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Mathematics and soft computing in music

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22.1 Introduction

Nowadays, emphasizing the strong relationship between music and mathematics seems unnecessary. Mathematics is the fundamental tool for dealing with the physical processes that explain music but it is also in the very essence of this art. How to choose the musical notes, the tonalities, the tempos and even some methods of composition is pure mathematics. In the sixth century B.C. the Pythagoreans completed and transmitted the Chaldean practice of selecting musical notes from the proportions between tight strings. They created a link between music and mathematics which has still not been broken. An example of this relationship is the use, sometimes intuitive, of the golden ratio in the sonatas of Mozart, Beethoven's Fifth Symphony or more recently in pieces by Bartok, Messiaen and Stockhausen. Besides, mathematicians throughout time have made music their object of study and now, both in musical and mathematical journals and in the Internet, many documents can be found in which mathematics is used in a practical way in the creation or analysis of musical pieces.

One question is to explore the common ground between these disciplines and quite another is the use that musicians make of the models and the solutions provided by mathematics. For example, the musical notes, the first elements which music works with, are defined for each tuning system as very specific frequencies, but the instrumentalist knows that a small change in these values does not have serious consequences. In fact, sometimes consensus is only reached if the entire orchestra alters the theoretical pitches. Does this mean that musicians must restrict their use of mathematics to the theoretical aspects? In our opinion, what happens is that musicians implicitly handle very complex mathematical processes involving some uncertainty in the concepts and this is better explained in terms of fuzzy logic (see [12]). Another example: why do two different orchestras and two directors offer such different versions of the same work? Our answer to this question is that a musical score is a very fuzzy system. Composers invent sound structures, they are creators. If they develop such musical ideas and notate them in a musical score, the expert re-creators (instrumentalists) interpret the notation and transform it into sound. The creator and the instrumentalist are not usually the same person and different types of uncertainty may be present at many stages of this process. One example of a fuzzy approach to music and art is provided by J.S. Bach, who did not prescribe the tempo, tuning or even the instrumentation of some of his compositions in the corresponding scores.

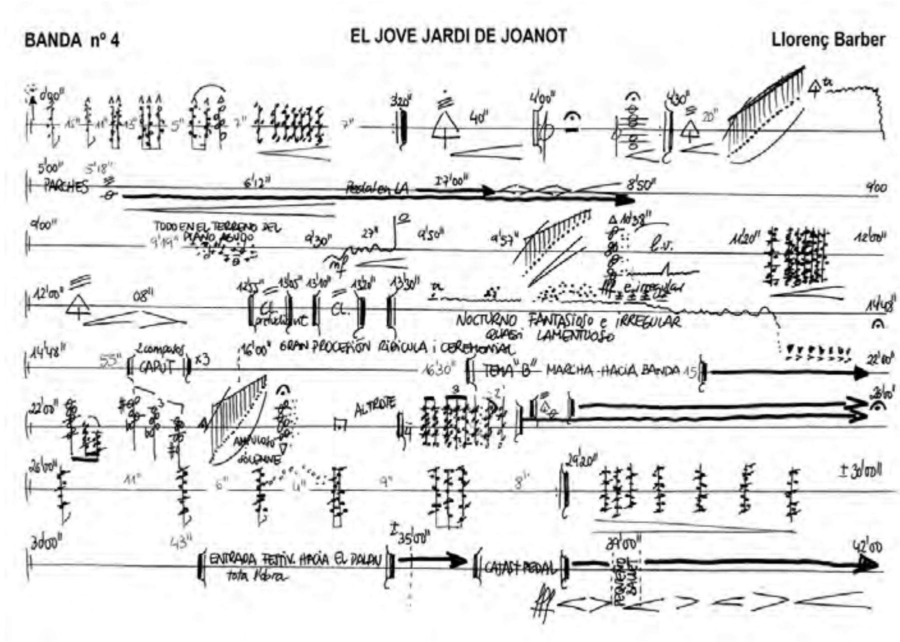

Fig. 22.1. Excerpt from "El jove jardí de Joanot" by Llorenç Barber.

El jove jardí de Joanot, a piece for dolçaina and drums by the Valencian composer Llorenç Barber (1948-), is an ideal example of the unconventional notation practices and performance expectations found in contemporary pieces. Figure 1 displays several bars from the piece in which the composer uses non-traditional notation. For instance, we can read the instruction "NOCTURNO FANTASIOSO E IRREGULAR, QUASI LAMENTUOSO".

