25 years since I first met Gabriel

Abstract: I will talk about Gabriel's influence on my work.

Alexander Moretó, University of Valencia

January 30th 2025

Richard Brauer

RICHARD DAGOBERT BRAUER 1901–1977

Santa Cruz CA, 1979

PREFACE

In the last year or so there have been widespread rumors that group theory is finished, that there is nothing more to be done. It is not so.

While it is true that we are tantalizingly close to that pinnacle representing the classification of finite simple groups, one should remember that only by reaching the top can one properly look back and survey the neighboring territory. It was the task of the Santa Cruz conference not only to describe the tortuous route which brings us so close to the summit of classification, but also to chart out more accessible paths–ones which might someday be open to the general mathematical public.

A third concern was the elucidation of topics in related fields, and it is to one of these three areas that the papers in this volume are devoted.

Just a quick glance at the table of contents will reveal a wide variety of topics with which the modern group theorist must contend. Some of these, for example, the connections with the theory of modular functions, have very recent origins, but they leave us with the clear impression that, far from being dead, group theory has only just come of age.

Geoffrey Mason
Chicago, June 1980

Santa Cruz CA, 1979

From Jon Alperin's paper in these proceedings:

CONJECTURE B. *If P is a Sylow p-subgroup of G then the number of characters of G of degree not divisible by p equals the number of characters of $N_G(P)$ of degree not divisible by p.*

This is a remarkable idea; its truth has been verified in many special cases. We keep P as a Sylow p-subgroup henceforth. Recall that P is said to be TI if it intersects each of its conjugates in the identity. This is a situation that arises often in studying groups. Here one also can make a good guess.

CONJECTURE C. *If P is TI then the number of characters of G not vanishing on the nonidentity elements of P equals the number of characters of $N_G(P)$.*

Santa Cruz CA, 1979

There is a famous conjecture of Brauer's concerning heights:

Conjecture D. *The defect group D is abelian if, and only if, every character in b has height zero.*

Brauer's First Main Theorem on Blocks also shows the critical role of the defect group: there is a canonical one-to-one correspondence between the blocks of G with defect group D and the blocks of $N = N_G(D)$ with defect group D. Hence, there is a block b_N of N corresponding to b. We can now state the refinement of McKay's conjecture.

Conjecture E. *The number of characters of height zero in b equals the number of characters of height zero in b_N.*

Santa Cruz CA, 1979

Everett Dade's paper in the same proceedings:

A CORRESPONDENCE OF CHARACTERS

EVERETT C. DADE

Bilbao, May 1999

Gabriel came to visit. He gave a talk, that started as follows:

Notes by Gabriel Navarro

Oberwolfach, June 20-26, 1999

ZEROS OF CHARACTERS

(I) Zeros of primitive characters in p-solvable groups.

If G is a finite group and p is a prime number, it is a standard fact that every character of G of degree not divisible by p never vanishes on any element of order a power of p.

THEOREM. *Suppose that χ is a character of G of degree not divisible by p. If $x \in G$ has p-power order, then $\chi(x) \neq 0$.*

Proof. Let $\mathbf{R}$ be the ring of algebraic integers and let M be a maximal ideal of $\mathbf{R}$ containing $p\mathbf{R}$, so that $F = \mathbf{R}/M$ is a field of characteristic p. If ϵ is a p-power root of unity, notice that $\epsilon \equiv 1 \bmod M$. Since $\chi(x)$ is a sum of $\chi(1)$ p-power roots of unity, we deduce that $\chi(x) \equiv \chi(1) \bmod M$ and the proof of the theorem follows. ∎

In some sense, it is natural to study to what extent we can replace p by not divisible by p. Can irreducible characters of G of p-power degree vanish on elements of p'-order?

The answer to this question is "yes", although it seems that not very often. For instance, the Mathieu group M_{11} has an irreducible character of degree 11 which vanishes on an element of order six. Also, although it is easy to find solvable examples none of the irreducible characters in these are primitive.

(Let ω be a cubic primitive. root of unity. Then

$$1 + 1 + (-\omega)^2 + (-\omega)^3 + (-\omega)^4 = 0\,.$$

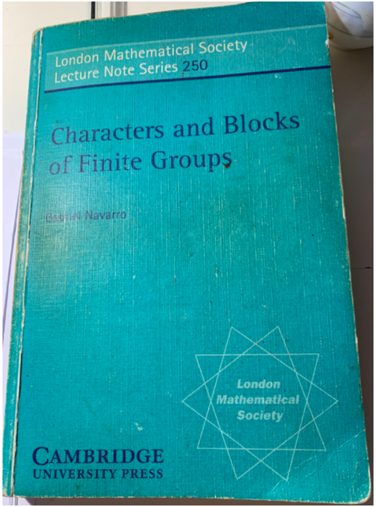
Bilbao, May 1999

Some sentences from Gabriel that I remember from that visit:

"I know a lot of Isaacs' π-theory"

"Do you know that if (G, N, θ) is a character triple with N a p-group, then there is an isomorphic character triple (G^*, N^*, θ^*) with N^* central cyclic p-group?"

