Elementary deformations and the HyperKaehler/Quaternionic Kaehler correspondence.

(Joint work with Prof. A.F. SWANN)

Oscar MACIA

Dept. Geometry Topology
University of Valencia (Spain)

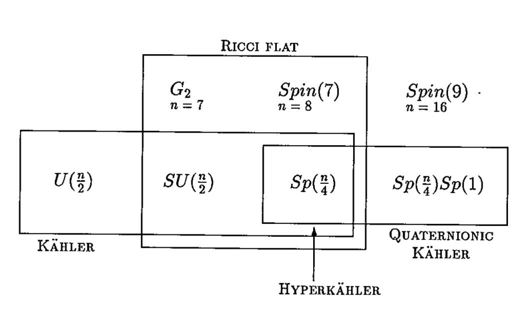
June 26, 2014

What do we have

Berger's list

Theorem

(Berger, 1955)
Let M be a oriented symply-connected n-dimensional Riemannian manifold which is neither locally a product nor symmetric. Then its holonomy group belongs to the following list.



Kaehler manifolds

Kaehler manifolds are Riemannian $2m$-dimensional manifold with holonomy group $Hol \subseteq \mathrm{U}(m)$.

- They admit mutually compatible Riemannian, Complex and Symplectic structures.
  1. acs: $I \in End(TM) : I^2 = -Id$
  2. Hermitian metric: $g(IX, IY) = g(X, Y)$
  3. Kaehler 2-form: $\omega(X, Y) = g(IX, Y)$ non-degenerate.
- Kaehler condition:

$$\nabla^{LC} I = 0 \qquad \text{or} \qquad d\omega = 0.$$

Hyperkaehler manifolds

HyperKaehler manifolds are $4k$-dimensional Riemannian manifolds with holonomy group $\subseteq \mathrm{Sp}(k)$.

- $H = K \cap Q$
- $H \Rightarrow \{r = 0,\ s = 0\}$.
- $H \Rightarrow \{I, J, K\} : I^2 = J^2 = K^2 = IJK = -Id$

$$\nabla^{LC} I = \nabla^{LC} J = \nabla^{LC} K = 0,$$
$$\mathrm{d}\omega_I = \mathrm{d}\omega_J = \mathrm{d}\omega_K = 0.$$

- eg., 4D Kaehler & Ricci flat, $K3$-surfaces, Beauville's $S^{[r]}$ manifolds.

Quaternionic Kaehler manifolds

Quaternionic Kaehler manifolds are $4k$-dimensional Riemannian manifolds with holonomy group $Hol \subseteq \mathrm{Sp}(k)\mathrm{Sp}(1)$.

- $Q \not\subset K$
- $s = constant.$
- $Q \Rightarrow \{I, J, K\} : I^2 = J^2 = K^2 = IJK = -Id$

$$\Omega = \omega_I^2 + \omega_J^2 + \omega_K^2$$
$$\nabla^{LC}\Omega = 0$$

- eg., Wolf spaces (compact,positive,symmetric), Alekseevsky spaces (non-compact, homogeneous not necessarily symmetric).

(First) motivating problem

1. FIND EXPLICIT METRICS WITH SPECIAL HOLONOMY

(First) motivating problem

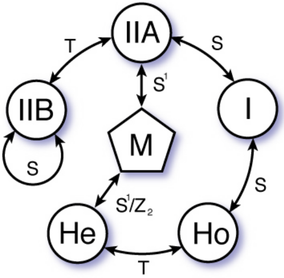
1. FIND EXPLICIT METRICS WITH SPECIAL HOLONOMY
2. HK and QK, in particular

Hint

"The mathematical problems that have been solved or techniques that have arisen out of physics in the past have been the lifeblood of mathematics."

