SO(3)-structures on AQH 8-manifolds

Óscar Maciá

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- O.M., A nearly quaternionic structure on SU(3), math.DG 0908.4183 (2009).
- S.Chiossi, O.M., SO(3)-structures on 8-manifolds, to appear.

INTRODUCTION: QK GEOMETRY AQH GEOMETRY

2 INTRINSIC TORSION AND IDEAL GEOMETRY

3 NEARLY QUATERNIONIC STRUCTURE

RIEMANNIAN HOLONOMY

Let $\{M, g\}$ be a Riemannian manifold.

Let $c : [0, 1] \to M$ a smooth curve on M from x to y. The Levi-Civita connection determines horizontal transport of vectors on TM along the curve c



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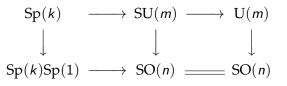
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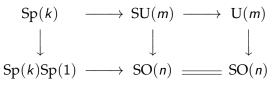
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- $Spin(9)^{\dagger}$: D=16. ALWAYS SYMMETRIC (ruled out from the list).



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- Indeed, QK manifolds are not 'quaternionic' they do not in general admit quaternionic coordinates.



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- NICE NAME...

groucho.png

Operational definitions for QK

In the following, when referring to QK manifolds we mean $\lambda \neq 0.$

Definition

A QK manifold is a **Riemannian 4n-manifold** $\{M^{4n}, g\}$ equipped with a family of **three compatible almost complex structures** $\mathcal{J} = \{J_i\}_1^3$

$$g(J_i\cdot, J_i\cdot) = g(\cdot, \cdot), \ i = 1, 2, 3$$

satisfying the algebra of imaginary quaternions

$$J_1^2 = J_2^2 = J_3^2 = J_1 J_2 J_3 = -1$$
,

such that it is preserved by the Levi-Civita connection

$$\nabla_X^{LC} J_i = \alpha_k(X) J_j - \alpha_j(X) J_k$$

for certain 1-forms α_i , α_j , α_k .

G-Structures

The theory of G-structures allows to work with general (non necessarily torsion-free) connections on Riemannian manifolds.

Definition

A G-structure is a reduction of the bundle of linear frames L(M) to a subbundle with structure group G.

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 ${SO(n) - structure on M} \leftrightarrow Riemannian manifold {M, g}$

• Example 2: A Riemannian 2*n*-manifold {2*n*} with U(*n*)-structure is equivalent to define a compatible almost complex structure J

 $\{U(n) - structure \text{ on } M\} \longleftrightarrow Almost \text{ Hermitian manifold } \{M, g, J\}$

Theorem

$$\nabla^{LC}\eta = \mathbf{0} \longleftrightarrow \Phi \subseteq \mathbf{G}$$

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If G is a closed and connected subgroup of SO(n) there is a <u>unique</u> (non-torsion free) metric G-connection (minimal G-connection) satisfying

$$\nabla^{\rm G} = \nabla^{\rm LC} + \xi$$

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•The decomposition of ξ in irreducible components w.r.t. the action of G classifies all possible minimal G-connections.

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Almost Quaternionic Hermitian (AQH) manifolds

Definition

A Riemannian 4n-manifold with Sp(n)Sp(1)-structure is called Almost Quaternionic Hermitian (AQH).

Definition

An AQH manifold is a **Riemannian** 4n-manifold $\{M, g\}$ equipped with a family three compatible almost complex structures $\{J_i\}_1^3$

$$g(J_i\cdot, J_i\cdot) = g(\cdot, \cdot), \ i = 1, 2, 3$$

satisfying the algebra of imaginary quaternions

$$J_1^2 = J_2^2 = J_3^2 = J_1 J_2 J_3 = -\mathbf{1}.$$

• The difference between AQH and QK definitions involves the relations between tensors {*J_i*} and the connection.

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• The AQH structure is defined by the distinguished Sp(k)Sp(1) invariant 4-form $\Omega \in \Lambda^4 M$.

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Theorem

An AQH manifold is QK if and only if $\nabla^{LC} \Omega = 0$.

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EH-Formalism

Representation theory notation

- $E \simeq \mathbb{C}^4$ irreducible basic complex representation of Sp(2). (Highest weight [1,0]).
- $H \simeq \mathbb{C}^2 \simeq \mathbb{H}$ irreducible basic complex represtentation of Sp(1). (Highest weight [1]).