Madison WI, September-December 1999



Madison WI, September-December 1999

Some sentences from Isaacs that I remember from that visit:

“The most important theorem is the Second Main.”

“I love Wielandt’s theory on subnormal subgroups.”

A consequence of Brauer's Second Main Theorem

(5.9) COROLLARY. *Let $\chi \in \mathrm{Irr}(G)$ and let $g \in G$. If g_p is not contained in any defect group of the block of χ, then $\chi(g) = 0$.*

Proof. Let B be the block of χ. Write $x = g_p$ and $y = g_{p'}$ and suppose that $b \in \mathrm{Bl}(\mathbf{C}_G(x))$ is such that $b^G = B$. By Theorem (4.14), we know that $\langle x \rangle$ is contained in some defect group of B. Since this is not possible by hypothesis, we see that such a block b cannot exist. By Corollary (5.8), $\chi(xy) = 0$ and the proof is complete. ∎

Madison WI, September 2000-May 2001

Both Gabriel and I spent the year visiting Isaacs.

Annals of Mathematics, **156** (2002), 333–344

New refinements of the McKay conjecture for arbitrary finite groups

By I. M. Isaacs and Gabriel Navarro

Abstract

Let G be an arbitrary finite group and fix a prime number p. The McKay conjecture asserts that G and the normalizer in G of a Sylow p-subgroup have equal numbers of irreducible characters with degrees not divisible by p. The Alperin-McKay conjecture is version of this as applied to individual Brauer p-blocks of G. We offer evidence that perhaps much stronger forms of both of these conjectures are true.

2001-2002: The year of the Galois-McKay conjecture

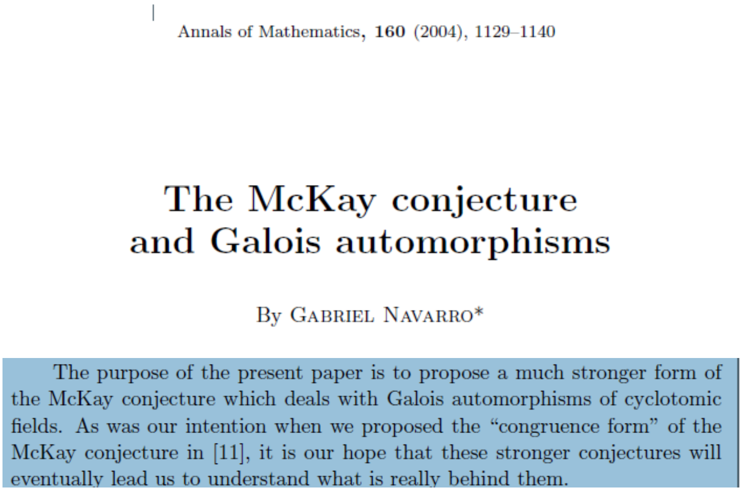
I spent the year travelling in the USA (Madison, Chicago, Athens OH, Urbana-Champaign IL,...). Gabriel was in Valencia, I think.

I tried to help him verify his conjecture for sporadic groups. Finally, Thomas Breuer had to compute several unknown character tables of $\mathbf{N}_G(P)/P'$.

2001-2002: The year of the Galois-McKay conjecture

Dade claimed that the conjecture was true for solvable groups. Gabriel and I were trying to understand Dade's arguments.

2001-2002: The year of the Galois-McKay conjecture


Annals of Mathematics, **160** (2004), 1129–1140

The McKay conjecture
and Galois automorphisms

By Gabriel Navarro*

The purpose of the present paper is to propose a much stronger form of the McKay conjecture which deals with Galois automorphisms of cyclotomic fields. As was our intention when we proposed the "congruence form" of the McKay conjecture in [11], it is our hope that these stronger conjectures will eventually lead us to understand what is really behind them.

2001-2002: The year of the Galois-McKay conjecture

Advances in Mathematics 217 (2008) 2170–2205

www.elsevier

Above the Glauberman correspondence

Alexandre Turull [1]

2001-2002: The year of the Galois-McKay conjecture

Theorem 7.4. *Let p be a prime number, let G be a finite p-solvable group and let P be a Sylow p-subgroup of G. Let F be a field of characteristic zero, and assume that, either $\mathbf{Q}_p \subseteq F$ or F is algebraically closed. Then there exists a bijection*

$$f : \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}\big(\mathrm{N}_G(P)\big)$$

satisfying all of the following conditions:

(1) *For every $\chi \in \mathrm{Irr}_{p'}(G)$, there is some $\epsilon \in \{1, -1\}$ such that $\chi(1) \equiv \epsilon f(\chi)(1) \pmod{p}$.*