Finally, we would like to cite a paper by J. Haluska [14] where a "type of uncertainty related to the creativity and psyche of the interpreter and listener of the composition" is mentioned. "Perhaps, the idea is best visible on Indian ragas when the same raga is played in different pitch systems depending on the mood, year season, occasion, place, etc."

If we perform a search in a database entering the keywords "music" and "fuzzy" we do not obtain so many retrievals as could be expected. This may be surprising due to the essentially subjective characteristics of music.

We do not intend to make an exhaustive review of the literature but to offer some insights into the use of soft computing techniques in music. In our search we find out that, in comparison to other areas, fuzzy logic has seldom been used in the artistic and creative fields. Only a few applications related to creative activities, such as musical composition and sound synthesis, have been reported in the literature. Some papers are related to the creative manipulation and transformation of digital sound. In [7], two applications of fuzzy logic in music are presented: a fuzzy logic-based mapping strategy for audiovisual composition and an audio synthesis technique based on sound particles and fuzzy logic. An extension of an existing performance system, capable of generating expressive musical performances based on case-based reasoning techniques (called SaxEx) is presented in [2] where the authors model the degree of different expressive parameters by means of fuzzy sets. Elsea [10] also applies concepts of fuzzy logic to problems in music analysis and composition.

Several digital audio processing applications based on fuzzy logic have been proposed. Some of them are mainly focused on technical questions such as digital signal restoration [8] or the design of adaptive filters in acoustic echo cancelation [6]. One of the most cited references describes an algorithm for the classification, search, and retrieval of audio files [21]. This paper presents a general paradigm and specific techniques for analyzing audio signals in a way that facilitates content-based retrieval. A recent paper [5] presents a soft computing procedure which automatically generates sequences of songs. Starting from a given seed song, a sequence is generated while listening to the music because the user can express his or her dislike for the song being played by pressing a skip button.

Some other references related to audio retrieval and soft computing techniques can be found in the literature (see for instance [16] and [18]), and other interesting references are reviewed in [22] and [7].

After this brief review of the literature, let us focus on our main concern. We are interested in a conceptual problem in music theory: What do we mean by a well-tuned note or passage? And we also deal with the apparently conflictive coexistence of multiple tuning systems in an orchestra.

In 1948, N. A. Garbuzov published a paper entitled “The zonal nature of the human aural perception”, where twelve bars of the aria from the suite in D by J. S. Bach played by three famous violinists: Oistrakh, Elman and Cimbalist were analyzed (see [15]). This article showed that, to an accuracy of five cents, most of the notes played by those violinists did not belong to the tuning system in which the musicians thought they were tuned, the twelve-tone equal temperament. Some of these notes correspond to other tuning systems but the rest of them did not belong to any system. However, when hearing the passage, the feeling was not only pleasant, but even persons endowed with a very sensitive ear rated their performance as well-tuned. Thus, a question arises that we believe deserves to be analyzed in greater detail. We could think that the result of Garbuzov’s experience was forced as it was performed with violins which have no frets and where the pitch depends largely on the instrumentalist. Certainly, such an experiment could not have been carried out with fixed tuning instruments. But what happens with most wind or fretless string instruments? Or maybe we should reformulate our question: What do we mean when we accept that a note or a passage is finely-tuned? This question has been and continues to be a matter for discussion among many researchers in Musical Acoustics. Actually in the late 20th century we can observe a revival of interest in tone systems among musicians and in the industry in connection with the development of computer music and the production of electronic musical instruments with computer control.

Different criteria have been used to select the sounds that music uses. A set containing these sounds (musical notes) is called a tuning system. Most of them have been obtained through mathematical arguments (see [3], [4], [11], [17]). The numerical nature of these systems facilitates their transmission and the manufacture of instruments. However, the harshness of the mathematical arguments relegated these tuning systems to theoretical studies while in practice musicians tuned in a more flexible way. Because of this, many musicians feel that the mathematical arguments that justify tuning systems are impractical.

Modelling the notes as fuzzy sets and extending the concept of tuning systems allows us to connect theory and practice, and understand how musicians work in real-life. The notes offered by a musician during a performance should be compatible with the theoretical ones, but not necessarily equal. A numerical experiment conducted with the help of a saxophonist illustrates our approach and also highlights the need for considering the sequential uncertainty previously studied by Garbuzov.