Locally

$$\mathbb{C}\otimes TM=E\otimes H.$$

Other important Sp(2) representations

- $K \simeq \mathbb{C}^{16}$ irreducible complex representation of Sp(2). (Highest weight [2, 1] in the basis of roots).
- $\Lambda_0^3 E \simeq \mathbb{C}^3$ irreducible complex representation of Sp(2). (Highest weight [3, 3] in the basis of roots).

$$\Lambda_0^n E = \operatorname{Coker} \{ L : \Lambda^{n-2} E \to \Lambda^n E : \alpha \mapsto \omega_E \wedge \alpha \}.$$

(Swann, 1989) The intrinsic torsion of an 4n-manifold, $n \ge 2$ can be identified with an element $\nabla \Omega$ in the space

$$(\Lambda_0^3 E \oplus K \oplus E) \otimes (H \oplus S^3 H)$$

ES ³ H	$\Lambda_0^3 ES^3 H$	KS ³ H
EH	$\Lambda_0^3 EH$	KH

For n = 2, the intrinsic torsion belongs to

 $E S^{3}H \oplus K S^{3}H \oplus KH \oplus EH$

$$AQH8 \stackrel{?}{\longrightarrow} QK8$$

Corollary

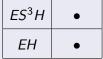
The fundamental 4-form Ω of an 8-manifold is closed $d\Omega = 0$, i.e., M is almost parallel if and only if $\nabla \Omega \in KS^{3}H$.

$$d\Omega = 0 \longleftrightarrow \boxed{ \begin{array}{c} \bullet \\ \bullet \end{array}} \xrightarrow{KS^3H} \\ \bullet \\ \bullet \end{array}$$

Corollary

The Kähler 2-forms $\{\omega_i\}$ of an 8-manifold generate a differential ideal if and only if $\nabla \Omega \in ES^3H \oplus EH$,

$$d\omega_i = \sum_j \beta_i^j \wedge \omega_j : \beta_i^j \in \Lambda^1 M \longleftrightarrow$$



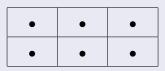
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 $AQH(4n) \xrightarrow{?} QK(4n)$

Theorem

(Swann, 1989) An AQH 4n-manifol, $4n \ge 12$ is QK if and only if

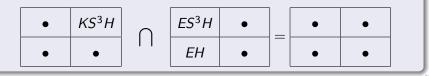
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An AQH 8-manifold is QK if and only if

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SO(3)-structures on AQH 8-manifolds

Theorem

(Salamon, 2001) There exists a closed 4-form Ω with stabilizer Sp(2)Sp(1) on a compact nilmanifold of the form $M^6 \times T^2$. The associated Riemannian metric g is reducible and is not therefore quaternionic Kähler.

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- \implies Relation between AQH & QK geometry in 8 dimensions is special.
- NON-QK AQH8 : $d\omega_i = \sum_j \beta_i^j \wedge \omega_j$ (Satisfies condition 2, not 1)

Ideal condition

$$\mathsf{d}\omega_i = \sum_j \beta_i^j \wedge \omega_j \; : \; \beta_i^j \in \Lambda^1 M$$

CHANGE OF BASE: $\{\omega_i\} \mapsto \{\widetilde{\omega}_i\}$

$$\widetilde{\omega}_i = \sum_{j=1}^3 A_i^j \omega_j, \qquad A = (A_i^j) \in \mathrm{SO}(3)$$

• The matrix β transforms as a connection

$$d\widetilde{\omega}_i = \sum_{j=1}^3 \widetilde{\beta}_i^j \wedge \widetilde{\omega}_j \qquad : \qquad \widetilde{\beta} = A^{-1} dA + \mathrm{Ad}(A^{-1})\beta.$$

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- Consider the decomposition

$$\beta = \alpha + \sigma, \qquad \alpha_i^j = \frac{1}{2}(\beta_i^j - \beta_j^i) \qquad \sigma_i^j = \frac{1}{2}(\beta_i^j + \beta_j^i)$$

• The symmetric part σ transforms as a tensor:

$$\widetilde{\sigma} = \mathrm{Ad}(A^{-1})\sigma = A^{-1}\sigma A.$$

• The tensor σ can be identified with the remaining non-zero components of intrinsic torsion

$$ES^{3}H \oplus EH$$
$$d\Omega = 2\sum_{i=1}^{3} d\omega_{i} \wedge \omega_{i} = 2\sum_{i,j=1}^{3} \sigma_{i}^{j} \wedge \omega_{i} \wedge \omega_{j}.$$

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Lemma

If an Sp(2)Sp(1)-structure satisfies the ideal condition then its intrinsic torsion belongs to ES³H if and only if $\operatorname{tr}(\beta) = \beta_1^1 + \beta_2^2 + \beta_3^3 = 0.$

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Corollary

Let $\{M, g, \mathcal{J}\}\$ be an AQH 8-manifold. It is QK if and only if generates a differential ideal with $\sigma = 0$, so that the ideal condition applies with $\beta_i^j = -\beta_j^i$.

