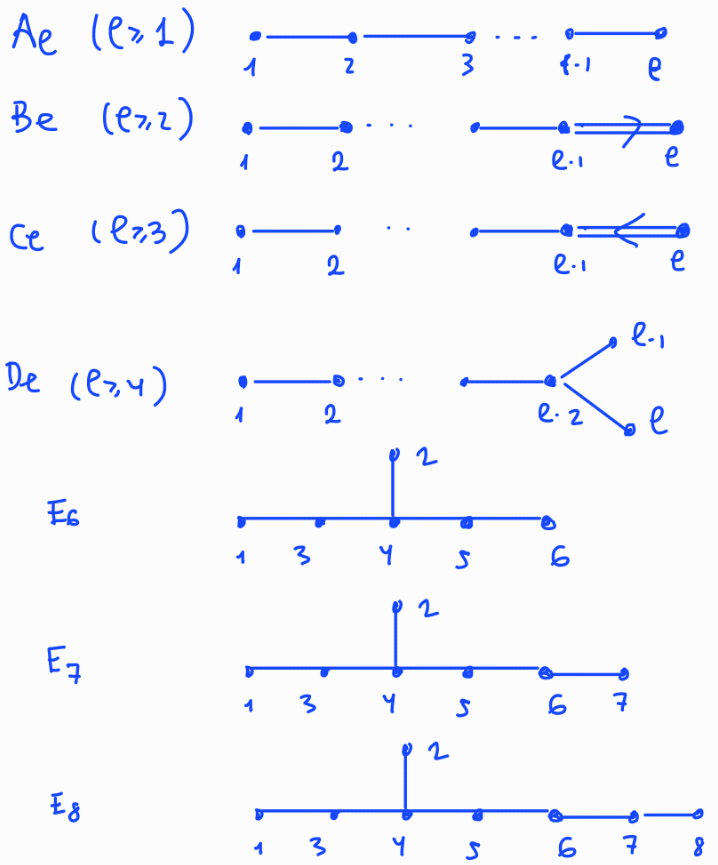
iii) $\forall \alpha \in \Phi$, $\exists \sigma \in W : \sigma(\alpha) \in \Delta$

iv) $\forall \alpha \in \Delta$, $s_\alpha(\Phi^+ \setminus \{\alpha\}) = \Phi^+ \setminus \{\alpha\}$.

I think this is enough but let me mention that it is relatively easy to classify root systems.

Sketch: • "It is enough to classify the irreducible ones".

Via Cartan integers one canonically associates Φ a graph (multiple edges and directed).
Isom root systems give "isom graphs". These graphs are called Dynkin diagrams.
A root system is irred iff Dynkin diagram is connected and the possible diagrams are



(this gives isom classes of simple Lie alg over a field of char. 0 $(k=\bar{k})$)

Root systems arise naturally in groups and Lie algebras and give a very combinatorial way to think of them!. (Discrete objects associated to LAGs, Lie alg...)

•Say smth about automorphisms.

2. GROUPS COMING FROM ALGEBRAIC GEOMETRY

Finite groups of Lie type arise as fixed points of certain automorphisms of certain groups that come in the context of algebraic geometry. Loosely speaking

(informal warning)

Group schemes (Group objects in the category of Schemes)

Algebraic groups (Group object in the category of varieties)

Abelian varieties (algebraic groups st the variety is complete and connected. It implies abel)

Linear Algebraic groups (algebraic groups st the variety is affine)

We care about this.

- Formally; fix $k=\bar{k}$ and let $\mathbb{A}^n = k^n$ (we forget v.s. structure).

Let $X \subseteq k[x_1,\dots,x_n]$, $V(X) := \{ x \in \mathbb{A}^n : f(x) = 0 \ \forall f \in S \}$; we declare a topology on $\mathbb{A}^n$ via declaring the closed subsets to be of this form. (One has to prove it is a topology)

consider it as a topological space with induced topology

Let $X \subseteq \mathbb{A}^n$ be closed. We say X is an affine variety (in a very classical sense). Note $X = V(S) = V(\langle S \rangle) = V(\{f_1,\dots,f_n\})$; so we are really considering solutions of systems of eqn.

Hilbert basis.

should be part of the defn!

The coordinate ring of $X \subseteq \mathbb{A}^n$ closed ($k[X]$) is the k-algebra of polynomial functions on X.
It is naturally isom to $k[x_1,\dots,x_n]/I(X)$; $I(X) = \{ f \in k[x_1,\dots,x_n] : f \text{ vanishes on } X \}$

- Why do I say "very classical"? A slightly more modern language is as follows. A ringed space is a topological space X with a sheaf of rings. What we have just built is a ringed space. Any ringed space isom to an affine variety is what we want to call affine variety.

