Decomposing the Truncated Conjugation Module

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Gabriel Navarro Birthday Meeting

Valencia

January 31,2025

Gabriel Navarro is a long-standing collaborator and friend. It is a privilege to be present at this meeting to mark his significant birthday. I hope to appeal to his well-known enthusiasm for character theory in this talk, and possibly to ignite the interest of others in some of the questions raised. We will discuss some conjectures involving block theory and character theory.

A consequence of the conjectures (if true) for Brauer character tables

Let G be a finite group which satisfies Conjecture A (to follow) for the prime p. Let Φ denote the matrix obtained from the Brauer Character table of G. Then the matrix

$$\Phi\overline{\Phi}^t = M$$

may be written in the form

$$M = \sum_{y \in G_{p'/G}} M_y,$$

where each M_y in an $\ell \times \ell$ symmetric positive semi-definite matrix with non-negative integer entries, and the rank of the matrix M_y is the number of p-regular conjugacy classes of G which have non-empty intersection with $C_G(y)$ (in particular, M_1 has full rank).

A generalized character

Let G be any finite group, and let p be a prime. We let G_p denote the set of elements of G whose orders are a power of p, and $G_{p'}$ denote the set of elements of G whose orders are coprime to p. We define an integer-valued class function $\Psi_{1,p,G}$ of G as follows: we set $\Psi_{1,p,G}(x) = 0$ whenever $x \in G$ has order divisible by p, and we set $\Psi_{1,p,G}(x) = |G_p \cap C_G(x)|$ whenever $x \in G_{p'}$.

Then $\Psi_{1,p,G}$ is always a generalized character of G. One way to see this is via Brauer's characterization of characters and a theorem of Frobenius. Another is via Adams operations and higher indicators: if we choose a power q of p such that $|G|_p$ divides q and $q \equiv 1 \pmod{|G|_{p'}}$, and we define the higher indicator $\nu_q(\chi)$ for each complex irreducible character χ of G via

$$\nu_q(\chi) = \frac{1}{|G|}\left(\sum_{g \in G} \chi(g^q)\right),$$

then each $\nu_q(\chi)$ is an integer and it is easy to check that

$$\Psi_{1,p,G} = \sum_{\chi \in \mathrm{Irr}(G)} \nu_q(\chi)\chi.$$

In other words, for this value of q, the integer $\Psi_{1,p,G}(x)$ is the number of q-th roots of x in G for each $x \in G$. We note also that for this value of q, we have $\chi(x^q) = \chi(x_{p'})$ for each $x \in G$.

Notice that each irreducible character μ of G occurs with multiplicity between $1 - \mu(1)$ and $\mu(1)$ in $\Psi_{1,p,G}$, and that μ occurs with multiplicity $\mu(1)$ if and only if $O^p(G) \leq \ker\mu$. Hence linear characters all occur with non-negative multiplicity.

The Truncated Conjugation Module

Let $(\mathbb{K}, R, \mathbb{F})$ be a p-modular system for the finite group G.

The *Truncated Conjugation Character* (with respect to p) for G is the class function $\Lambda_{c,p,G}$ which takes value 0 on all p-singular elements of G, and takes value $|C_G(x)|$ at each p-regular $x \in G$. Block orthogonality relations tell us that

$$\Lambda_{c,p,G} = \sum_{\phi \in \mathrm{IBr}_p(G)} \overline{\phi}\theta_\phi,$$

where θ_ϕ is the character of the projective cover (as RG-module) of the simple $\mathbb{F}G$-module affording Brauer character ϕ.

Hence $\Lambda_{c,p,G}$ agrees on p-regular elements with the Brauer character of the projective $\mathbb{F}G$-module

$$\bigoplus_{S} \mathrm{Hom}_{\mathbb{F}}(S, P(S)),$$

where S runs through the isomorphism types of simple $\mathbb{F}G$-modules, and $P(S)$ denotes the projective cover of S (as $\mathbb{F}G$-module). Since projective $\mathbb{F}G$-modules lift to projective RG-modules, it follows that $\Lambda_{c,p,G}$ is indeed a character of G, and is afforded by a projective RG-module, which we call the *Truncated Conjugation Module for* G, and denote by $T_{c,p,G}$.

A Conjecture in Three Equivalent Forms

We first propose :

Conjecture A: *The generalized character* $\Psi_{1,p,G}$ *is a character of* G*, and may be afforded by a projective* RG*-module* $P_{1,p,G}$.

Next, we propose:

Conjecture B: *Let $\{\phi_i : 1 \leq i \leq \ell\}$ be the Brauer characters of the absolutely irreducible $\mathbb{F}G$-modules. For each i, let ϕ_i^* be the unique extension of ϕ_i to a class function of G constant on p'-sections, and let $\langle , \rangle$ be the usual sesquilinear form on complex-valued class functions of G. (It is well-known that each ϕ_i^* is a generalized character of G). Then $\langle \phi_i^*, \phi_j^* \rangle$ is a non-negative integer for all $i, j \leq \ell$.*

Finally we propose:

Conjecture C: *The Truncated Conjugation Module $T_{c,p,G}$ of RG, is expressible in the form*

$$T_{c,p,G} \cong \bigoplus_{y \in G_{p'}/G} \mathrm{Ind}_{C_G(y)}^{G}(P_{1,p,C_G(y)}),$$

and each $P_{1,p,C_G(y)}$ is a projective $RC_G(y)$-module.