Geometry of the ideal condition

Consider the matrix $B = (B_i^j)$ of curvature 2-forms associated to the connection defined through β .

$$0 = d^2 \omega_i = \sum_j (d\beta_i^j - k\beta_i^k \wedge \beta_k^j) \wedge \omega_k = \sum_j B_i^j \wedge \omega_j$$

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• In particular, they have no S^2E component, because

$$S^2 E S^2 H \subset \Lambda^4 T^* M;$$

thus

$$B_i^j \in S^2 H \oplus \Lambda_0^2 E S^2 H \subset \Lambda^2 T^* M.$$

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$$B_i^j \in S^2 H \oplus \Lambda_0^2 E S^2 H \subset \Lambda^2 T^* M.$$

• In contrast to the QK case, there will in general be a component of B_i^j in $\Lambda_0^2 E S^2 H$.

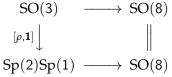
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 $SO(3)\subset SO(8)$ factors through $Sp(2)Sp(1)\equiv Sp(2)\times_{\mathbb{Z}_2}Sp(1)$ in a unique way.



$$\begin{split} \mathbf{1} : \mathrm{SO}(3) &\simeq \mathrm{SU}(2) \simeq \mathrm{Sp}(1) \\ \rho : \mathrm{SO}(3) &\simeq \mathrm{Sp}(1) \xrightarrow{irreducible} \mathrm{Sp}(2) \\ X &\in \mathrm{SO}(3) \longmapsto (\rho(X), \mathbf{1}(X)) \in \mathrm{Sp}(2) \times \mathrm{Sp}(1) \\ & [\rho(X), \mathbf{1}(x)] \in \mathrm{Sp}(2) \mathrm{Sp}(1) \end{split}$$

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SO(3) Intrinsic torsion from Sp(2)Sp(1)

• Let H denote the basic representation of SO(3) identified with Sp(1), the irreducible action ρ of Sp(1) embedded on Sp(2) gives the identification

$$E = S^3 H$$

• The Sp(2)-modules are reducible with respect to the action of SO(3). $EH \oplus KH \oplus ES^3H \oplus KS^3H$

$$\begin{array}{rcl} EH & \longrightarrow & S^4 \oplus S^2 \\ KH & \longrightarrow & S^8 \oplus 2S^6 \oplus S^4 \oplus S^2 \oplus S^0 \\ ES^3H & \longrightarrow & S^6 \oplus S^4 \oplus S^2 \oplus S^0 \\ KS^3H & \longrightarrow & S^{10} \oplus 2S^8 \oplus 2S^6 \oplus 3S^4 \oplus 2S^2 \end{array}$$

• The SO(3)-structure described has intrinsic torsion obstructions on $K\!H\oplus E\!S^3H.$

Action of SO(3) on SU(3)

• If
$$SO(3) \rightarrow Sp(2)Sp(1) \rightarrow SO(8)$$
 then,
 $\mathbb{C} \otimes TM = E \otimes H = S^3H \otimes H = S^4H \oplus S^2H$

• The SO(3) action leads to a $\mathfrak{so}(3)$ family of endomorphisms $\mathfrak{so}(3)\simeq S^2 H\subset End(T)$

Take the manifold

$$\begin{split} \mathcal{M} &= SU(3) \to \mathcal{T}_{\mathsf{x}} \mathcal{M} \simeq \mathfrak{su}(3) \\ \mathfrak{su}(3) &= \mathfrak{b} \oplus \mathfrak{p} : \left\{ \begin{array}{ll} \mathfrak{b} \simeq \mathfrak{so}(3) \subset \mathfrak{su}(3), & \mathfrak{b} \simeq S^2 \mathcal{H} \\ \\ \mathfrak{p} \simeq Span\{iS: \ S = S^t, \ Tr(S) = 0\} \simeq S^4 \mathcal{H}. \end{array} \right. \end{split}$$

• Then the action of SO(3) on SU(3) is given on tangent space as the action of $S^2H \simeq \mathfrak{so}(3) \subset End(T)$ on $\mathfrak{su}(3) = S^4H \oplus S^2H$

$$\phi:S^2H\otimes(\mathfrak{b}\oplus\mathfrak{p})\to\mathfrak{b}\oplus\mathfrak{p}$$

$$\phi = \phi_1 + \phi_2 + \phi_3 + \phi_4$$

$$\phi_1: (S^2 H \otimes \mathfrak{b}) = S^4 H \otimes S^2 H \otimes S^0 H \longrightarrow S^2 H = \mathfrak{b}$$

(A, B) \longmapsto [A, B]

$$\phi_2: (S^2 H \otimes \mathfrak{b}) = S^4 H \otimes S^2 H \otimes S^0 H \longrightarrow S^4 H = \mathfrak{p}$$

