Picky elements, subnormalisers, and character correspondences

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Picky elements

G a finite group

- Write $G = H_1 \cup \cdots \cup H_r$ for *proper* subgroups $H_i < G$.

Minimal such r: the *covering number* of G.

Known: $r > 2$.

Restrict to covering p-elements G_p of G (p a prime):

- write $G_p = P_1 \cup \cdots \cup P_r$ for $P_i \in \mathrm{Syl}_p(G)$.

Maróti–Martínez–Moretó (2024): Do we need all Sylow p-subgroups?

Else, G has *redundant Sylow p-subgroups*.

Picky elements, cont.

Easy: G has redundant Sylow p-subgroup $\iff$ all p-elements of G lie in at least two Sylow p-subgroups.

Definition

A p-element $x \in G$ is *picky* $:\iff$ x lies in unique Sylow p-subgroup of G.

Example

- *normal Sylow p-subgroups,*
- *TI (trivial intersection) Sylow p-subgroups*

are never redundant (all $1 \neq x \in G_p$ are picky).

Basic observation:

Lemma

Let $P \in \mathrm{Syl}_p(G)$ and $x \in P$ picky in G. Then $\mathrm{N}_G(\langle x \rangle) \leq \mathrm{N}_G(P)$.

Picky character correspondence

For $x \in G$ let

$$\mathrm{Irr}^x(G) := \{\chi \in \mathrm{Irr}(G) \mid \chi(x) \neq 0\}.$$

Conjecture (Moretó–Rizo (2024))

Let $x \in P \in \mathrm{Syl}_p(G)$ be picky $\Longrightarrow$ there exists a bijection

$$f_x : \mathrm{Irr}^x(G) \xrightarrow{1-1} \mathrm{Irr}^x(\mathrm{N}_G(P))$$

such that

(1) $\chi(1)_p = f_x(\chi)(1)_p$, *and*

(2) $\mathbb{Q}(\chi(x)) = \mathbb{Q}(f_x(\chi)(x))$.

In fact, Moretó–Rizo conjecture that the bijection can be taken independent of x (in a fixed Sylow p-subgroup P).

See talk of Alex for connections to other local-global conjectures.

Picky elements in nearly simple

[MMM]: symmetric groups $\mathfrak{S}_n$ contain picky p-elements for all p.

What about other simple groups?

Lemma

Let $N \trianglelefteq G$ with $[G:N]$ prime to p. Then:
A p-element $x \in G$ is picky $\iff$ x is picky in N.

Thus, for example, may consider $\mathfrak{S}_n$ instead of $\mathfrak{A}_n$ when $p \neq 2$.

Lemma

Let $N \trianglelefteq G$ where N is either a p-group or central. Then:
A p-element $x \in G$ is picky $\iff$ xN is picky in G/N.

Thus, may consider, $\mathrm{SL}_n(q)$ in place of $\mathrm{PSL}_n(q)$,
or even $\mathrm{GL}_n(q)$ when $p \nmid (q-1)$.

Unipotent picky elements

Now G of Lie type (e.g., $\mathrm{SL}_n(q)$, $\mathrm{Sp}_{2n}(q), \ldots, E_8(q)$)

First case: p is defining characteristic, so p-elements are *unipotent*.

Theorem

G quasi-simple of Lie type. A unipotent element $1 \neq x \in G$ is picky $\iff$ one of the following holds:

(1) *x is regular unipotent;*

(2) *$G = \mathrm{SU}_{2n+1}(q)$ and x has Jordan block sizes $(2n, 1)$;*

(3) *$G = {}^2B_2(q^2)$ is a Suzuki group;*

(4) *$G = {}^2G_2(q^2)$ is a Ree group; or*

(5) *$G = {}^2F_4(q^2)$ is a Ree group and $|\mathrm{C}_G(x)| = 2q^6$.*

Use: Sylow p-normalisers are *Borel subgroups*;
regular unipotent elements are contained in a *unique* Borel subgroup.

Unipotent picky elements, cont.

About proof of (2)–(5):

- For $\mathrm{SU}_3(q)$, ${}^2B_2(q^2)$ and ${}^2G_2(q^2)$: Sylow p-subgroups are TI $\Rightarrow$ all $1 \neq x \in G_p$ picky.

- For ${}^2F_4(q^2)$, $C = [x]$, use:

$$|C \cap P| = \frac{|C|}{[G : \mathrm{N}_G(P)]} \quad \Longrightarrow \quad x \text{ picky}$$

  (classes of $\mathrm{N}_G(P)$ know by work of Himstedt)

- For $\mathrm{SU}_{2m+1}(q)$ use induction, starting at $\mathrm{SU}_3(q)$.

Subnormalisers

For $H \leq G$, set

$$S_G(H) := \{g \in G \mid H \lhd\lhd \langle g, H\rangle\},$$

and for $x \in G$ the *subnormaliser* is

$$\mathrm{Sub}_G(x) := \big\langle S_G(\langle x\rangle)\big\rangle.$$

Proposition

Let $x \in G$ be a p-element $\Rightarrow \mathrm{Sub}_G(x)$ is generated by the normalisers of the Sylow p-subgroups of G containing x.

