UNDERSTANDING THE SYLOW-p DOUBLE COSETS OF S_n

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LET P_n BE THE SYLOW-p SUBGROUP OF S_n

CONSIDER $P_n \backslash S_n / P_n$

- HOW MANY DOUBLE COSETS?
- WHAT ARE THEIR SIZES?
- DO THEY HAVE 'NICE NAMES'?
- IS $L_k(P_n \backslash S_n / P_n)$ FROBENIUS? (CHAR $k = p$)

FOUR MOTIVATIONS

① LET $\mathcal{X}$ BE A FINITE SET, G ACTS ON $\mathcal{X}$

$$\mathcal{X} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \dots \cup \mathcal{O}_k$$

- HOW MANY ORBITS? TYPICAL SIZES? NICE NAMES? RANDOM GENERATION?

EX $\mathcal{X} = G$, $s^t = t^{-1} s t$, ORBITS – CONJUGACY CLASSES

$G = S_n$ ORBITS ARE PARTITIONS

$G = U_n(q) = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$, $* \in \mathbb{F}_q$, ORBITS ARE ???

EX $H, K \le G$, $H \times K$ ACTS, ORBITS ARE $H \backslash G / K$

$G = S_n$, $H = P_\lambda$, $K = P_\mu$ (PARABOLICS)

ORBITS ARE CONTINGENCY TABLES

$$\begin{array}{c} \begin{pmatrix} & t_{ij} & \end{pmatrix} \begin{matrix} \lambda_1 \\ \vdots \\ \lambda_I \end{matrix} \\ \mu_1 \cdots \mu_J \end{array} \qquad t_{ij} \in \mathbb{N}_+$$

BURNSIDE PROCESS ($\mathcal{X}$, G GENERAL)

- FROM x CHOOSE $s \in G_x = \{s : x^s = x\}$ (UNIFORM)
- FROM s CHOOSE $y \in \mathcal{X}_s = \{y : y^s = y\}$ "

$$K(x,y) = \frac{1}{|G_x|} \sum_{s \in G_x \cap G_y} 1/|\mathcal{X}_s|$$

$$\pi(x) = z^{-1}/|O_x| \quad (z = \#\text{ ORBITS})$$

(2) MICHAEL GELINE: WHAT ABOUT $P_n \backslash S_n / P_n$?

RECALL; SAWADA, CURTIS, GREEN, TINBERG

IF $G = (B, N, U)$ SPLIT B-N PAIR CHAR p

THEN · $E = L_k(U \backslash G / U)$ IS FROBENIUS

· EVERY SIMPLE RIGHT E MODULE IS 1-DIM.

GIVEN BY A MULTIPLICATIVE CHARACTER $\varphi : E \to k$.

· EACH SUCH φ IS DETERMINED BY VECTOR

$(\chi; \mu_1, \cdots, \mu_n)$, χ LIN CHARACTER OF B, $\mu_i \in k$

· THERE IS A BIJECTIVE CORRESPONDENCE

BETWEEN {IRREDUCIBLE k-G MODULES}, $\{\varphi\}$.

ALL WE WANT TO DO IS "LET $q \to 1$"

(3) ANY SIMPLE QUESTION ABOUT S_n IS WORTH ANSWERING

(4) WE ARE 'STUCK'. MAYBE SOMEONE HERE WILL HAVE AN IDEA.

BACK TO BUSINESS $P_n \backslash S_n / P_n$

SIMPLEST CASE ? $C_p \backslash S_p / C_p$?

RECALL $|HxK| = \dfrac{|H||K|}{|H \cap xKx^{-1}|}$

SO, DOUBLE COSETS FOR C_p IN S_p HAVE SIZES p OR p^2

$p! = pn_1 + p^2 n_2$, n_i = # DOUBLE COSETS SIZE p^i

IF $|HxH| = |H|$ THEN $x \in N_G(H) = C_p^* \ltimes C_p$

THE SIZE p DOUBLE COSETS BIJECT WITH $N_{S_p}(C_p)/C_p \cong C_p^*$

SO $n_1 = p-1$, $n_2 = \dfrac{(p-1)! + (p-1)^2}{p}$, $n = n_1 + n_2$

eg $p = 11$, $n_1 = 10$, $n_2 = 329{,}000$, $n = 329{,}010$

- ALMOST ALL DOUBLE COSETS AS BIG AS POSSIBLE
- ALL SIZES OCCUR

NOTE EVEN HERE, WE CAN'T 'NAME' THE BIG ONES!