Sir Michael F. Atiyah
Collected Works Vol. 1 (1988), 19, p.13

c-map as Superstring theory T-duality



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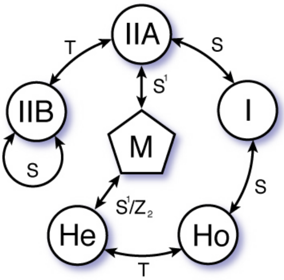

- (Cecotti, Ferrara, Girardello, 1989; Ferrara, Sabharwal, 1990)

$$S^{2n} \xrightarrow{c} Q^{4(n+1)}$$

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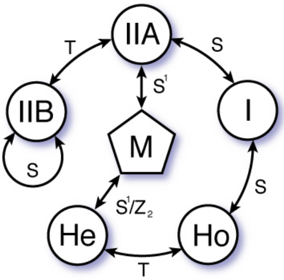

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$$S^{2n} \xrightarrow{c} Q^{4(n+1)}$$

- In the simplest $SUSY$ case

$$C^{2n} \xrightarrow{c} H^{4n}$$

Historic relevance of the c-map for homogeneous QK

- (Alekseevsky, 1975)
  1. Completely solvable Lie groups admitting QK metrics.

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- (Cortes, 1996)
  Complete classification (without c-map).

Special Kaehler manifolds I

- A Special Kaehler manifold is a K manifold (M, g, I, ω) together with a secondary connection ∇ (the *special connection*):

$$\mathrm{R}(\nabla) = \mathrm{T}(\nabla) = 0, \qquad \nabla\omega_I = 0, \qquad \nabla_X IY = -\nabla_Y IX.$$

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- A conic special Kaehler manifold C is a SK manifold $(M, g, I, \omega, \nabla)$ together with a distinguished vector field X (the conic isometry) satisfying:
  1. $g(X, X)$ is nowhere vanishing;
  2. $\nabla X = -I = \nabla^{LC} X$.

  The conic structure is periodic or quasiregular if it exponentiates to a circle action, regular if the circle action is free.

Special Kaehler manifolds II

- A projective special Kaehler manifold S is a Kaehler quotient $S = C//_c X = \mu^{-1}(c)/X$ of a conic SK manifold by a conic isometry X at some level $c \in \mathbf{R}$, together with the data necessary to reconstrut C up to equivalence.

$$\begin{array}{ccc} C_0 = \mu^{-1}(c) & \longrightarrow & C \\ \downarrow & & \\ C//_c X = S & & \end{array}$$

$N = 2,\ D = 4$ SIMPLE SUPERGRAVITY

- The field content of $N = 2, D = 4$ SUGRA:
  1. n-vector multiplets $(A^{i}_{\hat{\mu}},\ \lambda^{i\,\Lambda},\ \phi^{i})$
  2. the gravity multiplet $(V^{a}_{\hat{\mu}},\ \psi^{\Lambda},\ A^{0}_{\hat{\mu}})$.
- $i = 1, \ldots, n$ and $\hat{\mu} = 0, 1, 2, 3$.
- In the **σ-model apprach**, the scalar fields are interpreted as coordintes of some differentiable manifold.
  1. The manifold defined by the vector multiplet has n complex coordinates ϕ^{i} is **projective special Kaehler**.

Kaluza–Klein compactification of the metric field

- When we reduce from $D = 4$ to $D = 3$ by Kaluza–Klein compactification the space-time metric (a $4D$-tensor $g^{(4)}_{\hat{\mu}\hat{\nu}}$) splits as

$$g^{(4)}_{\hat{\mu}\hat{\nu}} = \left(\begin{array}{c|c} e^{i\sigma} g^{(3)}_{\mu\nu} + e^{2\sigma} A_\mu A_\nu & e^{2\sigma} A_\mu \\ \hline e^{2\sigma} A_\nu & e^{2\sigma} \end{array} \right)$$

  increasing the numer of independent fields:

  1. the $3D$-metric tensor $g^{(3)}_{\mu\nu}$;
  2. a $3D$-vector field A_μ;
  3. a new scalar field $\phi = e^{2\sigma}$.

- $\mu, \nu = 0, 1, 2.$

Kaluza-Klein compactification on vector fields

- The n $4D$-vector fields $A^i_{\hat{\mu}}$ (in vector mulitplets) and the *graviphoton* $A^0_{\hat{\mu}}$ also split into $(n+1)$ $3D$-vectors and $(n+1)$ scalars, called *axions*.

$$(A^i_{\hat{\mu}}, A^0_{\hat{\mu}}) = (A^i_{\mu}, A^i_3, A^0_{\mu}, A^0_3)$$

  1. Axions $\zeta^\Lambda = A^i_3, A^0_3$, $\Lambda = 0, \ldots, n$.
  2. Nevertheless, $3D$-vectors in $3D$ are T-dual to scalars therefore each A^i_μ, A^0_μ gives an extra scalar $\tilde{\zeta}_\Lambda$, $n+1$ in total.