Corollary

$x \in P \in \mathrm{Syl}_p(G)$ is picky $\iff \mathrm{Sub}_G(x) = \mathrm{N}_G(P)$.

Character correspondences

Conjecture (Moretó–Rizo (2024))

For any $x \in G_p$ there exists a bijection $f_x : \mathrm{Irr}^x(G) \xrightarrow{1-1} \mathrm{Irr}^x(\mathrm{Sub}_G(x))$ such that

(1) $\chi(1)_p = f_x(\chi)(1)_p$, *and*

(2) $\mathbb{Q}(\chi(x)) = \mathbb{Q}(f_x(\chi)(x))$ *for all* $\chi \in \mathrm{Irr}^x(G)$.

Specialises to "picky" conjecture.

Question

Does there exist f_x such that moreover

(3) $\chi(x)_p = f_x(\chi)(x)_p$, *and*

(4) $\mathbb{Q}_p(\chi) = \mathbb{Q}_p(f_x(\chi))$ *for all* $\chi \in \mathrm{Irr}^x(G)$ *?*

Subnormalisers in groups of Lie type

Again, G of Lie type in characteristic p, so "p-element"=unipotent.

Proposition

G of Lie type, $x \in G$ unipotent $\Rightarrow \mathrm{Sub}_G(x)$ is a parabolic subgroup of G.

Proof.

Clearly, $\mathrm{Sub}_G(x)$ contains a Sylow p-normaliser,
hence a Borel B of G.
Since G has a BN-pair $\Rightarrow$ all overgroups of B are parabolic
$\Rightarrow \mathrm{Sub}_G(x)$ is parabolic. □

Subnormalisers in groups of Lie type, II

Proposition

Let $G = \mathrm{SL}_n(q)$, $\mathrm{SO}_{2n}^+(q)$, $E_6(q)$, $E_7(q)$ *or* $E_8(q)$ *and* $x \in G$ *unipotent. Then:* $\mathrm{Sub}_G(x) = G \iff x$ *is not regular.*

Use that Dynkin diagram is simply laced.

Theorem

The Moretó–Rizo Conjecture holds for all unipotent elements of G *as above with* $\mathrm{Z}(G) = 1$, $p \geq 7$.

Use theorem of Green–Lehrer–Lusztig (1976):

$$x \in G \text{ regular} \quad \Longrightarrow \quad \mathrm{Irr}^x(G) = \mathrm{Irr}_{p'}(G).$$

Then done with McKay bijection.

Subnormalisers in rank 2

Proposition

Let $= B_2(q)$, $G_2(q)$, ${}^3D_4(q)$ *or* ${}^2F_4(q^2)$, *and* $x \in G$ *unipotent. Then* $\mathrm{Sub}_G(x) = G$ *unless one of*

(1) x *is picky, where* $\mathrm{Sub}_G(x) \sim_G B$;

(2) $G = B_2(q)$ *for* $p \neq 2$, $|\mathrm{C}_G(x)| = 2q^3(q+1)$, *and* $\mathrm{Sub}_G(x) \sim_G P_1$;

(3) $G = G_2(q)$ *for* $p \neq 3$, $|\mathrm{C}_G(x)| = 3q^4$, *and* $\mathrm{Sub}_G(x) \sim_G P_1$;

(4) $G = {}^3D_4(q)$ *with* $|\mathrm{C}_G(x)| = q^6$, *and* $\mathrm{Sub}_G(x) \sim_G P_2$; *or*

(5) $G = {}^2F_4(q^2)$ *with* $|\mathrm{C}_G(x)| \in \{3q^{12}, 2q^8, 4q^8\}$, *and* $\mathrm{Sub}_G(x) \sim_G P_1$.

Proposition

The Moretó–Rizo Conjecture holds for all unipotent elements of G *as above.*

The case $G = {}^2F_4(q^2)$, $q^2 = 2^{2f+1}$, $\bar{q} := q/\sqrt{2}$

Table: 2-Parts of character values for G ...

	#	1	u_9	u_{10}	$u_{11} = u_{12}^{-1}$	$u_{13,14}$	u_{15-18}
$\chi_{2,3,23,24}$	$2q^2$	$\bar{q}$	$\bar{q}$	$\bar{q}$	$\bar{q}$	$\bar{q}$	$\bar{q}$
$\chi_{4,27,30,33}$	q^2	q^2	q^2	q^2	q^2	.	.
$\chi_{5,6,8,9}$	4	$\bar{q}^4$	$\bar{q}^4$	$\bar{q}^4$	$\bar{q}^4$	$\bar{q}^3$	$\bar{q}^2$
χ_{11-14}	4	$\bar{q}^4$	$\bar{q}^4$	$\bar{q}^4$	$(\bar{q}^4 \pm 2i\bar{q}^3)_2$	$\bar{q}^3$	$\bar{q}^2$
$\chi_{7,10}$	2	$2\bar{q}^4$	$2\bar{q}^4$	$2\bar{q}^4$	$2\bar{q}^4$	.	.
χ_{15-17}	3	q^4	$2q^4$	.	.	.	.