22.2 Some Concepts and Notation

The word "tone" is used with different meanings in music. In [1] we can read that a tone is "a sound of definite pitch and duration, as distinct from noise and from less definite phenomena, such as the violin portamento." In this dictionary we find that notes are "the signs with which music is written on a staff. In British usage the term also means the sound indicated by a note". A pure tone can be defined as the sound of only one frequency, such as that given by an electronic signal generator. The fundamental frequency of a tone has the greatest amplitude. The other frequencies are called overtones or harmonics and they determine the quality of the sound. Loudness is a physiological sensation. It mainly depends on sound pressure but also on the spectrum of harmonics and physical duration. Although timbre and loudness are very important, we are focusing on pitch. Pitch is a psychological concept depending on the frequency of the tone. A higher frequency is perceived as a higher pitch.

In music only a small choice of possible sounds is used and a tuning system is the system used to define which tones to use when playing music; these tones are the tuned notes.

We will identify each note with the frequency of its fundamental harmonic (the frequency that chromatic tuners measure). The usual way to relate two frequencies is through their ratio; this number is called the interval. Actually, some authors (see [13]) identify the note with its relative frequency to the frequency of a fundamental, fixed tone (conventionally, such a tone is usually taken as A=440 Hz)

It is well known that in the middle zone of the audible field, the "pitch sensation" changes somewhat according to the logarithm of the frequency, so the distance between two sounds whose frequencies are f_1 and f_2 can be estimated by means of the expression

$$d(f_1, f_2) = 1200 \times \left| log_2 \left(\frac{f_1}{f_2} \right) \right| . \tag{22.1}$$

where the logarithm in base 2 and the factor 1200 have been used in order to express d in cents.

Let us define the well-known concept of an octave mathematically: given two sounds with frequencies f_1 and f_2, we say that f_2 is one octave higher than f_1 if f_2 is twice f_1.

Two notes one octave apart from each other have the same letter-names. This naming corresponds to the fact that notes which are one octave apart sound like the same note produced at different pitches and not like entirely different notes. Based on this idea, we can define in R^+ (the subset of all the frequencies of all the sounds) a binary equivalence relation, denoted by R, as follows:

$$f_1 R f_2 \text{ if and only if } \exists n \in Z \text{ such that } f_1 = 2^n \times f_2.$$

Therefore, instead of dealing with R^+, we can analyze the quotient set R^+/R, which for a given fixed note f_0 (diapason) can be identified with the interval $[f_0, 2f_0[$. For the sake of simplicity, we can assume that $f_0 = 1$ and work in the interval $[1,2[$. Octave equivalence has been an assumption in tonal music; however, the terminology used in atonal theory is much more specific.

Let us offer an outline of the main tuning systems used in Western music. For an overview of the topic, including historical aspects we recommend interested readers the book by Benson [3]. The Pythagorean system is so named because it was actually discussed by Pythagoras, who in the sixth century B.C. already recognized the simple arithmetical relationship involved in the intervals of octaves, fifths, and fourths. He and his followers believed that numbers were the ruling principle of the universe, and that musical harmonies were a basic expression of the mathematical laws of the universe. Pythagorean tuning was widely used in Medieval and Renaissance times. All tuning is based on the interval of the pure fifth: the notes of the scale are determined by stacking perfect fifths without alterations. The Just Intonation (Zarlinean version) can be viewed as an adaptation of the Pythagorean system to diminish the thirds; it can be obtained by replacing some fifths of the Pythagorean system 3:2, by syntonic fifths 40:27 (see [20]). For these two tuning systems the circle of fifths is not closed, hence to establish an appropriate number of notes in an octave, some additional criteria are necessary. In order to avoid this question and also to permit transposing, the temperaments were introduced. If every element in the tuning system is a rational number, we say that it is a tuned system, whereas if some element is an irrational number then the system is a temperament. The most-used temperaments are the equal cyclic temperaments that divide the octave into equal parts. Hölder's temperament divides the octave into 53 parts, providing a good approximation to the Pythagorean system. The Twelve-Tone Equal Temperament 12-TET is today's standard on virtually all western instruments. This temperament divides the octave into twelve equal half steps. Tuning systems based on a unique interval (like the Pythagorean) admit a direct mathematical construction. However, the definition of systems generated by more than one interval requires specifying when and how many times each interval appears. Next, we give a general definition of a tuning system (see [19])

Definition 32. *(tuning system) Let* $\Lambda = \{\lambda_i\}_{i=1}^{k} \subset [0,1[$ *be a family of functions* $F = \{h_i : Z \to Z\}_{i=1}^{k}$. *We call the tuning system generated by the intervals* $\{2^{\lambda_i}\}_{i=1}^{k}$ *and* F *the set*

$$S_{\Lambda}^{F} = \left\{ 2^{c_n} : c_n = \sum_{i=1}^{k} \lambda_i h_i(n) - \left\lfloor \sum_{i=1}^{k} \lambda_i h_i(n) \right\rfloor, n \in Z \right\}$$

where $\lfloor x \rfloor$ *is the integer part of* x *(which is added to gain octave equivalence).*

The advantage of expressing the tuned notes as 2^{c_n} is that if our reference note is 2^0, in accordance with (1), the exponent c_n provides the pitch sensation. Let us mention that the family of integer-valued functions F marks the "interval locations". For those systems generated by one interval (for instance the Pythagorean) they are not really necessary, while they are for the other systems. For instance, in the Just Intonation $h_1(n)$ and $h_2(n)$ indicate the position of the fifths and the thirds considered as tuned. Table 22.2 displays some examples of tuning systems.