- Say something about sheafs
- Prevarieties (ringed space with finite open cover by affine varieties)
- Varieties (prev with closed diagonal)
- Schemes. (Ringed space with open cover by affine schemes)

DEFn Let $X \subseteq \mathbb{A}^n$, $Y \subseteq \mathbb{A}^m$ be affine varieties. Let $\varphi : X \to Y$ be a map; we say that it is a morphism of affine varieties if $\exists f_1, \dots, f_m \in k[X]$: $\varphi(x) = (f_1(x), \dots, f_m(x))$ $\forall x \in X$

DEFn A LAG is an affine variety G with group structure st

$\mu : G \times G \to G$ $\quad$ $i : G \to G$ $\quad$ are morphisms of affine varieties.
$(x,y) \mapsto xy$ $\quad$ $x \mapsto x^{-1}$

Ex

i) $\mathbb{G}_a = k$ $\quad (k,+)$ addition $\quad (V(0))$.

ii) $\mathbb{G}_m = k \setminus \{0\}$ with mult.

- Pure way: $\mathbb{G}_m = \{(x,y) \in k^2 : xy = 1\} = V(I)$ where $I = \langle xy - 1 \rangle$

$\mu((x_1,y_1),(x_2,y_2)) = (x_1x_2, y_1y_2)$ ...

as an abstract group this is canonically k^* (we identify)

Why? k^* not closed in k

- Ringed spaces : With the above abstraction one can really say that k^* is an honest LAG. (This is absolutely worth it in my opinion)

iii) GL_n

iv) $SL_n = \{A \in \mathbb{A}^{n \times n} : \det A - 1 = 0\}$ ⟶

v) PGL_n? Quotient varieties?
Needs work. But yes!

vi) Sp_{2n}, GO_{2n}, GO_{2n+1}

(Symplectic and orthogonal groups).

note $\det(A) - 1$ is a poly in $\mathbb{Z}$, considering solutions of this poly can be done over any field so really

$$\text{Fields} \xrightarrow{SL_n} \text{Groups}$$
$$k \longmapsto SL_n(k)$$

This is related to functor perspective of Schemes. See (Jantzen). We do not need this abstraction

(of course in the next thm we fix $k = \bar{k}$).

Theorem Let G be a LAG, then G is isom to some closed subgroup of GL_n.

↳ $\varphi : G \to \varphi(G) \leq GL_n$, morphism, bij, and inverse is a morphism (the fact that $\exists$ bij morphisms that are not isoms makes life harder).

The good subgroups of a LAG are the closed subgroups (naturally LAGs). It is nontrivial to show that if $\varphi: G \to H$ LAG morphism then $\ker\varphi$, $\operatorname{Im}\varphi$ are closed subgroups ($\ker\varphi$ is obviously closed).

<u>Thm</u> Let G be a LAG, then G connected iff G irred. In this case $k[G]$ is an integral domain.

(a top space is irred if can't be written as union of two proper closed subsets)

I think is worth mentioning that if X affine variety one sets $\dim(X)$ = Krull dimension of $k[X]$ (and when G connected LAG, it is $\operatorname{tr.deg}_k(k(G))$). fraction field.

<u>DEF</u> Let R be a comm ring, we define the Krull dimension of R to be $\dim(R) = \sup\{r : \exists P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_r\}$ with $P_i \in \operatorname{Spec}(R)$.

dimension theory is a business that requires care.

(of course this is either $<\infty$ or ∞)

- It is intuitive;

We have LAGs although they are usually infinite, have finite dimension and we can use dimension for inductive arguments; thanks to

<u>Prop</u> If $Y \subsetneq X$ closed, X irred affine variety then $\dim(Y) < \dim X$.

(finally let me just say that if $\varphi: G \to H$ LAG morphism, $\dim G = \dim\ker\varphi + \dim\operatorname{Im}\varphi$)

3. SOLVABLE AND UNIPOTENT LAGS.

To study these classes we will first introduce the Jordan-Chevalley decomposition

Let $g \in \operatorname{End}(V)$ when $V = k^n$. We say that g is semisimple if diagonalizable.

$\operatorname{spec}(g) = 1$ $\qquad$ $\operatorname{spec}(b) = 0$.

We say that g is unipotent if $g = 1 + b$; b nilpotent ($b^m = 0$). <u>Exercise</u> Char $k = p > 0$; g unip iff g is a p-element. $\in GL(V)$

<u>Thm (Jordan-Chevalley)</u> Let G be a LAG, let $g \in G$, then $\exists!\ g_s, g_u \in G$: $g = g_s g_u = g_u g_s$ and such that for any $\pi: G \to GL_n$ 1-1 morphism, $\pi(g_s)$ is ss $\pi(g_u)$ is unip. (also if $\varphi: G \to H$ LAG morphism, $\varphi(g_s) = \varphi(g)_s$ $\varphi(g_u) = \varphi(g)_u$)
(familiar in abstract groups)

<u>DEF</u> We say that a LAG is unipotent if $G = G_u = \{g \in G : g \text{ is unipotent}\}$
(In general G_u is closed; not necessarily subgroup. Any simple LAG is gen by unip. elmts)

We already know what solvable means.