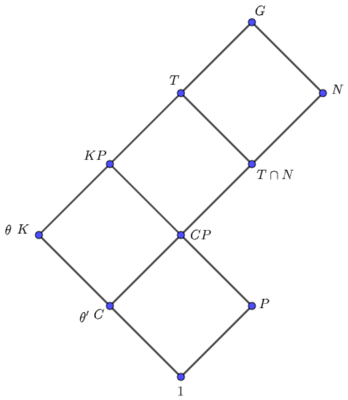
(2) *f commutes with the action of $\mathrm{Gal}(\overline{F}/F)$, so, in particular, $F(\chi) = F(f(\chi))$ for every $\chi \in \mathrm{Irr}_{p'}(G)$.*

(3) *For every $\chi \in \mathrm{Irr}_{p'}(G)$, we have $[f(\chi)] = [\chi] \in \mathrm{Br}(F(\chi))$, so that $f(\chi)$ and χ have the same Schur index over every field containing F.*

(4) *Let U be a normal p-subgroup of G, let $\zeta \in \mathrm{Irr}(U)$, and assume that ζ is P-invariant. Then,*

$$f\big(\mathrm{Irr}_{p'}(G; \zeta, F)\big) = \mathrm{Irr}_{p'}\big(\mathrm{N}_G(P); \zeta, F\big).$$

The Dade-Turull theorem



- $K \lhd G$, $|K| = p'$,
- $P \leq G$ p-subgroup and $KP \trianglelefteq G$,
- $N = \mathbf{N}_G(P)$, $C = \mathbf{C}_G(P)$,
- $\theta \in \mathrm{Irr}(K)$ P-invariant and $\theta' \in \mathrm{Irr}(C)$ its P-Glauberman correspondent,
- $T = G_\theta = I_G(\theta)$, so $T \cap N = N_{\theta'} = I_N(\theta')$,
- (T, K, θ) and $(T \cap N, C, \theta')$ are isomorphic.

2001-2002: The year of the Galois-McKay conjecture

Seeing Gabriel's excitement with his strengthenings of McKay's conjecture I tried to find some strengthening involving zeros of characters. I failed.

In September 2002, I got a research position. I could go anywhere I wanted in Spain, and came to Valencia because Gabriel was here. This position started in September 2003.

June 2004: Soccer eurocup

Gabriel thought that it was possible to reduce McKay conjecture to questions on simple groups. Isaacs and Malle came to work on that during the soccer eurocup.

Invent. math. 170, 33–101 (2007)
DOI: 10.1007/s00222-007-0057-y

Inventiones mathematicae

A reduction theorem for the McKay conjecture

I.M. Isaacs[1], Gunter Malle[2], Gabriel Navarro[3,⋆]

Fall 2004: Gainesville FL

Gabriel was visiting Tiep and Turull in Gainesville. I was also there. We tried to extend the Gluck-Wolf theorem to arbitrary groups. We failed, but Gabriel is persistent, and many years later he and Tiep succeeded.

I already had in mind that it should be possible to extend Brauer's height zero conjecture to blocks with non-abelian defect group.

Valencia, September 2005

Eaton and Robinson, independently, came to visit Gabriel at the same time. I ended up talking to Eaton on my ideas to extend Brauer's height zero conjecture to blocks with non-abelian defect groups.

Isaacs conference (Valencia), June 2009

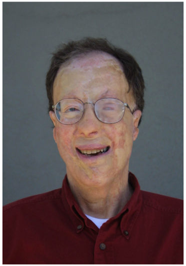
ISAACS CONFERENCE

Conference on Character Theory of Finite Groups

in honor of Martin Isaacs

June 3, 2009 - June 5, 2009

Universitat de Valencia (SPAIN)



Main Speakers: E. Dade, P. Diaconis, P. Fong, S. Gagola, G. Glauberman, D. Gluck, R. Gow, R. Guralnick, T. Keller, A. Mann, A. Moretó, R. Solomon, J. Thompson, P. Tiep and A. Turull.

Organizing Committee: M. Lewis, G. Navarro, D. Passman and T. Wolf.

A. Moretó. *Heights of Characters and Defect Groups.*

There are a number of results and conjectures that relate the characters of height zero of a block B of a finite group G with the characters of height zero of groups related to the defect group D of B. But what happens for characters of positive height? It is definitely not true that the sets of heights of B and D coincide but, for instance, we consider the question of whether the smallest positive member of both sets coincide.

The Eaton-M conjecture

Our paper was finally published in 2014.

There has been significant recent progress.

New math

Second title: Local representation theory, picky elements and subnormalizers

Abstract: I hope to convince Gabriel that there are still many interesting open problems in character theory. Character theory is not finished!

Joint work **in progress** with Noelia Rizo.