Remark: The equivalence of Conjectures A and C is routine. Also, it is easy to check that $\langle \phi_i^*, \phi_j^* \rangle = \langle \phi_i \Psi_{1,p,G}, \phi_j \rangle$ for all $i, j \leq \ell$. If Conjecture B holds, then taking ϕ_1 to be the trivial Brauer character and allowing j to vary shows that $\langle \Psi_{1,p,G}, \phi_j \rangle \geq 0$ for $1 \leq j \leq \ell$, so that $\Psi_{1,p,G}$ is a non-negative integer combination of characters of projective indecomposable RG-modules and Conjecture A holds. On the other hand, if Conjecture A holds, then we have

$$\langle \phi_i^*, \phi_j^* \rangle = \langle \Psi_{1,p,G}, \overline{\phi_i} \phi_j \rangle \geq 0$$

for all i, j as the product of Brauer characters is a Brauer character, so Conjecture B holds. Hence Conjectures A and B are equivalent.

Some Special Cases and some Partial Cases

If G is a p'-group, then $\Psi_{1,p,G}$ is the trivial character and the trivial RG-module is projective, so Conjecture A holds for G. If G is a p-group, then $\Psi_{1,p,G}$ is the regular character, and the regular RG-module is projective, so Conjecture A holds for G.

If Conjecture A holds for $G/O_p(G)$, then Conjecture A holds for G. If $\Psi_{1,p,G/O_p(G)}$ is afforded by a projective $RG/O_p(G)$-module $P_{1,p,G/O_p(G)}$ then $\Psi_{1,p,G}$ is afforded by the projective cover of $P_{1,p,G/O_p(G)}$ as RG-module, so $P_{1,p.G}$ is that projective cover. In particular, we have:

Lemma 1: *If G has a normal Sylow p-subgroup, then Conjecture A holds for G and $P_{1,p,G}$ is the projective cover of the trivial RG-module.*

Lemma 2: *If G has a normal p-complement, then Conjecture A holds for G.*

Proof: We know that $\Psi_{1,p,G}$ is a virtual projective. Hence it suffices to prove that

$$\langle \Psi_{1,p,G}, \phi_i \rangle \geq 0$$

for each Brauer irreducible ϕ_i of G. But

$$\langle \Psi_{1,p,G}, \phi_i \rangle = \sum_{x \in G_p/G} \frac{1}{|C_G(x)|_p} \left(\langle 1, \mathrm{Res}^G_{O_{p'}(C_G(x))}(\phi_i) \rangle \right)$$

which is certainly non-negative, because ϕ_i restricts to a character on p'-subgroups.

In fact, using Brauer's Second and Third Main Theorems, we may prove:

Lemma 3: *If $C_G(x)$ has a normal p-complement for each non-identity p-element $x \in G$, then $\Psi_{1,p,G}$ is a character of G. If, in addition, p is odd, then Conjecture A holds for G.*

Remark: Notice that Conjecture A holds for G if

$$G = O_{p,p',p}(G).$$

To prove Conjecture A for p-solvable G, it is sufficient, by Fong theory, to prove that $\Psi_{1,p,G}$ is a character of G, but this seems to be open at present.

Lemma 4: *If G is a finite group with a characteristic p BN-pair, then $\Psi_{1,p,G}$ is a character afforded by the projective module*

$$\mathrm{St}_p(G) \otimes \mathrm{St}_p(G),$$

where St_p denotes the Steinberg module.

Proof: This follows immediately from general properties of the Steinberg character. For we have

$$\Psi_{1,p,G}(y) = |C_G(y)|_p^2$$

for every p-regular element $y \in G$.

An easy consequence of this viewpoint is the following version of a Theorem of G. Lusztig :

Corollary 5: *Let G be as in Lemma 4, and $\{\phi_i : 1 \leq i \leq \ell\}$ be the irreducible Brauer characters of G and let χ_s denote the Steinberg character of G. Then $\{\chi_s\phi_i : 1 \leq i \leq \ell\}$ is* *a $\mathbb{Z}$-basis (consisting of characters) for the $\mathbb{Z}$-module of generalized characters of G vanishing on all p-singular elements of G.*

Theorem (T. Scharf): *If $G \cong S_n$ for n a positive integer, then $\Psi_{1,p,G}$ is a character of G for each prime p.*

Remark : Scharf proved a much more general theorem for S_n than this particular case, but the more general version of the Theorem does not hold for all finite groups.