Let $\boxed{T_n} = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ & & * \end{pmatrix} \right\} \leq GL_n$; this is a connected solvable LAG.
upper triangulars

In fact it is maximal among the closed connected solvable subgroups of GL_n.

<u>DEF</u> Let G be a LAG, let $B \leq G$ be maximal among the closed connected solvable groups of G. Then we say that B is a **Borel subgroup of G**.

In view of the thm that embeds a LAG $G \hookrightarrow GL_n$ the next is very expected

<u>Thm</u> Let G be a connected solvable LAG, then $G \hookrightarrow T_n$. (Lie-Kolchin) It becomes trivial with

Let us look closer at T_n; note $T_n = \boxed{U_n} \rtimes \boxed{D_n}$

$$U_n = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} \right\} \leq GL_n \qquad D_n = \left\{ \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \right\} \leq GL_n$$

This is a very important dec. to have in mind. We generalize it;

<u>DEFn</u> A LAG G is said to be a **torus** if $G \cong \underbrace{\mathbb{G}_m \times \cdots \times \mathbb{G}_m}_{\text{direct products of LAGs are naturally LAGs.}} \overset{\text{obvious (explain)}}{\cong} D_n := \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \in GL_n \right\}$

Diagonal invertible matrices.

Let G be a LAG, we say that T is a **max torus** if T is a torus and it is not contained in another one (properly).

<u>Thm</u>. Let G be a unipotent LAG, then $G \hookrightarrow U_n$. (So unipotent $\longrightarrow$ nilpotent)

· Let G be a connected solvable LAG; then G_u closed connected normal. In fact if T max torus $G = G_u \rtimes T$, and all max torus are conj.

Let me mention that one can show that all the Borels (respec. max torus) in a LAG G are conjugate.

Let me give an idea of how one shows conjugacy of Borels. This also allows me to introduce quotients. Let $H \leq G$ closed; then $\{Hg : g \in G\}$ can naturally be identified with an open subset of a projective variety. In short, $\{Hg : g \in G\}$ have a variety structure (If $H \trianglelefteq G$, affine and G/H LAG).

Now one shows: ① G connected LAG, X proj var; $G \curvearrowright X$ (morphically) then $\exists$ fixed point

G conn. $\left\{ \right.$ ② B Borel then G/B proj. (not nec a group)

③ $B_1 \times G/B \to G/B$ left mult. $\exists\, x \in G : g \times B = xB$ $\forall g \in B_1$ thus $g \in xBx^{-1}$; $B_1 = xBx^{-1}$.

④ Go to disconnected case (I promise easy and irrelevant for us)

<u>Cor</u> Let G be a LAG, any two max torus are conj. $(\mathrm{rk}(G) := \dim(\text{max torus}))$

Pf/ T_1, T_2 max torus; take $\underset{\underset{T_1}{\cup\!\shortmid}}{B_1}, \underset{\underset{T_2}{\cup\!\shortmid}}{B_2}$ Borels; $B_1^x = B_2$ hence T_1^x max torus of B_2. Use solvable result. □

• <u>Convention</u> For us it is convenient to from now on take the underlying field of any affine variety to be $k = \overline{\mathbb{F}_p} = \bigcup_{n=1}^{\infty} \mathbb{F}_{p^{n!}}$. This way we see immediately that if G LAG, $g \in G$, $o(g) < \infty$; g p-elem iff g unipotent. Also semisimple elements have p'-order. So the above theorem is parallel to Schur Zassenhaus.

4. REDUCTIVE GROUPS.

<u>DEF</u> Let G be a LAG.

i) Consider the (next result) maximal among closed connected normal solvable subgroup of G. We call it the radical of G, $R(G)$, $Rad(G)$.

ii) Consider the maximal among closed connected normal unipotent subgroup of G. We call it the unipotent radical of G, $R_u(G)$.

$(R_u(G) \leq R(G))$

iii) G is ==reductive== if $R_u(G) = 1$,

G is ==semisimple== if G connected and $R(G) = 1$ (one would expect to define G semisimple if all elements are ss. NO! G_s is weird).

G connected then $G/R(G)$ is ss

$G/R_u(G)$ c.r

That's why study of LAGs is solv/unip then ss/c.r.