The main problem of block theory

According to Alperin:

> **2. Characters.** Brauer's work on block theory, stretching over decades, strongly suggests the following problem as a reasonable choice for the main problem of the subject.
>
> *Problem A. Give rules which determine the values of the characters of G in terms of the p-local subgroups of G.*

As pointed out by Alperin, McKay's conjecture deals with the values at the identity.

But what happens with all other character values?

Picky elements

The origin of this is a paper that I wrote with Attila Maróti and Juan Martínez Madrid.

Journal of Algebra 638 (2024) 840–861

Contents lists available at ScienceDirect

Journal of Algebra

journal homepage: www.elsevier.com/locate/jalgebra

Covering the set of p-elements in finite groups by Sylow p-subgroups ☆

Attila Maróti [a], Juan Martínez [b], Alexander Moretó [b,*]

Picky elements

Definition.

We say that a p-element $x \in G$ is picky if it belongs to a unique Sylow p-subgroup.

In MMM, we saw that "most" groups have picky elements. For instance, symmetric groups and groups of Lie type in defining characteristic always have picky elements.

Picky elements

In Maróti-Martínez Madrid-M, there is a small comment on block theory and zeros of characters.

Note that $x \in G$ belongs to a unique Sylow p-subgroup of G if and only if x does not belong to any intersection of two different Sylow p-subgroups. Since defect groups are Sylow intersections, this suggests that this is related to block theory. For instance, there is the following connection with zeros of characters.

Corollary 2.5. *Let G be a group without a redundant Sylow p-subgroup. Then there exists $x \in G_p$ such that $\chi(x) = 0$ for every $\chi \in \mathrm{Irr}(G)$ that does not belong to a p-block of full defect.*

Proof. Let B be a p-block with defect group D such that $|D| < |G|_p$. By Lemma 2.1, there exists $x \in G_p$ that does not belong to the intersection of any two different Sylow subgroups. By Corollary 4.21 of [31], x does not belong to any conjugate of D. Now, Corollary 5.9 of [31], implies that $\chi(x) = 0$ for any $\chi \in \mathrm{Irr}(B)$, as wanted. □

Alperin's Conjecture C

Given a non-empty subset $\mathcal{S}$ of a finite group G, we set

$$\mathrm{Irr}^{\mathcal{S}}(G) = \{\chi \in \mathrm{Irr}(G) \mid \chi(S) \neq \{0\}\}.$$

If $S = \{x\}$, we write $\mathrm{Irr}^{\mathcal{S}}(G) = \mathrm{Irr}^{x}(G)$.

With this notation, Alperin's Conjecture C, proved by Blau and Michler in 1990, asserts that if G has a TI-Sylow p-subgroup P then

$$|\mathrm{Irr}^{P}(G)| = |\mathrm{Irr}^{P}(\mathbf{N}_G(P))|.$$

Recall that P is TI if $P^g \cap P = 1$ or P for any $g \in G$.

Towards the picky conjecture

Let $\mathcal{P}$ be the set of picky elements in P. Note that $\mathcal{P} = P - \{1\}$ if and only if P is TI.

Could it be true that

$$|\mathrm{Irr}^{\mathcal{P}}(G)| = |\mathrm{Irr}^{\mathcal{P}}(\mathbf{N}_G(P))|?$$

I started doing GAP, and did not find counterexamples... In fact, much stronger results seemed to hold.

The picky conjecture

Conjecture (Global picky conjecture)

If G has picky elements, then there exists a bijection $f : \mathrm{Irr}^{\mathcal{P}}(G) \longrightarrow \mathrm{Irr}^{\mathcal{P}}(\mathbf{N}_G(P))$ such that for every $\chi \in \mathrm{Irr}^{\mathcal{P}}(G)$

1. $\chi(1)_p = f(\chi)(1)_p$;
2. $\mathbb{Q}(\chi(x)) = \mathbb{Q}(f(\chi)(x))$ for every $x \in \mathcal{P}$;
3. $f(\mathrm{Irr}^x(G)) = \mathrm{Irr}^x(\mathbf{N}_G(P))$ for every $x \in \mathcal{P}$;
4. $\chi(x)_p = f(\chi)(x)_p$ for every $x \in \mathcal{P}$.

Note that $\mathrm{Irr}_{p'}(G) \subseteq \mathrm{Irr}^x(G)$ for any $x \in \mathcal{P}$. Therefore, this conjecture is a strong form of McKay's conjecture for groups with picky elements.

We had noticed that Condition 4 seems to hold when $\chi(x)$ is rational. After checking examples in groups of Lie type, Malle pointed out that $\chi(x)_p$ makes sense even when $\chi(x)$ is not rational and asked whether this could hold in general.

