Other Invariants

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Characters and blocks of finite groups

Conference in honor of Gabriel Navarro's 60-th birthday

València, Spain

Per molts anys, Gabriel!

Happy Birthday, Gabriel!

I met Gabriel at an international conference in Manchester in 1988.

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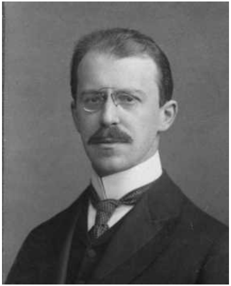
Gabriel Navarro in 1988



Issai Schur 1875–1941

Richard Brauer 1901–1977

John Thompson and Walter Feit circa 1960

John Thompson and Walter Feit at Turull's house 2003

McKay Conjecture (1972)

Let G be any finite simple group, $p = 2$, let P be a Sylow p-subgroup of G. Then, there is a bijection

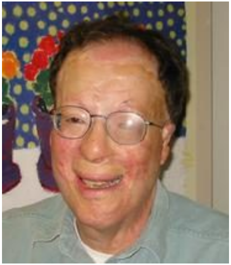
$$f : \mathrm{Irr}_{p'}(G) \to \mathrm{Irr}_{p'}(\mathrm{N}_G(P)).$$

Theorem (Ferguson – Turull (1985))

Let G be a finite group and let χ be some quasi-primitive non linear irreducible character of G. Then in some central extension of G there exist an admissible set of prime characters $\pi_1, \pi_2, \ldots, \pi_n$ such that

$$\chi = \pi_1 \cdots \pi_n$$

and the $\pi_1, \pi_2, \ldots, \pi_n$ are basically unique.



Marty Isaacs suggested that I consider Schur indices

Theorem (Turull (1992))

Describes the Schur index of each irreducible character of each double cover of S_n or A_n.

This question was open since Schur asked it in 1927.

When Gabriel heard my talk about the previous theorem in Berkeley, he said to me:

Sandre, parles de coses molt estranyes.

Alex, you talk about strange things.

At another time, Gabriel encouraged me to better our understanding of Clifford theory.

This combined with my interest in small fields lead me to study Clifford theory for small fields and to obtain as a result some reduction theorems for certain conjectures.

Schur indices

Let G be a finite group;

Let $\chi \in \mathrm{Irr}(G)$;

Let K be a field in characteristic 0 such that $K(\chi) = K$.

Then the Schur index of χ over K is $m_K(\chi)$ the smallest positive integer such that some KG-module affords the character $m_K(\chi)\chi$.

The Schur index is invariant under Galois conjugation:

Let $\sigma \in \mathrm{Gal}(K/\mathbf{Q})$ then

$$m_K(\sigma\chi) = m_K(\chi).$$

Brauer groups

Let $\mathrm{Br}(K)$ denote the Brauer group of K.

We can associate to χ an element of $\mathrm{Br}(K)$ as follows:

Let M be a KG-module affording $n\chi$ for some positive integer n.

Then $\mathrm{End}_{KG}(M)$ is a central simple algebra over K and its class in $\mathrm{Br}(K)$ does not depend on the choice of M. We set

$[[\chi]]_K$ to be the class of $\mathrm{End}_{KG}(M)$.

The Schur index $m_K(\chi)$ is the order of $[[\chi]]_K$ in $\mathrm{Br}(K)$.

However the element is not necessarily invariant under Galois conjugation:

There exist χ and $\sigma \in \mathrm{Gal}(K/\mathbf{Q})$ such that

$$[[\sigma\chi]]_K \neq [[\chi]]_K.$$

Invariants related to p

Some invariants

- Let p be any prime.
- Let G be any finite group.
- Then G has a character table.
- and G has a p-Brauer character table.

Theorem

Suppose that p-Brauer characters for finite groups are "defined" then the characters of finite groups "have" values in $\overline{\mathbf{Q}_p}$ *the algebraic closure of the field of p-adic numbers.*

p-adic numbers

- Let $\mathbf{Q}_p$ be the field of p-adic numbers.
- We fix an algebraic closure $\overline{\mathbf{Q}_p}$ of $\mathbf{Q}_p$.
- Then because p-Brauer characters are defined, every character and every p-Brauer character has values in $\overline{\mathbf{Q}_p}$.