Lemma $R(G)$, $R_u(G)$ are well defined and $R_u(G) = R(G)_u$

Finite group parallel: $R(G) \sim Sol(G)$

$R_u(G) \sim O_p(G)$

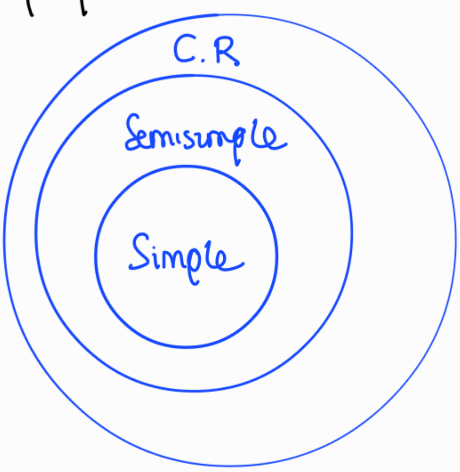
Example GL_n is connected reductive not semisimple; $R(G) = Z(G)$

SL_n is ss;

- Conn: $k[G] = k[T_{ij}]_{(\det T_{ij})}$. We localise a domain; domain. So the ideal is prime so G irred $\to$ G conn.
- $Z(G)$ closed connected solvable; so not ss
- $R(G) = \underset{\text{always true}}{\downarrow}\left(\bigcap_{\substack{B \leq G \\ \text{Borel}}} B\right)^\circ$; so $R(G) \subseteq D_n$; so $R_u(G) = 1$

For connected reductive groups, $R(G) = Z(G)^\circ$ Identity comp.

A LAG is ==simple== if it is semisimple and has no non-trivial proper closed connected normal subgroup.



A semisimple LAG G determines $G_1, \dots, G_k \trianglelefteq G$ closed connected normals that are simple, commute with each other and $G = G_1 \cdots G_n$, $|G_i \cap \prod_{i \neq j} G_j| < \infty$.

So if one wants to study simple LAGs it is perhaps better to study semisimple LAGs. Connected reductive is a larger class in which we can argue. C.R. groups have a very nice theory and they can be classified combinatorially (this gives a classification of simple LAGs). One key piece of the combinatorial data that allows their classification is the root system.

• Let G be a LAG (let us take C.R. over $k = \overline{\mathbb{F}_p}$). Let T be a maximal torus (of dim ≥ 1; else G is a point; explain). We set $X(T) := \mathrm{Hom}(T, \mathbb{G}_m) = \{ \varphi : T \to \mathbb{G}_m : \varphi \text{ morphism} \}$
If $T \cong D_n$, $X(T) \cong \mathbb{Z}^n$ (abelian group). "Character group"

$\{ \delta : k[G] \to k : \delta(fg) = f(1)\delta(g) + \delta(f) g(1) \ \forall f, g \in k[G] \}$

G always has a finite dimensional k-vector space $\mathfrak{g}$ $= \{\ldots\}$ (dim $\mathfrak{g}(G) = \dim G$) attached to it. This vector space is always endowed with a form $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that is

- Bilinear
- $[x, x] = 0 \quad \forall x \in \mathfrak{g}(G)$
- $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

So $\mathfrak{g}$ is a Lie algebra. Moreover G acts on its Lie algebra naturally, giving a rep. of G

$$G \xrightarrow{Ad} GL(\mathfrak{g}) \qquad \text{"adjoint representation"}$$

This representation is a LAG morphism (we do not need it but how can we make $GL(\mathfrak{g}(G))$ into a LAG? Think)

<u>Ex</u> If $G = GL_n$, $\mathfrak{g} = \mathfrak{gl}_n = \{ n \times n \text{ matrices over } k \}$; $G \xrightarrow{Ad} GL(\mathfrak{gl}_n)$

$$g \longmapsto Ad(g) : \mathfrak{gl}_n \to \mathfrak{gl}_n$$
$$X \mapsto gXg^{-1}$$

It plays well with subgroups so we can think of it as this.

<u>Exercise</u> Let k be a field, $x, y \in \mathrm{End}(V)$ (V fdim k-vs) commuting ss. Then they are simultaneously diagonalisable. Let $S \subseteq \mathrm{End}(V)$ a family of commuting ss endomorphisms then they are also simultaneously diagonalisable.

Now $Ad(T) \subseteq GL(\mathfrak{L})$ is a commuting family of ss hence we can simultaneously diagonalize ; this way we obtain

$$\mathfrak{L} = \bigoplus_{\chi \in X(T)} \mathfrak{L}_\chi \quad \text{where} \quad \mathfrak{L}_\chi = \{ v \in \mathfrak{L} : Ad(t)v = \chi(t)v \ \forall t \in T \}$$

We set $\Phi(G) := \{ \chi \in X(T) : \mathfrak{L}_\chi \neq 0 \text{ and } \chi \text{ nontrivial} \}$. We call this set of roots of G wrt T. We set $W = W_G(T) := N_G(T)/T$ (if G is C.R $T = C_G(T)$) and call it the Weyl group of G wrt T

(theorem)

finite group (theorem)

So $\mathfrak{L} = \mathfrak{L}_0 \oplus \bigoplus_{\chi \in \Phi} \mathfrak{L}_\chi$.