In fact, it is very often the case that for $x \in \mathcal{P}$,

$$\chi(x) = \pm f(\chi)(x).$$

For instance, we expect this to hold when one of the following holds:

- G has abelian Sylow p-subgroups.
- G is p-solvable.
- The principal p-block of G is not controlled in the sense of Alperin-Broué. (?)

If $\chi(x) = \pm f(\chi)(x)$ holds for every $x \in \mathcal{P}$ we say that G satisfies the strong global picky conjecture.

The paradigmatic counterexamples to the strong global picky conjecture are the simple groups with non-abelian TI Sylow p-subgroups.

The picky conjecture: one element at a time

Conjecture (Picky conjecture)

Let $x \in G$ be a picky p-element. Then there exists a bijection $f : \mathrm{Irr}^x(G) \longrightarrow \mathrm{Irr}^x(\mathbf{N}_G(P))$ such that for every $\chi \in \mathrm{Irr}^x(G)$

1. $\chi(1)_p = f(\chi)(1)_p$;
2. $\mathbb{Q}(\chi(x)) = \mathbb{Q}(f(\chi)(x))$;
3. $\chi(x)_p = f(\chi)(x)_p$.

If, in addition, $\chi(x) = \pm f(\chi)(x)$, we say that (G, x) satisfies the *strong* picky conjecture.

Cyclic Sylow subgroups

Theorem

The strong global picky conjecture holds for groups with cyclic Sylow p-subgroups. Furthermore, the bijection is compatible with the Isaacs-Navarro and Navarro strengthenings of McKay's conjecture.

We prove this using Dade's cyclic defect theory. Our conjectures also admit a block version. I will not discuss them in this talk.

We expect the conjectures to be compatible with all strengthenings of McKay (Isaacs-Navarro, Navarro, Turull,...).

Sporadic groups

We checked the global picky conjecture for the small sporadic groups. Finally, Thomas Breuer was able to check it for all sporadic groups. Several of them are not straightforward. There are 6 cases of sporadic groups with picky p-elements and non-cyclic Sylow p-subgroups for which the character table of the p-Sylow normalizer is still not known. In these cases, he used clever ad hoc arguments to prove the conjecture.

Perhaps, the most complicated of these is the Baby Monster $G = B$ for $p = 2$. This group has 4 picky classes of 2-elements, all of them of order 32. The characters of $2'$-degree have value ± 1 at these classes. There are 64 of them. In addition, there are two characters with $\chi(1)_2 = 64$ such that if x is picky, $\chi(x) = \pm\zeta_8 \pm \zeta_8^3$, where ζ_8 is a primitive 8-th root of unity.

Symmetric groups

Theorem (Martínez Madrid)

The strong global picky conjecture holds for symmetric groups.

If $p = 2, n \geq 3$ and the cycle structure of $x \in S_{2^n}$ is $(2^{n-1}, 2^{n-2}, \ldots, 2^2, 2, 1^2)$ then x is picky and

$$\{\chi(1)_2 \mid \chi \in \mathrm{Irr}^x(S_{2^n})\} = \{1, 2, 2^2, \ldots, 2^{n-2}\} = \{\chi(1)_2 \mid \chi \in \mathrm{Irr}^x(P)\},$$

where P is the Sylow 2-subgroup that contains x.

It is worth remarking that, as far as we know,

$$\{\chi(1)_2 \mid \chi \in \mathrm{Irr}(S_{2^n})\}$$

is not known, but it has "jumps". On the other hand, the set of character degrees of the Sylow 2-subgroups is a set of consecutive powers of 2.

$\mathrm{Irr}^x(G)$ **vs** $\mathrm{Irr}_{p'}(G)$

Our general philosophy is that if x is a picky p-element, $\mathrm{Irr}^x(G)$ works much better than $\mathrm{Irr}_{p'}(G)$ or $\mathrm{Irr}(G)$. For instance, the picky conjecture implies the following.

Corollary

Let $x \in G$ be a picky p-element. Assume that the picky conjecture holds for (G, x). Then the multisets $(\chi(1)_p \mid \chi \in \mathrm{Irr}^x(G))$ and $(\chi(1)_p \mid \chi \in \mathrm{Irr}^x(\mathbf{N}_G(P)))$ coincide.

It seems likely that even $\{\chi(1)_p \mid \chi \in \mathrm{Irr}^x(G)\} = \{\chi(1)_p \mid \chi \in \mathrm{Irr}^x(P)\}$.

Compare it with the Eaton-M conjecture! In that setting we need blocks and we can only talk about the smallest positive height.