(A, B) $\longmapsto i\left(\{A, B\} - \frac{2}{3}Tr(AB)\mathbf{1}\right)$

$$\phi_3: (S^2 H \otimes \mathfrak{p}) = S^6 H \otimes S^4 H \otimes S^2 H \longrightarrow S^2 H = \mathfrak{b}$$

(A, C) $\longmapsto i\{A, C\}$

 $\phi_4: (S^2 H \otimes \mathfrak{p}) = S^6 H \otimes S^4 H \otimes S^2 H \longrightarrow S^4 H = \mathfrak{p}$ (A, C) \longmapsto [A, C]

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AQ Action of SO(3) on SU(3)

• Denote the action defined by ϕ with the dot-product

$$A \cdot X = \lambda_1[A, X^a] + i\lambda_2\left(\{A, X^a\} - \frac{2}{3}Tr(AX^a)\right) + i\lambda_3\{A, X^s\} + \lambda_4[A, X^s].$$

for $A \in \mathfrak{so}(3)$, $X \in \mathfrak{su}(3)$.

• Taking $\mathcal{J} = \{J_1, J_2, J_3\} \in S^2 H$ and asking the previous equation to satisfy

$$J_i \cdot (J_i \cdot X) = -X$$
 $J_1 \cdot (J_2 \cdot X) = J_3 \cdot X$

one obtains

$$\lambda_1 = \frac{1}{2}, \qquad \lambda_3 = -\frac{3}{4}\lambda^{-1}, \qquad \lambda_4 = -\frac{1}{2}$$

where $\lambda = \lambda_2$ is a real parameter. This is a 1-parameter family of almost quaternionic actions of SO(3) on SU(3).

AQH structure on SU(3)

• Let $\{e_i\}_1^8$ be a base for $\mathfrak{su}(3)$, orthonormal for a multiple of the Killing metric.

•
$$\mathfrak{b} = \mathfrak{so}(3) = \operatorname{Span}\{e_6, e_7, e_8\}.$$

- Identify $\mathcal{J}_{\lambda} = \{e_6, e_7, e_8\}$ acting through the 1-parameter family of AQ SO(3) actions defined by ϕ .
- Define a new metric by rescaling the $\mathfrak b$ subspace

$$g_{\lambda} = \sum_{i=1}^{i=5} e^i \otimes e^i + \frac{4\lambda}{3} \sum_{i=6}^{i=8} e^i \otimes e^i.$$

Theorem

 \mathcal{J}_{λ} is compatible with g_{λ}

 $\{\operatorname{SU}(3),\mathcal{J}_{\lambda},g_{\lambda}\}$ is a 1-parameter family of AQH 8-manifolds

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Ideal AQH structure on SU(3)

Theorem

A set of λ -dependent Kähler 2-forms $\{\omega_i\}_{\lambda}$ associated to the AQH 8-manifold $\{SU(3), \mathcal{J}_{\lambda}, g_{\lambda}\}$ is given by

$$\begin{split} \omega_1 &= \frac{1}{2}(e^{15} + \sqrt{3}e^{25} + e^{34}) + \lambda(\frac{1}{\sqrt{3}}e^{28} - e^{46} + e^{37} - e^{18}) - \frac{2}{3}\lambda^2 e^{67}, \\ \omega_2 &= -e^{14} - \frac{1}{2}e^{35} + \lambda(\frac{2}{\sqrt{3}}e^{27} - e^{38} - e^{56}) - \frac{2}{3}\lambda^2 e^{68}, \\ \omega_3 &= \frac{1}{2}(e^{13} - \sqrt{3}e^{23} + e^{45}) + \lambda(\frac{1}{\sqrt{3}}e^{26} - e^{48} + e^{57} + e^{16}) - \frac{2}{3}\lambda^2 e^{78} \end{split}$$

Theorem

AQH {SU(3), \mathcal{J}_{λ} , g_{λ} } satisfies the ideal condition $d\omega_i = \sum_j \beta_i^j \wedge \omega_j$ if and only if

$$\lambda^2 = \frac{3}{20}.$$

Corollary

 $\{SU(3), \mathcal{J}_{\lambda}, g_{\lambda}\}$ is not QK for any choice of λ .

Due to the topology of SU(3),

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Hence, for $\lambda^2 = rac{3}{20}$, $\{\mathrm{SU}(3), \mathcal{J}_{\lambda}, g_{\lambda}\}$

• NON-QK AQH8 : $d\omega_i = \sum_j \beta_i^j \wedge \omega_j$ (Satisfies condition 2, not 1) \checkmark

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ES ³ H	•
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THANK YOU.