- The following is a long story with many actors explained backwards and omitting key scenes : (G LAG , $k = \overline{\mathbb{F}_p}$, T max torus , X, Φ, W as above)

- The group $W = N_G(T)/T$ acts on $X(T)$ faithfully via $w \cdot \chi = T \to \mathbb{G}_m$, $t \mapsto \chi(n^{-1} t n)$

where $w = nT = nC_G(T)$. It stabilizes Φ .

Let $E = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$, extend the action of W on X to E (flatness). Use the Weyl unitarian trick to define a W-invariant form on E (euclidean product).
We have that seeing $\Phi \subseteq E$, Φ is an abstract root system in $E' = \langle \Phi \rangle_{\mathbb{R}}$ (indep of form ; which is unique up to scalar on each W-inv. subspace) with Weyl group isom to W. Namely,

$$\begin{aligned} W &\longrightarrow GL(E') \\ w &\longmapsto \text{extend action of } w \text{ on } X(T) \text{ to } E \text{ (or } E' \text{; recall } \Phi \text{ is invariant)} \end{aligned}$$

is a faithful rep of W whose image is $W(\Phi)$. In fact G is ss iff $E' = E$ and G is simple iff Φ irred .

Recap One starts with G CR, fixes T max torus, with character group X, $W = \frac{N_G(T)}{T}$ one determines roots $\Phi \subseteq X$ and there is a canonical way to make this into an abstract root system (which as an abstract root system is indep of T! essentially by conj of max torus)
In fact with a bit more combinatorial data (root datum) one classifies CR groups.

- Let G be CR, T max torus, B a Borel containing T. Let Φ be the set of roots determined by T. We want to "see the roots from the group". Let $\alpha \in \Phi$ then

$\exists$ 1-1 morphism $u_\alpha : \mathbb{G}_a \to G$ st $\forall_{\substack{t \in T \\ c \in \mathbb{G}_a}}$, $t u_\alpha(c) t^{-1} = u_\alpha(\alpha(t)c)$.

  - Unique up to scalar; if u'_α satisfies the same, $\exists a \in k^*$: $u'_\alpha(c) = u_\alpha(ac)$
  - So the image $u_\alpha(\mathbb{G}_a) =: U_\alpha$ only depends on α. (Root subgroups)
  - $U_\alpha \underset{u_\alpha}{\cong} \mathbb{G}_a$; $\underbrace{\text{Closed connected unip normalized by } T \text{ and } 1 \text{ dim}}$

    If V is any such, $\exists! \alpha \in \Phi$: $V = \mathrm{Im}(u_\alpha)$)

  - Let $w \in W$, $w = nT$, $w(U_\alpha) = U_{w \cdot \alpha}$

This gives a very nice way to think about Borels $\supseteq T$. In fact fixed T, we have

$$\{B \supseteq T : B \text{ Borel in } G\} \longleftrightarrow \{\Delta \text{ base of } \Phi\}$$

How? Let B be a Borel, since it is solvable and T is a maximal torus in T, $B = B_u \rtimes T$; The theorem is that $\exists! \Delta$ base st $\underset{R_u(B)}{\overset{\shortparallel}{B_u}} = \prod_{\alpha \in \Phi^+} U_\alpha$)

Moreover, $B = T \prod_{\alpha \in \Phi^+} U_\alpha$ with uniqueness of expression wrt a fixed order.

(The only U_α's in B are $U_\alpha : \alpha \in \Phi^+$).

This gives enough info about conn red for us. ($\Phi(SL_n) = A_{n-1}$; •—•—• . . . •)

5. FROBENIUS MAPS ; FINITE GROUPS OF LIE TYPE.

Until here the only reason why I said that char $k>0$ is to have the unipotent elmts as p-elmts .... But everything holds. Now we really need $k=\overline{\mathbb{F}_p}$.

- $F_q : GL_n \longrightarrow GL_n$, $(a_{ij}) \longmapsto (a_{ij}^q)$ is called standard Frobenius map on GL_n

Bijective morphism :
- 1-1 : $a_{ij}^q = b_{ij}^q$; $(a_{ij}-b_{ij})^q = 0$. (char p is crucial)
- Onto : k is closed.
- Group map : Obvious.

The inverse is not a LAG morphism. (Group iom).