Self-normalizing Sylow p-subgroups

Theorem

Let G be a p-solvable group such that $\mathbf{N}_G(P) = P$. Let $x \in G$ be a picky p-element. Then there exists a bijection $f : \mathrm{Irr}^x(G) \longrightarrow \mathrm{Irr}^x(P)$ such that for any $\chi \in \mathrm{Irr}^x(G)$

1. $\chi_P = f(\chi) + \Delta$, where Δ is a character of P or zero.
2. $\chi(1)_p = f(\chi)(1)_p$ and $\chi(x) = f(\chi)(x)$.
3. For every φ irreducible constituent of Δ, $\varphi(1)_p > \chi(1)_p$ and $\varphi(x) = 0$

The proof of this theorem involves all the topics that Gabriel mentioned to me in his 1999 visit to Bilbao. In particular, we use Isaacs B_π-characters. In fact, we show that $f(\chi)$ is the unique Fong character associated to χ.

Souza has proved a "going up" version of this theorem.

Normal p-complement

Theorem

Let G be a group with a normal p-complement N and let $C = \mathbf{C}_N(P)$. Let $x \in P$ be picky. Then

1. $\mathrm{Irr}^x(G) = \{\tilde{\varphi}\mu \mid \tilde{\varphi} \in \mathrm{Irr}(G) \text{ is the canonical extension of } \varphi \in \mathrm{Irr}(N), \mu \in \mathrm{Irr}^x(G/N)\}$.
2. $\mathrm{Irr}^x(\mathbf{N}_G(P)) = \{\mu \times \nu \mid \mu \in \mathrm{Irr}^x(P), \nu \in \mathrm{Irr}(C)\} = \mathrm{Irr}^x(P) \times \mathrm{Irr}(C)$.
3. The map

$$\begin{array}{rccc} \Phi : & \mathrm{Irr}^x(G) & \to & \mathrm{Irr}^x(\mathbf{N}_G(P)) \\ & \tilde{\varphi}\mu & \mapsto & \varphi^* \times \mu \end{array}$$

   is a bijection, where φ^* is the Glauberman correspondent of φ. In particular the following holds:

   3.1 $\Phi(\chi)(1)_p = \chi(1)_p$ for every $\chi \in \mathrm{Irr}^x(G)$,

   3.2 $\chi(x) = \pm\Phi(\chi)(x)$.

Abelian Sylow subgroups

Theorem

Let G be a finite group with abelian Sylow p-subgroups and let $x \in P$ be picky. Then $\mathrm{Irr}_{p'}(G) = \mathrm{Irr}^x(G)$.

Proof.

Let $\chi \in \mathrm{Irr}^x(G)$ and let B be the p-block of G containing χ. By Corollary 5.9 of Gabriel's book, we have that $x \in D$ for some defect group D of B. By Green's theorem, there is $g \in G$ with $P \cap P^g = D$. Since x is picky, $P = P^g$ so $P = D$. But then $\chi \in \mathrm{Irr}_{p'}(G)$ by the Kessar-Malle solution to one-half of Brauer's height zero conjecture.

Abelian Sylow p-subgroups

We hope to be able to prove the strong picky conjecture in this case, or at least to reduce it to proving an "inductive picky condition".

For p-solvable groups we can prove that the strong picky conjecture holds for groups of p-length 1, modulo some details.

These details are related to the Dade-Turull theorem. We need some additional information on the properties of the Dade-Turull character correspondence.

This is not only about McKay!

Next, we discuss several other fields where picky elements are relevant.

1. Strengthening the Itô-Michler theorem:

Theorem

Let G be a finite group and let $x \in G$ be picky. If $\mathrm{Irr}^x(G) = \mathrm{Irr}(G)$ then $P \trianglelefteq G$.

2. Character tables and the size of the Sylow normalizer.
   Let K be a conjugacy class of G. Does the character table of G know if K is picky? We have proved that this would imply that the character table of G knows the size of a Sylow normalizer, at least for p-solvable groups.

3. The Navarro-Tiep paper on fields of values (Forum Math. Pi, 2021).
   The results and conjectures in that paper seem to admit a picky version for groups that satisfy the picky conjecture.

The Navarro-Tiep Condition on fields of values

Navarro and Tiep noticed that the following holds often:

Navarro-Tiep's Condition D

There exists a bijection $f : \mathrm{Irr}_{p'}(G) \longrightarrow \mathrm{Irr}_{p'}(\mathbf{N}_G(P))$ such that $\mathbb{Q}(\chi_P) = \mathbb{Q}(f(\chi)_P)$ for every $\chi \in \mathrm{Irr}_{p'}(G)$.

Sambale discovered that PrimitiveGroup(64,38) is a counterexample when $p = 3$. This is a solvable group with abelian Sylow p-subgroups.

Note that a consequence of the picky conjecture is the following.

Corollary

If the global picky conjecture holds for G then there exists a bijection $f : \mathrm{Irr}^{\mathcal{P}}(G) \longrightarrow \mathrm{Irr}^{\mathcal{P}}(\mathbf{N}_G(P))$ such that $\mathbb{Q}(\chi_{\mathcal{P}}) = \mathbb{Q}(f(\chi)_{\mathcal{P}})$ for every $\chi \in \mathrm{Irr}^{\mathcal{P}}(G)$.