Table: ... and for $\mathrm{N}_G(P_1)$

	#	1	$c_{1,33}$	$c_{1,34}$	$c_{1,35} = c_{1,36}^{-1}$	$c_{1,37-38}$	$c_{1,39-42}$
$\chi_{7,8}(k), \chi_{9,10}$	$2q^2$	$\bar{q}$	$\bar{q}$	$\bar{q}$	$\bar{q}$	$\bar{q}$	$\bar{q}$
$\chi_2(k), \chi_{11}$	q^2	q^2	q^2	q^2	q^2	.	.
χ_{21-24}	4	$\bar{q}^4$	$\bar{q}^4$	$\bar{q}^4$	$\bar{q}^4$	$\bar{q}^3$	$\bar{q}^2$
χ_{14-17}	4	$\bar{q}^4$	$\bar{q}^4$	$\bar{q}^4$	$(\bar{q}^4 \pm 2i\bar{q}^3)_2$	$\bar{q}^3$	$\bar{q}^2$
$\chi_{13,25}$	2	$2\bar{q}^4$	$2\bar{q}^4$	$2\bar{q}^4$	$2\bar{q}^4$	.	.
χ_{18-20}	3	q^4	$2q^4$	.	.	.	.

Subnormalisers in non-simply laced types

Lemma

Let $G = \mathrm{SO}_{2n+1}(q)$*, q odd, and* $x \in G$ *unipotent non-regular. Then:* $\mathrm{Sub}_G(x) = G$ *unless x has Jordan form* $J_{2n-1} \oplus J_1^2$*.*

Lemma

Let $G = \mathrm{Sp}_{2n}(q)$ $(n \geq 3)$ *and* $x \in G$ *unipotent non-regular. Then:* $\mathrm{Sub}_G(x) = G$ *unless q is odd and x has Jordan form* $J_{2n-2k} \oplus J_{2k}$ *for some* $1 \leq k \leq n/2$*.*

Lemma

Let $G = \mathrm{SU}_{2m}(q)$ *and* $x \in G$ *unipotent non-regular. Then:* $\mathrm{Sub}_G(x) = G$ *unless x has Jordan form* $J_{2m-1} \oplus J_1$*.*

Odd-dimensional unitary groups: even more tricky.

Picky semisimple elements

Now, $G = G(q)$ of Lie type in characteristic $\neq p$ (so $p \nmid q$), and
$d :=$ order of q modulo p.

Theorem

Assume Sylow p-subgroups of G are abelian. Then:
$x \in G_p$ is picky $\iff$ $\mathrm{C}_G(x)$ is the centraliser of a Sylow d-torus.

Theorem

Assume Sylow p-subgroups of G are non-abelian and $p > 3$.
Then G possesses no picky p-elements.

Proof in non-abelian case

Proof.

Use: for $P \in \mathrm{Syl}_p(G)$ have

(1) $\mathrm{N}_G(P) \leq N := \mathrm{N}_G(T_d)$ for T_d a Sylow d-torus (as $p > 3$)

(2) p divides order of $W_d := \mathrm{N}_G(T_d)/T_d$ (as Sylow non-abelian)

(3) W_d has ≥ 2 Sylow p-subgroups (as $p > 3$, by inspection)

Now, if $x \in G$ is p-element $\overset{(1)}{\Longrightarrow}$ may assume $x \in N$.

If $x \in T_d \Rightarrow xT_d$ lies in all Sylow p-subgroups of $W_d \overset{(3)}{\Longrightarrow} x$ not picky.

If $x \notin T_d$, show $|\mathrm{C}_G(x)|$ is divisible by some prime not dividing $|N|$
$\Longrightarrow \mathrm{C}_G(x) \not\leq \mathrm{N}_G(P) \Longrightarrow x$ not picky
(as otherwise $\mathrm{C}_G(x) \leq \mathrm{N}_G(\langle x \rangle) \leq \mathrm{N}_G(P)$). □

For $p = 2, 3$, both (1) and (3) above may fail.

Picky semisimple 3-elements

For $p = 3$, still complete classification:

Theorem

Assume Sylow 3-subgroups of G are non-abelian. Then G possesses a picky 3-element x if and only if one of:

1. $G = \mathrm{SU}_3(8)$ *or* $G_2(8)$ *with* $|\mathrm{C}_G(x)| = 81$;
2. $G = {}^3D_4(2)$ *with* $|\mathrm{C}_G(x)| = 54$; *or*
3. $G = G_2(2) \cong \mathrm{PSU}_3(3).2$.

Proposition

The "picky" Moretó–Rizo Conjecture holds for G as above at $p = 3$.

Picky semisimple 2-elements

Work in progress.....

I expect:

1. no picky 2-elements for groups of rank at least 8,
2. but quite a few for small rank groups.

So further interesting cases for Alex and Noelia's conjectures!