DEF Let G be a LAG (over k) (let us say connected), and $F: G\to G$ be a morphism. We say that G is defined over $\mathbb{F}_q$ with Frob map F if $\exists\; G \xrightarrow{\varphi} GL_n$ closed embedding ($\varphi|_G : G \to \varphi(G)$ LAG iom, $\varphi(G)$ closed) st

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "F"'] \arrow[dr, phantom, "="] & GL_n \arrow[d, "F_q"] \\ G \arrow[r, "\varphi"'] & GL_n \end{tikzcd}$$

commutes

- F is an automorphism of abstract groups.
  - Hom : $F(xy)=F(x)F(y)$ iff $\varphi(F(xy)) = \varphi(F(x)F(y)) = \varphi(F(x))\varphi(F(y)) = F_q(\varphi(x))F_q(\varphi(y))$; $\varphi(F(xy)) = F_q(\varphi(xy)) = F_q(\varphi(x))F_q(\varphi(y))$
  - 1-1 comes from φ, F_q being 1-1
  - Onto, $F(G)$ connected by topology, closed by sec 2. By last prop of sect 2. If $F(G)<G$, $\dim G > \dim F(G)$; but $\dim G = \underbrace{\dim \operatorname{Ker} F}_{=0} + \dim F(G)$ ✓.

DEF Let G be a connected LAG, $F: G\to G$ a morphism (of LAGs) is called a Steinberg map if $\exists\, n\geq 1$ st G is defined over $\mathbb{F}_q$ (for some $q=p^m$) with Frob. map F^n.

Note auton of abstract groups.

Let G be connected reductive, $F: G \to G$ Steinberg map; let $G^F = \{g \in G : F(g) = g\}$ then G^F is called finite group of Lie Type.

↳ note it is finite, and a group.

Ex: $GL_n(q)$, $SL_n(q)$, ...

- If G is simple LAG, $F: G \to G$ Steinberg map, G^F perfect then $G^F/Z(G^F)$ is simple (so G^F quasisimple) (Theorem 24.14 [MT])

Moral: Let G^F be a finite group of Lie type, then the fact that we can view it through the lens of $F: G \to G$ allows us to understand G^F much better. For example, if T F-stable max torus, $W = N_G(T)/T$, $w \in W$, $w = n_w T$ one can define affine varieties Y_{n_w} on which G^F acts and hence $G^F \curvearrowright H^i_c(Y_{n_w}, \overline{\mathbb{Q}}_\ell)$ (vector space); we get "natural" representations of G^F.

($g: Y_{n_w} \to Y_{n_w}$ induces $g^*: H^i_c(Y_{n_w}, \overline{\mathbb{Q}}_\ell) \to H^i_c(Y_{n_w}, \overline{\mathbb{Q}}_\ell)$ $\forall i \in \mathbb{Z}$)

- Let G be C.R. over k, $F: G \to G$ Steinberg. Then $\exists$ F-stable $\underbrace{T \leq B}_{\text{this guarantees } F|_T, F|_B \text{ are also Steinberg}}$ Borus. Let us consider $G^F \geq B^F \geq T^F$. Note that the p-elmts of G^F are exactly the unipotent elmts of G^F.

Note that if $U = B_u$, U is F-stable (Jordan-Chev) and $U^F \trianglelefteq B^F$; in in fact one can show $U^F \in Syl_p(G^F)$ and $N_{G^F}(U^F) = B^F$.

(24.11 MT)

| In SL_2, $F = F_q$; $U^F = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in \mathbb{F}_q \right\}$

Such pair exists; $G = GL_n$, $F = F_q$, $T = D_n$, $B = T_n$, and T^F = Diag matrices over $\mathbb{F}_q$; B^F = upper triang...

6. ROOTS OF G^F AND PICKY ELEMENTS

$U = B_u = R_u(B)$.

- Let G be c.r. over k, $F: G \to G$ Steinberg, $T \leq B$ F-stable Borus, Φ root system (wrt T), W Weyl group. Note that $\exists\, \Delta \subseteq \Phi$ base st

$$B = T \prod_{\alpha \in \Phi^+} U_\alpha ;$$

Since $F(U) = U$, $F(U_\alpha) = U_{\rho(\alpha)}$ for some $\rho \in S_{\Phi^+}$. Define

$F: X(T) \to X(T)$

$\chi \longmapsto F(\chi): T \xrightarrow{F} T \xrightarrow{\chi} \mathbb{G}_m$

($\rho(\alpha) \neq F(\alpha)$ in general; for isoms σ, $\sigma(\rho(\alpha)) = \alpha$; so $\sigma = \rho^{-1}$ on Φ)

We can extend by linearity $F: E = X \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow E = X \otimes_{\mathbb{Z}} \mathbb{R}$

<u>Fact</u> $\exists\, n \geq 1$: F^n = positive integral multiple of $\mathrm{Id}\ X$.

Let δ be the smallest such, let $k \in \mathbb{Z}_{\geq 1}$, $F^\delta = k\,\mathrm{Id}$ on X. One shows $k = p^m$. Let q (positive real) be defined by $q^\delta = k$. Let $F_0: E \to E$ linear def by $F = q F_0$.
Note $F_0^\delta = \mathrm{Id}\ E$.