NEXT SIMPLEST CASE $n = hp$, $1 \le h \le p-1$

THEN $P_n \cong C_p^h$ ABELIAN

POSSIBLE SIZES: $p^h, p^{h+1}, \ldots, p^{2h}$ LET $n_a = \#$ SIZE p^a

THEOREM $\forall p$ AND $1 \le h < p-1$

(1) $$\frac{(hp)!}{p^{2h}}\left(1 - \frac{1}{2(p-1)!}\right) \le n_{2h} \le \frac{(hp)!}{p^{2h}}$$

SO ALMOST ALL ARE AS LARGE AS POSSIBLE (SUPER EXP. ERROR)

(2) $$\frac{n_a}{n_{a-1}} \sim \frac{(hp)^p}{p(p-1)}\left(\frac{2h+1}{h}\right)^2$$ (p LARGE, UNIFORM IN h)

SO ALL OCCUR AND n_a DECREASES SUPEREXPONENTIALLY.

PROOF IN THIS CASE WE HAVE FORMULAS, eg

$$n_{2n} = \frac{1}{p^{2n}} \sum_{j=0}^{n} (-1)^j ((n-j)p)!\, j! \binom{n}{j}^2 (p(p-1))^j$$

BUT

- EVEN FOR S_{p^2}, NO FORMULAS
- EVEN FOR S_{np}, NO NAMES
- WE EXPECT SAME PATTERN, ALL n.

SO HOW DO WE NAME THINGS?
HOW DO WE UNDERSTAND {DOUBLE COSETS}?

A GENERAL THEOREM

Fix p, $2 < p \leq n$

Th 1. LET $f(n,p)$ = PROBABILITY $P_n \cap P_n^x > \{id\}$

THEN $f(n,p) \to 0$, UNIFORMLY IN p.

SO MOST DOUBLE COSETS ARE AS LARGE AS POSSIBLE ($|P_n|^2$)

BUT THIS FAILS WHEN $p=2$ (!)

$f(n,2) \sim 1 - e^{-1/2}$

WHY? WE PROVE SOMETHING STRONGER

LET $x \in S_n$ HAVE ORDER p WITH FEWER THAN p FIXED POINTS

LET $f'(n,p)$ = PROBABILITY TWO RANDOM CONJUGATES OF x HAVE A COMMON CENTRALIZER OF ORDER DIVISIBLE BY p.

THEOREM 2 $f'(n,p) \to 0$ (UNIFORMLY IN p)

Th 2 $\Rightarrow$ Th 1 LET P BE A SYLOW-p SUBGROUP OF S_n

SAY $w \in P \cap P^y$ SOME $y \notin 1$, $w \neq 1$

$Z(P)$ CONTAINS ELEMENT x OF ORDER p, LESS THAN p F.P.

w COMMUTES WITH x AND WITH x^y SO $f'(n,p) \geq f(n,p)$.

TO PROCEED, USE SPECIAL CASE OF Th EBERHARD - GARZONI

THEOREM 3 LET $x_n \in S_n$ HAVE ODD ORDER AND $|FP(x_n)|/\sqrt{n} \to 0$

THEN THE PROBABILITY THAT TWO RANDOM CONJUGATES OF

x_n GENERATE A_n TENDS TO 1 AS $n \to \infty$.

TH 3 $\Rightarrow$ Th 2 ($p^{5/2} < n$) LET P BE A p-SYLOW OF S_n

PICK $z \in Z(P)$ WITH $r \leq p-1$ FIXED POINTS ($n = kp + r$)

AS ABOVE, PROB $P \cap P^x = 1$ IS AT MOST PROBABILITY $\exists$

AN ELEMENT CENTRALIZING BOTH z AND z^x BY E-G THIS TENDS

TO ZERO AS LONG AS $(p-1)/\sqrt{n} \to 0$

COUNTER EXAMPLES FOR $p=2$ ARE EASY

- SAY $n = 2^m$, $Z(P_n)$ IS AN ELEMENTARY ABELIAN 2-GROUP

  $Z(P_n^4)$ IS TOO.

- PICK TWO RANDOM PARTITIONS OF n INTO $n/2$ TWO ELEMENT SUBSETS AT RANDOM: $(i_1, i_2), (i_3, i_4) \cdots (i_{n/2-1}, i_{n/2})$

  $(j_1, j_2), (j_3, j_4) \cdots (j_{n/2-1}, j_{n/2})$

  WHATS # MATCHING PAIRS? $(i_a, i_{a+1}) = (j_b, j_{b+1})$ CALL IT W

  $P(W = \ell) \to e^{-1/2} (1/2)^\ell / \ell!$ WHEN n IS LARGE.

  $P(W>0) \sim 1 - e^{-1/2}$

(CAN SHOW INTERSECTING CENTERS IS MAIN OBSTRUCTION)

REFERENCES

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