S	Λ	F
Pythagorean	$\lambda_1 = \log_2(3/2)$	$h_1(n) = n$
12-TET	$\lambda_1 = 7/12$	$h_1(n) = n$
Zarlinean (Just Intonation)	$\lambda_1 = \log_2(3/2)$ $\lambda_2 = \log_2(5/4)$	$h_1(n) = n - 4h_2(n)$ $h_2(n) = \left\lfloor \frac{n+1}{7} \right\rfloor + \left\lfloor \frac{n+4}{7} \right\rfloor$
Neidhart's temperament (1/2 & 1/6 comma)	$\lambda_1 = \log_2(3/2)$ $\lambda_2 = 1/6 \cdot \log_2\left(2^{12}/3^6\right)$ $\lambda_3 = 7/12$	$a_n = n - 12\lfloor n/12 \rfloor$ $h_1(n) = \left\lfloor \frac{a_n+2}{12} \right\rfloor + \left\lfloor \frac{n+3}{12} \right\rfloor + \left\lfloor \frac{a_n+10}{12} \right\rfloor + \left\lfloor \frac{n+11}{12} \right\rfloor$ $h_2(n) = \left\lfloor \frac{a_n+6}{12} \right\rfloor + \left\lfloor \frac{n+7}{12} \right\rfloor + \left\lfloor \frac{a_n+8}{12} \right\rfloor + \left\lfloor \frac{n+9}{12} \right\rfloor$ $h_3(n) = \left\lfloor \frac{a_n+1}{12} \right\rfloor + \left\lfloor \frac{n+4}{12} \right\rfloor + \left\lfloor \frac{a_n+5}{12} \right\rfloor$

Fig. 22.2. Table 1. Examples of generators of some tuning systems.

Although we only analyze Pythagorean, Zarlinean and 12-TET systems, the study of other tuning systems would be similar.

22.3 Notes as fuzzy sets

If we take the note $A = 440Hz$ (*diapason*) as our fixed note, then a note offered with a frequency of $442Hz$ would be out of tune from the point of view of Boolean logic. However, anybody that hears it would consider it to be in tune.

While not trying to delve into psychoacoustic issues, we need to make some brief remarks about hearing sensitivity.

According to J. Piles (1982) (see [20]), there is no unanimity among musicologists about aural perception. Roughly speaking, we could distinguish between two great tendencies: those who, following the work of Hermann von Helmholtz (1821 - 1894), consider that a privileged and educated ear can distinguish a difference of two cents, and those who fix the minimum distance of perception at 5 or 6 cents. For instance, Haluska states that the accuracy of an instrumentalist is not better than 5or 6 cents and that this accuracy is between 10 and 20 cents for non-trained listeners.

Nonetheless, this threshold of the human aural perception depends on many factors: the sensitivity of the ear, the listener's age, education, practice and mood, the intensity and duration of the sounds, etcetera. As the human ear is not "perfect", a musical note should be understood as a band of frequencies around a central frequency and it is appropriate to express it as a fuzzy number.

Therefore the modal interval corresponding to the pitch sensation of a tone with frequency f should be expressed as $[log_2(f)-\varepsilon, log_2(f)+\varepsilon]$, where $\varepsilon = 3/1200$ (for instrumentalists) or $\varepsilon \in [5/1200, 10/1200]$ for non-trained listeners (see [20], [14]).

Accordingly, we define the band of unison as: $[f2^{-\varepsilon}, f2^{\varepsilon}]$, where $\varepsilon > 0$, and where 1200ε expresses, in cents, the accuracy of the human ear to the perception of the unison.

Next, let us focus on its support. If the number of notes per octave is q, the octave can be divided into q intervals of widths $1200/q$ cents. So, if we represent it as a segment, the (crisp) central pitch would be in the middle, and the extremes would be obtained by adding and subtracting $1200/(2\times q)$ cents. In fact, chromatic tuners assign $q = 12$ divisions per octave, suggesting that a tolerance of $\delta = 50/1200 = 1/24$ is appropriate. Therefore, the support of the pitch sensation should be expressed as $[log_2(f)-\delta, log_2(f)+\delta]$, where $\delta = 1/(2\times q)$. Therefore, we can express the interval of the note f as: $[f2^{-\delta}, f2^{\delta}]$.