For convenience, we assume that all finite groups take their characters and p-Brauer characters to have values in $\overline{\mathbf{Q}_p}$.

Hasse invariants

- Let K be a finite extension field of $\mathbf{Q}_p$.
- There is a group isomorphism

$$\mathrm{inv}_K\colon \mathrm{Br}(K) \to \mathbf{Q}/\mathbf{Z}$$

The values $\mathrm{inv}_K(A)$ is called the Hasse invariant of A.

- The order of $\mathrm{inv}_K(A)$ is the Schur index of A.

Hasse invariants and characters

- Let K be a finite extension field of $\mathbf{Q}_p$.
- For each $\chi \in \mathrm{Irr}_K(G)$, we have $\mathrm{inv}_K(\chi) \in \mathbf{Q}/\mathbf{Z}$.
- We also have $m_K(\chi)$ the p-local Schur index of χ.

Galois action and invariants

- Let σ be a Galois automorphism and $\chi \in \mathrm{Irr}(G)$. Then $\sigma\chi$ and χ have the same Schur index, but not necessarily the same Hasse invariant.
- If B is a p-block, $\sigma(B)$ may be different from B.
- If λ is a linear character $\sigma\lambda$ may be different from λ.

Dade et al.'s Refined Projective Conjecture

Let p be any prime, let G be any finite group, let $Z \leqslant \mathrm{Z}(G)$ be a p-subgroup, let $\lambda \in \mathrm{Irr}(Z)$, B be a p-block of G, and let d be a non-negative integer. Let $r \in \{1, \ldots, p-1\}$, let $F \subseteq \overline{\mathbf{Q}_p}$ be a finite Galois extension of $\mathbf{Q}_p$, and let $s \in \mathrm{Br}(F)$. Assume that Z is not a defect group of B. Then

$$\sum_{C \in \mathcal{N}(G,Z)/G} (-1)^{|C|} k(\mathrm{N}_G(C), B, \lambda, d, r, F, s) = 0.$$

Note on the last two variables:

F fixes the field of values of the character to be F (giving the "Galois" or "Navarro" version of the Conjecture).

s fixes the element of $\mathrm{Br}(F)$ for the character, or alternatively, it fixes the value of inv_F on the character.

Theorem (Turull (2017))

The above conjecture is true whenever G is p-solvable.

Weaker versions of this conjecture for p-solvable groups were known at the time, notably the one by Geoffrey Robinson (2000).

Blocks with cyclic defect group

Let B be a p-block with non trivial cyclic defect D. Say $|D| = p^a > 1$.

Everett C. Dade (1966) described all the ordinary and modular irreducible characters in B. There is a $e \mid p-1$. $\mathrm{Irr}(B)$ has exactly $e + (p^a - 1)/e$ ordinary irreducible characters divided into e non-exceptional characters and $(p^a - 1)/e$ exceptional characters.

There is also the Brauer tree of B. The vertices of the tree correspond roughly to the irreducible characters of B and the edges correspond to the non exceptional irreducible characters of B.

The p-local Schur index for p-blocks with cyclic defect:

Theorem (Benard (1976))

Let $\chi \in \mathrm{Irr}(B)$ *be exceptional and* $\phi \in \mathrm{Irr}(B)$ *be non exceptional. Then*

$$m_{\mathbf{Q}_p(\chi)}(\chi) = [\mathbf{Q}_p(\chi, \phi) : \mathbf{Q}_p(\chi)] .$$

and

$$m_{\mathbf{Q}_p(\phi)}(\phi) = 1.$$

The Hasse invariant for p-blocks with cyclic defect:

Theorem (Nebe (2005))

Let $\chi \in \mathrm{Irr}(B)$. For χ non exceptional it follows from Benard that the invariant is

$$\mathrm{inv}_{\mathbf{Q}_p(\chi)}(\chi) = \mathbf{Z} \in \mathbf{Q}/\mathbf{Z}.$$

For χ non exceptional. The Hasse invariant $\mathrm{inv}_{\mathbf{Q}_p(\chi)}(\chi)$ can be calculated as follows. First compute the action of a certain element of a Galois group on the Brauer tree of B. Second calculate the action of certain permutation δ associated with the reduction modulo p of the characters. She then gives a formula for $\mathrm{inv}_{\mathbf{Q}_p(\chi)}(\chi)$ on the basis of both actions.