Let $E^{F_0} = \{ v \in E : F_0(v) = v \} \leq E$. Now let $\theta: E \longrightarrow E^{F_0}$ linear; a projection

$$v \longmapsto \frac{1}{\delta} \sum_{i=0}^{\delta-1} F_0^i(v)$$

Define an equivalence relation on Φ, $\alpha \sim \beta$ iff $\theta(\alpha), \theta(\beta)$ are positive multiples of each other. For each class A, set α_A = vector of max length among the $\theta(\alpha)$

$\{\alpha_A : A \text{ class of } \Phi\}$ =: Set of roots of G^F; This satisfies the axioms (R1)-(R3) of a root system in E^{F_0} (it almost always satisfies R4 as well) with base

$\Delta_F = \{\alpha_A : A \subseteq \Delta\}$. For each root α_A the corresponding root subgroup is

U_A^F, where $U_A := \prod_{\alpha \in A} U_\alpha$

(after fixing order)

<u>Theorem</u> In the above setup, each elmt $x \in U^F$, can be expressed uniquely as $x = \prod_{A>0} x_A$ where $x_A \in U_A^F$, ($A>0$ means that all roots in A are positive)

In this equivalence rel either all roots in a class are positive or negative

• If $F: W \to W$, $nT \mapsto F(n)T$ is trivial, (for example $F = F_q : SL_2 \to SL_2$) one shows

$E^{F_0} = E$ so the set of roots is really Φ (and the root system) and the root subgroups are U_α^F ($\alpha \in \Phi$). "So we are just putting an F".

↳ (additive groups of finite fields)

<u>Prop (Malle)</u> G cr, F Steinberg, $T \leq B$ F-stable Borus.

Let $x \in G^F$ be ^not 1^ unipotent (this is a p-elmt, after conj WMA $x \in U^F$). Express x as above. Then x is picky iff in the expression all simple roots appear.

(ie $\forall$ class A: $A \subseteq \Delta$, $x_A \neq 1$)

This allows to check directly (he classifies unip picky).

Directly; in $SL_2(\mathbb{F}_q)$, $\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{F}_q \setminus \{0\} \right\}$ is picky. Why?

Only one root!

Appendix : Example of root system, root subgroups ....

Let $G = SL_2$, Max torus? $T = D_2 \cap SL_2 = \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} : x \in k^* \right\}$, $X(T) = \mathbb{Z}$

$(\dim SL_2 = 3)$ Borel? $B = T_2 \cap SL_2 = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} : x \in k^* \right\}$

$\left\{ \chi : \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \to x^m \right.$ so $\chi \equiv m \in \mathbb{Z}$

Why? B is closed $\left(\{ A \in SL_2 : a_{21} = 0 \} \right)$, connected, solvable (a bit of work). Hence in SL_2 $\exists \tilde{B}$ Borel st $B \subseteq \tilde{B}$; but $\tilde{B} \subseteq T_2^g$ thus

$T_2 \cap SL_2 \subseteq \tilde{B} \subseteq T_2^g \cap SL_2 \underset{\text{normality}}{=} (T_2 \cap SL_2)^g$. By dimensions $\tilde{B} = SL_2 \cap T_2$.

Let S be a torus in SL_2, we can simultaneously diagonalize it so clearly T max torus.

$\mathrm{Lie}(G) = \mathfrak{sl}_2 = \{ X \in M_2(k) : \mathrm{tr}(X) = 0 \}$; $\left(\text{basis of } \mathfrak{sl}_2 \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \right)$

$$\begin{aligned} \mathrm{Ad} : SL_2 &\longrightarrow GL(\mathfrak{sl}_2) \\ g &\longmapsto \mathrm{Ad}(g) : \mathfrak{sl}_2 \longrightarrow \mathfrak{sl}_2 \\ &\qquad\qquad\qquad X \longrightarrow g X g^{-1} \end{aligned}$$

$\mathrm{Ad}(T)$ is a family of commuting ss elements of $GL(\mathfrak{sl}_2)$. Let $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in T$

$$\mathrm{Ad}\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) : \mathfrak{sl}_2 \longrightarrow \mathfrak{sl}_2$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 0 & t^2 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ t^{-2} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hence $\mathfrak{sl}_2 = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle_k \oplus \left\langle \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\rangle_k \oplus \left\langle \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\rangle_k$

$$\Vert \qquad\qquad \Vert \qquad\qquad \Vert$$

$\left\{ v \in \mathfrak{sl}_2 : \mathrm{Ad}\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) v = v \right\}$ $\quad$ $\left\{ v \in \mathfrak{sl}_2 : \mathrm{Ad}\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) v = t^2 v \right\}$ $\quad$ $\left\{ v \in \mathfrak{sl}_2 : \mathrm{Ad}\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) v = t^{-2} v \right\}$

$$\Vert \qquad\qquad \Vert \qquad\qquad \Vert$$

$\mathfrak{g}_0$ $\quad$ $\mathfrak{g}_\alpha$; $\alpha : T \to \mathbb{G}_m$, $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \to t^2$ $\quad$ $\mathfrak{g}_\beta$: $\beta : T \to \mathbb{G}_m$, $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \to t^{-2}$