PrimitiveGroup(64,38) has picky elements for $p = 3$ and the picky conjecture holds for this group.

The subnormalizer

So far I have only talked about how to compute the values of characters at picky elements. What happens with all other elements $x \in G$? The key is to replace $\mathbf{N}_G(P)$ by the subnormalizer of x.

But what is the subnormalizer?

The subnormalizer

Around 1990, Casolo wrote several papers using the following definition of subnormalizer.

Definition

For $H \leq G$, the subnormalizer of H is

$$S_G(H) = \{g \in G \mid H \triangleleft\triangleleft \langle H, g \rangle\}.$$

This set does not need to be a subgroup.

We define the **subnormalizer (subgroup)** of $x \in G$ to be $\mathbf{Sub}_G(x) = \langle S_G(\langle x \rangle) \rangle$.

The subnormalizer

From one of Casolo's papers, from 1990, on the subnormalizer subset:

> My purpose is to prove some results which I hope are at the beginning of a study of the subsets $S_G(H)$ in a finite group G. M

Lemma

Let $x \in P$ be a p-element. Then:

1. $\mathbf{N}_G(P) \subseteq \mathbf{Sub}_G(x)$.
2. $\mathbf{N}_G(P) = \mathbf{Sub}_G(x)$ if and only if x is picky.

The subnormalizer conjecture for p-elements

Conjecture (Subnormalizer conjecture)

Let $x \in G$ p-element. Then there exists a bijection $f : \mathrm{Irr}^x(G) \longrightarrow \mathrm{Irr}^x(\mathbf{Sub}_G(x))$ such that for every $chi \in \mathrm{Irr}^x(G)$

1. $\chi(1)_p = f(\chi)(1)_p$;
2. $\mathbb{Q}(\chi(x)) = \mathbb{Q}(f(\chi)(x))$;
3. $\chi(x)_p = f(\chi)(x)_p$.

If, in addition, $\chi(x) = \pm f(\chi)(x)$, we say that (G, x) satisfies the *strong* subnormalizer conjecture.

Symmetric groups

As in the picky case, we expect the strong subnormalizer conjecture to hold for symmetric groups.

Martínez Madrid is working on this. One case that he has proved is when $G = S_{2^n}$ and x is a 2^{n-1}-cycle. In this case, $\mathbf{Sub}_G(x) = S_{2^{n-1}} \wr C_2$.

For $n = 4$, the list of values at x and 2-parts of the degrees for the characters in $\mathrm{Irr}^x(S_{16})$ is the following:

Symmetric groups

[[-90, 4], [-90, 4], [-90, 4], [-90, 4], [-70, 4], [-70, 4], [-70, 4], [-70, 4], [-70, 4], [-70, 4], [-70, 4], [-70, 4], [-64, 128], [-64, 128], [-64, 128], [-64, 128], [-64, 128], [-64, 128], [-64, 128], [-64, 128], [-56, 2], [-56, 16], [-56, 16], [-56, 16], [-56, 16], [-56, 16], [-56, 16], [-56, 16], [-56, 16], [-42, 2], [-42, 4], [-42, 4], [-42, 4], [-42, 4], [-36, 2], [-35, 1], [-35, 1], [-34, 2], [-28, 2], [-28, 2], [-28, 8], [-28, 8], [-28, 8], [-28, 8], [-28, 8], [-28, 8], [-28, 8], [-28, 8], [-22, 2], [-21, 1], [-21, 1], [-20, 2], [-20, 8], [-20, 8], [-20, 8], [-20, 8], [-20, 8], [-20, 8], [-20, 8], [-20, 8], [-14, 2], [-14, 2], [-14, 4], [-14, 4], [-14, 4], [-14, 4], [-14, 4], [-14, 4], [-14, 4], [-14, 4], [-8, 2], [-7, 1], [-7, 1], [-6, 2], [-1, 1], [-1, 1], [1, 1], [1, 1], [6, 2], [7, 1], [7, 1], [8, 2], [14, 2], [14, 2], [14, 4], [14, 4], [14, 4], [14, 4], [14, 4], [14, 4], [14, 4], [14, 4], [20, 2], [20, 8], [20, 8], [20, 8], [20, 8], [20, 8], [20, 8], [20, 8], [20, 8], [21, 1], [21, 1], [22, 2], [28, 2], [28, 2], [28, 8], [28, 8], [28, 8], [28, 8], [28, 8], [28, 8], [28, 8], [28, 8], [34, 2], [35, 1], [35, 1], [36, 2], [42, 2], [42, 4], [42, 4], [42, 4], [42, 4], [56, 2], [56, 16], [56, 16], [56, 16], [56, 16], [56, 16], [56, 16], [56, 16], [56, 16], [64, 128], [64, 128], [64, 128], [64, 128], [64, 128], [64, 128], [64, 128], [64, 128], [70, 4], [70, 4], [70, 4], [70, 4], [70, 4], [70, 4], [70

Non-prime-power order elements

Subnormalizers of non-prime-power order seem to be **very** small subgroups.