Lemma

One can reduce this calculation to the case when $D \trianglelefteq G$.

Use the Brauer correspondence.

Two linear characters

Let G be a finite and let B a p-block of G with defect group D, where D is cyclic and non trivial.

For simplicity we assume that $D \trianglelefteq G$.

$|\mathrm{IBr}(B)| \mid p - 1$.

There is a unique subgroup E with $\mathrm{C}_G(D) \subseteq E \subseteq G$ such that $[E : \mathrm{C}_G(D)] = |\mathrm{IBr}(B)|$.

Action character

$E/\operatorname{C}_G(D)$ acts faithfully on D.

For each $g \in E$ there is some $n \in \mathbf{Z}$ prime to p such that, for all $d \in D$, we have $d^g = d^n$.

We let $\epsilon_n \in \mathbf{Q}_p$ be the unique $(p-1)$-th root of 1 corresponding to n.

We let $\alpha_B(g) = \epsilon_n$. This gives us the action character α_B of B with values in $\mathbf{Q}_p$. $\alpha_B \in \operatorname{Irr}(E/\operatorname{C}_G(D))$.

Inertial character

Let K be a finite extension of $\mathbf{Q}_p$ such that $F(B) \subseteq K$.

Let $\chi \in \mathrm{Irr}(B)$ be non exceptional. Set $L = K(\chi)$.

Then L/K is an unramified extension. Let $\sigma \in \mathrm{Gal}(L/K)$ be the Frobenius element.

Then the inertial character of B with respect to K is $\mu_{B,K} \in \operatorname{Irr}(E/\operatorname{C}_G(D))$.

It satisfies

$$\sigma\chi = \mu_{B,K}\chi.$$

New formula for Hasse invariants

Since α_B is faithful there exists some $a \in \mathbf{Z}$ such that $\alpha_B^a = \mu_{B,K}$.

Then if $\chi \in \mathrm{Irr}(B)$ is exceptional and $e = \left|\mathrm{IBr}(B)\right|$ then

$$\mathrm{inv}_K(\chi) = \frac{a}{e} + \mathbf{Z} \in \mathbf{Q}/\mathbf{Z}.$$

If χ is not exceptional then

$$\mathrm{inv}_K(\chi) = \mathbf{Z} \in \mathbf{Q}/\mathbf{Z}$$

Since α_B is faithful, there exists some $a \in \mathbf{Z}$ such that $\mu_{B,K} = \alpha_B^a$, and we fix any such a. Let f be the ramification index of $K/\mathbf{Q}_p$. Let $\chi \in \mathrm{Irr}(B)$ be such that $K = K(\chi)$. Then $\mathrm{inv}_K(\chi) \in \mathbf{Q}/\mathbf{Z}$ and

$$\mathrm{inv}_K(\chi) = \begin{cases} \frac{fa}{p-1} + \mathbf{Z} = \frac{ef}{p-1}\frac{a}{e} + \mathbf{Z} & \text{if } \frac{ef}{p-1} \in \mathbf{Z} \\ \\ \mathbf{Z} & \text{if } \frac{ef}{p-1} \notin \mathbf{Z} \end{cases}$$

Conclusion

We introduced two linear characters α_B and μ_B associated with B. We have $\alpha_B, \mu_B \in \mathrm{Irr}(E)$. The order of μ_B is the p-local Schur index of all the $\chi \in \mathrm{Irr}(B)$, giving an alternative description of the p-local Schur indices calculated by Benard (1976). The two characters α_B and μ_B together give us a descriptions of the invariants of each $\chi \in \mathrm{Irr}(B)$, giving an alternative description to the one given by Nebe (2005).