Notice that the quantity $\Delta = 1200\delta$ expresses, in cents, the tolerance that we admit for every note, and for $q = 12$, we have $\Delta = 1200\frac{1}{2\times 12} = 50$ cents.

These arguments justify the expression of a musical note as a trapezoidal fuzzy number with peak $[f2^{-\varepsilon}, f2^{\varepsilon}]$ and support $[f2^{-\delta}, f2^{\delta}]$.

For notational purposes let us recall the definition of a *LR*-fuzzy number (see [9]).

Definition 33. *(LR-fuzzy number)* $\tilde{M}$ *is said to be a LR-fuzzy number,* $\tilde{M} = (m^L, m^R, \alpha^L, \alpha^R)_{LR}$ *if its membership function has the following form:*

$$\mu_{\tilde{M}} = \begin{cases} L\left(\frac{m^L - x}{\alpha^L}\right), x < m^L \\ 1, m^L \le x \le m^R \\ R\left(\frac{x - m^R}{\alpha^R}\right), x > m^R \end{cases}$$

where L and R are reference functions, i.e. $L, R : [0, +\infty[\to [0,1]$ are strictly decreasing in $supp\tilde{M} = x : \mu_{\tilde{M}}(x) >=$ and upper semi-continuous functions such that

$L(0) = R(0) = 1$. If $supp\tilde{M}$ is a bounded set, L and R are defined on $[0,1]$ and satisfy $L(1) = R(1) = 0$.

Moreover, if L and R are linear functions, the fuzzy number is called *trapezoidal*, and is defined by four real numbers, $\tilde{A} = (a^L, a^R, \alpha^L, \alpha^R)$.

As notes are expressed as powers of two, it is not only more practical to express the fuzzy musical notes using their exponent but it also makes more sense, because as we have already mentioned, the exponent reflects the pitch sensation. Therefore we represent the pitch sensation of a note $2^{\tilde{t}}$ as the trapezoidal fuzzy number $\tilde{t} = (t-\varepsilon, t+\varepsilon, \delta-\varepsilon, \delta-\varepsilon)$.

Now that notes are modelled as fuzzy numbers, the concept of fuzzy tuning system arises naturally:

Definition 34. *(fuzzy tuning system) Let* $\delta \in [0,1], \Lambda = \{\lambda_i\}_{i=1}^k \subset [0,1[$, *and a family of functions* $F = h_i : Z \to Z_{i=1}^k$. *A fuzzy tuning system generated by the intervals* $2^{\lambda_i}{}_{i=1}^{k}$ *and* F *is the set:*

$$\tilde{S}_\Lambda^F(\delta) = \left\{ 2^{\tilde{c}_n} : \tilde{c}_n = \left(\sum_{i=1}^k \lambda_i h_i(n) - \left\lfloor \sum_{i=1}^k \lambda_i h_i(n) \right\rfloor, \delta \right) n \in Z \right\}$$

In [19] the compatibility between two fuzzy notes is defined as the Zadeh consistency index between their pitch sensations. Figure 22.3 illustrates the definition of ?compatibility.

Definition 35. *(compatibility) Let* $2^{\tilde{t}}$ *and* $2^{\tilde{s}}$ *be two musical notes, where* $\tilde{t} = (t-\varepsilon, t+\varepsilon, \delta-\varepsilon, \delta-\varepsilon)$ *and* $\tilde{s} = (s-\varepsilon, s+\varepsilon, \delta-\varepsilon, \delta-\varepsilon)$. *The degree of compatibility between* $2^{\tilde{t}}$ *and* $2^{\tilde{s}}$ *is defined as*

$$comp[2^{\tilde{t}}, 2^{\tilde{s}}] = max_x \mu_{\tilde{s}\cap\tilde{t}}(x),$$

and we say that $2^{\tilde{t}}$ *and* $2^{\tilde{s}}$ *are* □*-compatible,* $\alpha \in [0,1]$, *if* $comp[2^{\tilde{t}}, 2^{\tilde{s}}] \geq \alpha$.