$\alpha = 2 \in \mathbb{Z}$ $\quad$ $\beta = -2 \in \mathbb{Z}$

So we have extracted from $X(T)$, $\{\alpha, \beta\} = \{\alpha, -\alpha\} = \Phi$.

$W \curvearrowright \Phi$; $W = N_{SL_2}(T)/T = \left\{ \overset{1}{\begin{pmatrix}1&0\\0&1\end{pmatrix}}, \overset{w}{\begin{pmatrix}0&1\\1&0\end{pmatrix}} \right\}$; acts on $\overset{m}{\chi} \in X(T)$,

$w\cdot\chi : T \to \mathbb{G}_m$

$$\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix} \longmapsto \chi\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}t&0\\0&t^{-1}\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right) = \chi\left(\begin{pmatrix}0&t^{-1}\\t&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\right) =$$

$$= \chi\begin{pmatrix}t^{-1}&0\\0&t\end{pmatrix} = (t^{-1})^m = t^{-m}$$

so $w\cdot\chi \equiv -m$.

W acts here by changing sign

$\mathbb{Z}$ ($X(T)$ think of it as $\mathbb{Z}$)

$\{\alpha, -\alpha\} = \Phi$.

tensor $X \otimes_{\mathbb{Z}} \mathbb{R} = E \simeq \mathbb{R}$; extend action of W to $E = \mathbb{R}$; $\{1 = Id, w = -1\}$

Consider on E a W-inv $\langle\, , \rangle$; $\langle\, , \rangle : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$
$(x,y) \mapsto xy$

A_1

$(E \supseteq X)$

$\mathbb{R} \supseteq \mathbb{Z}$

Now $\{-\alpha, \alpha\}$ is a root system in E. ("•" Dynkin diagram)

Comment: $\mathbb{Z}\Phi \subseteq X$ as

If we go to PGL_2, same rank, same Φ ("•"). But $\mathbb{Z}\Phi = X$.

This distinguishes PGL_2 & SL_2 combinatorially.

Choice

as abstract groups they are isom
$GL_2 = Z(GL_2)SL_2$

How do we reflect roots on the group? The U_α's.

$U_\alpha := \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} : c \in k \right\}$; $U_{-\alpha} := \left\{ \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} : c \in k \right\}$.

unique subgroup isom to $\mathbb{G}_a$ such that it is $u_\alpha(c)$ $\left(t u_\alpha(c) t^{-1} = u_\alpha(\alpha(t) c) \right)$

$u_\alpha : \mathbb{G}_a \longrightarrow SL_2$

$c \longmapsto \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$

Note all our results make sense

- $T_2 \cap SL_2 = \underset{\text{Borel}}{B} = B_u \rtimes T = U_\alpha \rtimes T \left(= T \prod_{\alpha \in \Phi^+} U_\alpha . \right)$
- $w \in W \setminus \{1\}$, $w U_\alpha w^{-1} = U_{-\alpha} = U_{w \cdot \alpha}$
- $\{ \text{Bases of } \Phi \} \longleftrightarrow \{ \text{Borels} \supseteq T \}$

$\{\alpha\} \longmapsto B$

$\{-\alpha\} \longmapsto T_{2n}^- \cap SL_2 = B^-$ (opposite Borel)

$\qquad\qquad\qquad \left\{ \begin{pmatrix} 1 & 0 \\ * & * \end{pmatrix} \right\}$

"Auts of Dynkin diagrams $\longrightarrow$ aut of LAG"

$\rho : \Delta \to \Delta$ $\qquad\qquad$ $\sigma : G \to G$

$\sigma(U_\alpha) = U_{\rho(\alpha)}$

• Bibliografía

• Para Root Systems ; Ch3 de Humphreys "Intro to Lie Algebras & Rep thy"
- leyendo Ch1-4 de Humphreys uno clasifica las álgebras de Lie semisimples complejas de acuerdo a su root system. (en LAGs necesitamos más data ; G LAG determines root system but non-isom LAGs can give the same)

• Para LAGs lo que he dicho está en Malle Testerman con un pelín mas de detalle. "Full" detail en Humphreys "LAGs". Para Lie Type Carter "FG of Lie Type".
Digne - Michel

• El resultado de Malle, está 3.1 de
"Picky elements, subnormalisers and character correspondences".

• Para group schemes Jantzen