The Strongly p-Embedded Subgroup Case

We outline in some detail the proof of the following:

Lemma 6: *Let G be a finite group with a proper strongly p-embedded subgroup H. Then if Conjecture A holds for H, it also holds for G.*

Proof: We work over $\mathbb{F}$ for convenience. Suppose that $\Psi_{1,p,H}$ is a character and is afforded by a projective RH-module $P_{1,p,H}$ which is the lift of a projective $\mathbb{F}H$-module Q. Then $\Psi_{1,p,H}$ agrees with

$$\sum_{x \in H_p/H} \mathrm{Ind}_{C_H(x)}^{H}(1)$$

on p-regular elements of H, and vanishes elsewhere, and similarly for G. Since H is strongly p-embedded in G, we see that

$$\mathrm{Ind}_H^G(\Psi_{1,p,H} - 1) = \Psi_{1,p,G} - 1.$$

It follows that in the Green ring for $\mathbb{F}G$. we have

$$\text{Ind}_H^G(Q) - M = N,$$

where M is a projective complement to the trivial $\mathbb{F}G$-module in $\text{Ind}_H^G(\mathbb{F})$ and N is a virtual projective $\mathbb{F}G$ module lifting to the virtual projective RG-module $P_{1,p,G}$. Now the projective cover of the trivial $\mathbb{F}H$-module is a summand of Q with multiplicity one. Hence $\text{Ind}_H^G(Q)$ has a submodule isomorphic to $\mathbb{F} \oplus M$, so since M is projective, M is isomorphic to a direct summand of $\text{Ind}_H^G(Q)$ and N is a projective $\mathbb{F}G$-module. Hence $P_{1,p,G}$ is a projective RG-module, and Conjecture A holds for G, as required.

Corollary 7: *Let G be a finite group with a TI-Sylow p-subgroup S , and let $H = N_G(S)$ and T be a Hall p'-subgroup of H. Then*

$$\Psi_{1,p,G} = 1 + \text{Ind}_T^G(1) - \text{Ind}_H^G(1)$$

which is a character afforded by a projective RG-module.

Proof: Use the proof of Lemma 6, and the fact that $\Psi_{1,p,H}$ is afforded by the projective cover of the trivial module, which is $\mathrm{Ind}_T^H(R)$. Then $\Psi_{1,p,G}$ is as stated.

Using similar methods, we may prove:

Theorem 8: *Conjecture A holds for G if G has a cyclic Sylow p-subgroup, or if the Sylow p-subgroup of G is a Klein 4-group or a quaternion group of order 8. Hence Conjecture A holds for all prime divisors of $|G|$ if $G \cong \mathrm{PSL}(2,\mathrm{q})$ with $q \not\equiv \pm 1 \pmod 8$.*

The general case

The strongly p-embedded case uses the fact that the virtual module $P_{1,p,G} - R$ is visibly p-local in that case. In general, we can prove that $P_{1,p,G} - R$ is p-local. This allows an inductive procedure to construct $P_{1,p,G}$ for an arbitrary finite group G. Loosely speaking, we have seen how to construct $P_{1,p,G}$ using projective covers when $O_p(G) \neq 1$. The fact that $P_{1,p,G} - R$ is p-local means we can construct it by taking signed sums of modules induced from known modules of p-local (proper) subgroups of G when $O_p(G) = 1$.

Using a Möbius inversion argument and the Steinberg (virtual) module for general finite groups, as considered by P.J. Webb, we may obtain an explicit formula (as virtual projective module) for $P_{1,p,G}$ in the Green ring for RG.

The Steinberg (virtual) module in the Green ring for RG was defined (up to sign) by Webb as

$$\mathsf{St}_p(G) = \sum_{\sigma \in \mathcal{N}_p(G)/G} (-1)^{|\sigma|} \mathsf{Ind}_{G_\sigma}^{G}(R),$$

where $|\sigma|$ denotes the number of non-identity (p)-subgroups in the chain σ (we include the empty chain). This can be considered as the "non-p-local" part of the trivial module R.

Then we may see by an inversion argument that (in the Green ring for RG), the virtual module

$$P_{1,p,G} = \sum_{Q/G} \mathsf{Ind}_{N_G(Q)}^{G} \left(P[\mathsf{St}_p(N_G(Q)/Q)]\right),$$

where $P[]$ denotes that we are taking the projective cover of a given (virtual) projective $N_G(Q)/Q$-module when considered as (virtual) $N_G(Q)$-module. We include the case $Q = 1$ in this sum.

In conjunction with Webb's inversion formula for St_p, we may write this (in the Green ring) as

$$(P_{1,p,G} - R) =$$

$$\sum_{1 \neq Q/G} \mathsf{Ind}_{N_G(Q)}^{G} \left(P[\mathsf{St}_p(N_G(Q)/Q)] - \mathsf{St}_p(N_G(Q)/Q)\right)$$

Remark: We consider $P_{1,p,G}$ as an analogue for general finite groups of the endomorphism ring of the Steinberg module for finite groups with a characteristic p BN-pair.