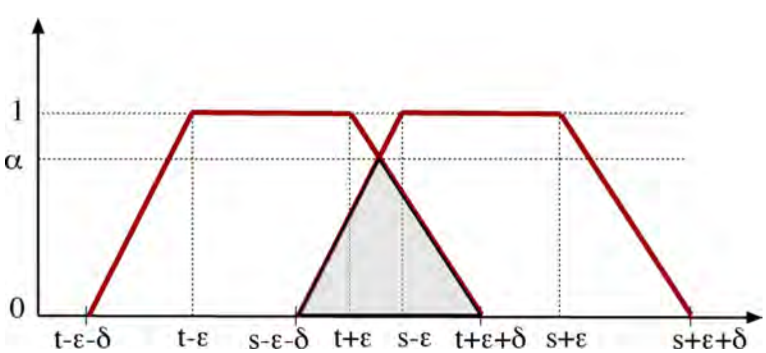

Fig. 22.3. Graph illustrating the concept of?compatibility between two notes.

By a direct calculus we can obtain the formula which allows us to calculate the compatibility between two notes.

$$comp[2^{\tilde{t}}, 2^{\tilde{s}}] = \begin{cases} 1, if\, |t-s| < 2\varepsilon \\ 1 - \frac{|t-s|-2\varepsilon}{2\delta-\varepsilon}, if 2\varepsilon \le |t-s| < 2\delta \\ 0 if\, |t-s| \ge 2\delta \end{cases}$$

Moreover, the concept of compatibility is also extended for fuzzy tuning systems. The definition of compatibility between two tuning systems reflects both the idea of proximity between their notes and also whether their configuration is similar.

Definition 36. *(compatibility between two tuning systems) Let $\tilde{S}_q(\delta)$ and $\tilde{T}_q(\delta)$ be two tuning systems with q notes. We say that $\tilde{S}_q(\delta)$ and $\tilde{T}_q(\delta)$ are □-compatible, if for each $2^{\tilde{s}_i} \in \tilde{S}_q(\delta)$ there is a unique $2^{\tilde{t}_i} \in \tilde{T}_q(\delta)$ such that*

$$comp[2^{\tilde{s}_i}, 2^{\tilde{t_j}}] \ge \alpha.$$

The quantity α is the degree of interchangeability between $\tilde{S}_q(\delta)$ and $\tilde{T}_q(\delta)$ and the uniqueness required in the definition guarantees that these systems have a similar distribution in the cycle of fifths.

Note that the α-compatibility does not define a binary relation of equivalence in the set of tuning systems because the transitive property is not verified.

Example

Let us analyze the compatibility between the 12-TET and the Pythagorean system with 21 notes. Our data are the exact (crisp) frequencies, displayed in Table 22.4. The following pairs of notes are said to be enharmonic: $(C\#?, D\#), (D\#, E\#), (F\#, G\#), (G\#, A\#) and (A\#, B\#)$ because although they have different names they sound the same in the 12-TET. For a better visualization, instead of the exact compatibilities between the notes, we show their graphical representation in Figure 22.5. We have set $\delta = 50/1200$ and $\varepsilon = 3/1200$ (suitable for trained listeners).

The minimum compatibility between the notes is equal to 0.84, however the systems are not 0.84-compatible because the uniqueness property does not hold. Nonetheless, if we consider the 12-TET and the Pythagorean system with 12 notes, C, $C^{\#}$, D, E^b, E, F, $F[\#$, G, $G[\#$, A, B^b, B, they are α-compatible for $\alpha \le 0.84$.

22.4 A numerical experiment and sequential uncertainty

The purpose of the experiment described in this section is to study the different variations of a note that usually occur in a wind instrument (a baritone saxophone) where the pitch may be subject to the interpretation of the performer or the characteristics of the instrument.

In order to set one of the parameters of the experiment, the same saxophonist performed five interpretations of the excerpt represented in Figure 22.6. The recordings took place on the same day without changing the location of the recording or its

Note	Pythagorean	12-TET
C	260,74074	261,6265
$B^{\#}$	264,29809	261,6265
D^{b}	274,68983	277,1826
$C^{\#}$	278,4375	277,1826
D	293,33333	293,6648
E^{b}	309,02606	311,127
$D^{\#}$	313,24219	311,127
F^{b}	325,55832	329,6275
E	330	329,6275
F	347,65432	349,2282
$E^{\#}$	352,39746	349,2282
G^{b}	366,25311	369,9944
$F^{\#}$	371,25	369,9944
G	391,11111	391,9954
A^{b}	412,03475	415,3047
$G^{\#}$	417,65625	415,3047
A	440	440
B^{b}	463,5391	466,1638
$A^{\#}$	469,86328	466,1638
C^{b}	488,33748	493,8833
B	495	493,8833

Fig. 22.4. Table 2. Exact frequencies of the notes.

physical characteristics such as temperature and humidity. The measurements were made with the free software Audacity®. The saxophone brand name is Studio. We considered two possible conceptual frameworks: “static tuning”, in which each note is treated separately, and "dynamic tuning" where notes are studied in their context. In this section we will describe our results for the second approach, which seems more relevant for this study.