For instance, if x is an element of order 15 in A_9, then the following is the character table of the subnormalizer of x.

Non-prime-power order elements

```
gap> T:=CharacterTable(S);
CharacterTable( C15 : C4 )
gap> Display(T);
CT7

     2  2  2  1  2  2  2  1   .   .   .
     3  1  1  .  .  1  1  1   1   1
     5  1  1  .  .  .  .  1   1   1

       1a 3a 4a 4b 2a 6a 5a 15a 15b
    2P 1a 3a 2a 2a 1a 3a 5a 15a 15b
    3P 1a 1a 4b 4a 2a 2a 5a  5a  5a
    5P 1a 3a 4a 4b 2a 6a 1a  3a  3a
    7P 1a 3a 4b 4a 2a 6a 5a 15b 15a
   11P 1a 3a 4b 4a 2a 6a 5a 15b 15a
   13P 1a 3a 4a 4b 2a 6a 5a 15b 15a

X.1     1  1  1  1  1  1  1   1   1
X.2     1  1 -1 -1  1  1  1   1   1
X.3     1  1  A -A -1 -1  1   1   1
X.4     1  1 -A  A -1 -1  1   1   1
X.5     2 -1  .  . -2  1  2  -1  -1
X.6     2 -1  .  .  2 -1  2  -1  -1
X.7     4  4  .  .  .  . -1  -1  -1
X.8     4 -2  .  .  .  . -1   B  /B
X.9     4 -2  .  .  .  . -1  /B   B

A = -E(4)
  = -Sqrt(-1) = -i
B = E(15)^7+E(15)^11+E(15)^13+E(15)^14
  = (1-Sqrt(-15))/2 = -b15
```

Non-prime-power order elements

There is also a subnormalizer conjecture for elements that do not have prime power order, but I will not go into details. I will only discuss one particular case, related to a problem Gabriel was interested in.

Sets of primes

Let π be a set of primes. For π-separable groups it is known that

$$|\mathrm{Irr}_{\pi'}(G)| = |\mathrm{Irr}_{\pi'}(\mathbf{N}_G(H))|,$$

where H is a Hall π-subgroup. It was believed that this could be true for arbitrary G with nilpotent Hall π-subgroups.

Finally, Navarro and Tiep discovered that $G = J_4$ for $\pi = \{5, 7\}$ is a counterexample:

$$|\mathrm{Irr}_{\pi'}(G)| = 30 \neq 25 = |\mathrm{Irr}_{\pi'}(\mathbf{N}_G(H))|.$$

Sets of primes

One particular case of the subnormalizer conjecture for non-prime-power order elements is the following.

Let $H \leq G$. Let $x \in H$. We say that x is H-picky if $x \in H^g$ implies $g \in \mathbf{N}_G(H)$.

Conjecture (Picky conjecture for nilpotent Hall subgroups)

Let π be a set of primes. Let G be a group with a nilpotent Hall π-subgroup H. Suppose that $x \in H$ is H-picky. Then there exists $f : \mathrm{Irr}^x(G) \longrightarrow \mathrm{Irr}^x(\mathbf{N}_G(H))$ bijection such that for every $\chi \in \mathrm{Irr}^x(G)$

1. $\chi(1)_\pi = f(\chi)(1)_\pi$;
2. $\mathbb{Q}(\chi(x)) = \mathbb{Q}(f(\chi)(x))$.

As usual, in many cases, we expect $\chi(x) = \pm f(\chi)(x)$. For instance if H is abelian.

Sets of primes

When $G = J_4$ and $\pi = \{5, 7\}$, the Hall π-subgroups are cyclic of order 35. Let $H = \langle x \rangle$ be a Hall π-subgroup. Note that x is H-picky.

It turns out that the non-zero values of the characters of G and $\mathbf{N}_G(H)$ at x are as follows:

1. 15 times ± 1;
2. 5 times $\pm(-1 + \sqrt{-7})/2$;
3. 5 times $\pm(-1 - \sqrt{-7})/2$.

All of them have π'-degree.

The reason why this "fixes" the McKay conjecture for sets of primes is that $\mathrm{Irr}_{\pi'}(G) \not\subseteq \mathrm{Irr}^x(G)$. In this example there are 5 characters of π'-degree that vanish at x.

Concluding remarks

There are stronger versions of these conjectures that we have not announced yet. We need time to write a readable version of this. But then, we think, there will be many exciting open problems to work on for many years. We would be surprised to see a reduction of these conjectures to questions on simple groups. Going back to the techniques used before the classification was completed could be useful.

Many thanks!!