Firstly we obtained the compatibility between the notes recorded and the notes tuned in the fuzzy 12-TET. We fixed $\delta = 50/1200$ and $\varepsilon = 6/1200$. Figure 22.6 is a graphic representation of the compatibility values. We can observe that the worst compatibilities with the theoretical notes occur for the notes *D*#4, *C*#4, G4, *D*#4. A first conclusion is that the saxophonist should make an effort to improve his interpretation of these notes. However, our analysis should be completed by taking “sequential uncertainty” into account.

We have already mentioned the musicologist N. A. Garbuzov in the introduction. According to J. Haluska [15], “... (he) revolutionized the study of musical intervals suggesting a concept of musical “zones” in the 1940s. This theory can be characterized in the present scientific language as an information granulation in the sense of Zadeh”. Table 22.9 was obtained by Garbuzov from hundreds of measurements. We are taking it as a reference, although bearing in mind that it should be recomputed to take into account the higher precision of the present measurement instruments.

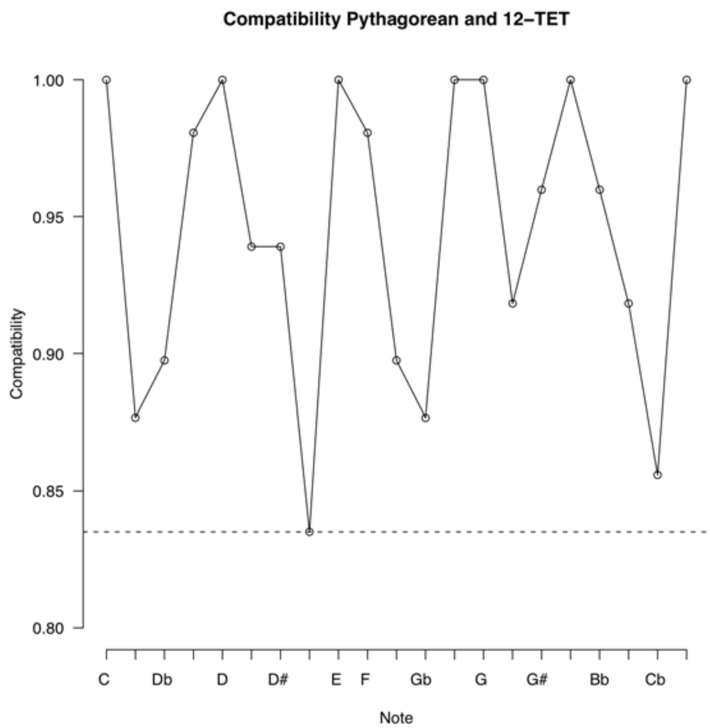

Fig. 22.5. Compatibility between the 12-TET and the Pythagorean tuning system.

Fig. 22.6. Score for the excerpt interpreted by the musician.

How should we interpret Table 22.9? As an example we will comment on the second row, as the reasoning is similar for all of them. For the 12-TET, two notes which differ by a semitone are exactly 100 cents apart. However, according to Garbuzov's

Note	Rep1	Rep2	Rep3	Rep4	Rep5
C#3	139.7	138.77	139.3	137.75	137.12
F#3	185.12	184.19	185.38	184.84	183.14
B3	247.36	249.17	247.83	247.48	244.55
F3	174.34	175.52	175	174.8	174.84
A3	218.26	218.38	218.86	218.73	217.41
D#3	156.49	156.64	157.27	156.61	155.93
B3	247.41	248.65	246.99	245.28	242.06
D#4	323.38	324.19	323.33	321.43	322.31
C#4	284.48	284.75	284.57	283.43	284.49
G4	401.2	401.1	401	399.31	399.79
D#4	322.08	322.03	320.03	319.9	320.01
A#3	234.48	234.14	234.11	233.56	233.56
A3	219.11	219.19	219.12	219.06	218.75
F3	175.46	175.8	175.21	174.97	174.82
E3	164.3	164.46	164.61	165.05	165.15
D#3	156.13	156.11	155.65	156.42	156.74

Fig. 22.7. Table 3. Exact (crisp) frequencies of the notes offered.

experiments if two notes differ between 48 and 124 cents and they are played consecutively, the human ear perceives them to be a semitone apart. When these two notes are played simultaneously (for instance two instrumentalists are playing together) and they differ between 66 and 130 cents, they are perceived to be a semitone apart. Clearly, for our experiment we should only take into account the second column because our saxophonist is playing "a solo". Column 1 in Table 5 contains the "low compatibility notes", the second column the corresponding Garbuzov zones, Column 3 the distance in semitones from an offered note to its previous one and Colums 4-8 the distances in cents from an offered note to its previous one.

We can see that the distances are out of the Garbuzov zone for only two notes. In the other cases, the saxophonist and those listening to his interpretation would probably perceive them as correct.

22.5 Conclusions

Defining tuning systems as comprised of fuzzy notes allows us to include the daily reality of musicians and their theoretical instruction in a mathematical structure. We can justify that the adjustments the musicians make to play together constitute a method for increasing the compatibility level among systems. Complex tuners indicating precisely the difference between the note offered and the desired pitch could

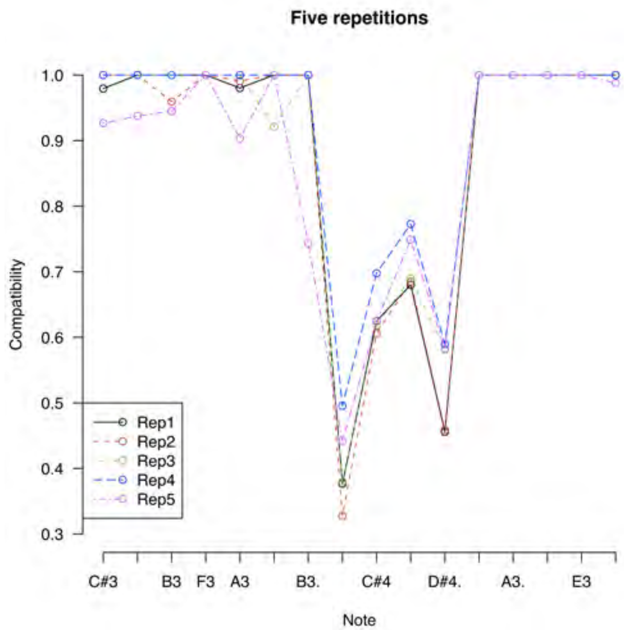

Fig. 22.8. Compatibility between the theoretical notes and those offered by the musician.

Granule	Melodic	Harmonic
Unison (octave)	(-12, 12)	(-30,30)
Minor second	(48, 124)	(66, 130)
Major second	(160, 230)	(166, 230)
Minor third	(272, 330)	(266, 330)
Major third	(372, 430)	(372, 430)
Fourth	(472, 530)	(466, 524)
Tritone	(566,630)	(566, 630)
Fifth	(672,730)	(672, 730)
Minor sixth	(766, 830)	(766, 830)
Major sixth	(866, 930)	(866, 924)
Minor seventh	(966, 1024)	(966, 1024)
Major seventh	(1066, 1136)	(1066, 1136)

Fig. 22.9. Table 4. Garbuzov zones in cents: sequential and simultaneous uncertainty (source [15]).

suggest to musicians that they should aspire to achieving “perfect tuning”; however, getting a high degree of compatibility or similarity with the score is a more achievable and reasonable goal. On the other hand, knowing the compatibility between notes allows musicians to improve their performance by choosing between different

Note	Zone	semitones	Rep1	Rep2	Rep3	Rep4	Rep5
$D^{\#}4$	[372,430]	2	**463.59**	**459.27**	**466.27**	**468.09**	**495.7**
$C^{\#}4$	[160,230]	1	221.88	224.57	221.07	217.815	216.09
G4	[566,630]	3	595.19	593.12	593.78	593.42	589.04
$D^{\#}4$	[372,430]	2	380.28	380.12	390.47	383.866	385.35
$A^{\#}3$	[472,530]	2.5	**549.55**	**551.79**	**541.22**	**544.593**	**545.19**

Fig. 22.10. Table 5. Low compatibility notes (5 repetitions).

tuning positions, increasing lip pressure, etcetera. The numerical example that we have presented causes us to reflect: it is not only important to consider compatibility with the theoretical notes (allowing the coexistence of different instruments in an orchestra), but also that a new concept of sequential compatibility should be considered to better explain instrumentalists' performances. In addition, we should not forget "simultaneous uncertainty". We intend to define the concepts of sequential compatibility and simultaneous compatibility in order to aggregate them with the compatibility between the theoretical notes and the notes offered. The weights of these quantities in the aggregation should depend on whether a musician is playing a solo, a duet or playing with the orchestra.

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