- The $3D$-gravitational vector A_μ also by T-duality gives rise to an extra scalar a.

Scalar fields in $D = 3$.

After the Kaluza–Klein compactification we have the following list of scalar fields:

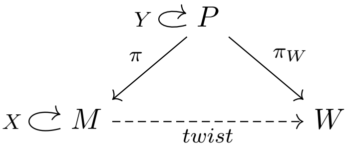
	SUGRA	SUSY
ϕ^i	$2n$	$2n$
ϕ	1	
ζ^Λ	$n+1$	n
$\widetilde{\zeta}_\Lambda$	$n+1$	n
a	1	
TOTAL	$4(n+1)$	$4n$
	Q	H

The general picture arising from physics

$$\begin{array}{ccc} C^{2(n+1)} & \xrightarrow{\quad c \quad} & H^{4(n+1)} \\ {\scriptstyle //_c X}\big\downarrow & & \big\downarrow{\scriptstyle ?} \\ S^{2n} & \xrightarrow[\quad c \quad]{} & Q^{4(n+1)} \end{array}$$

The Twist Construction (sketch)

- The Twist Construction associates to a manifolds M with a S^1-action (generated by X) a new space W of the same dimension with distinguished vector field Y. This construction fits in to a double fibration



so W is M TWISTED by the S^1-bndle P.

Twists & HK/QK correspondence

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$$\left\{ \begin{array}{c} HK \\ \text{symmetry fixing one } \omega \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} QK \\ \text{cirlce action} \end{array} \right\}$$

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- (Alekseevsky, Cortes, Dyckmanns, Mohaupt, 2013)

$$\text{c-map } QK^{4(n+1)} \subset HK/QK \text{ correspondence}$$

Objective

To give an explanation to the HK/QK correspondence arising in the c-map ...

$$\begin{array}{ccc} C^{2(n+1)} & \xrightarrow{\;c\;} & H^{4(n+1)} \\ \Big\downarrow /\!/_{c} X & & \Big\downarrow ? \\ S^{2n} & \xrightarrow[\;c\;]{} & Q^{4(n+1)} \end{array}$$

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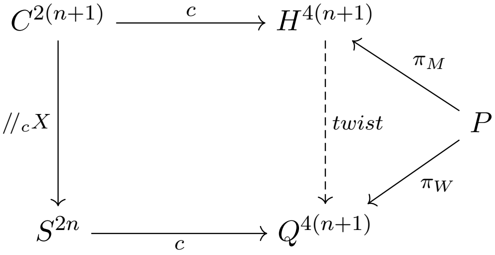
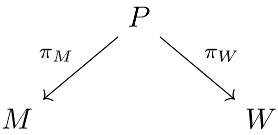


... using the Twist Construction.

The Twist Construction in detail

(Swann, 2007,2010)



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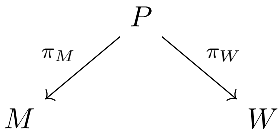

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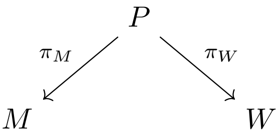

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3. $X' = \hat{X} + aY \in \mathfrak{X}_P$ **preserving** θ **and** Y: $\hat{X} \in \mathcal{H} = ker\ \theta$, $\pi_{M*}\hat{X} = X$ and $\boxed{da = -X \lrcorner F}$.

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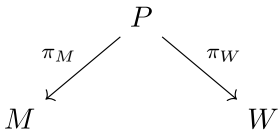

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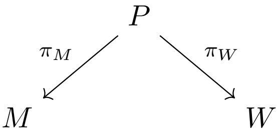
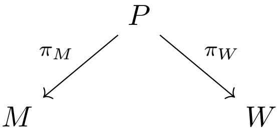
4. $W = P/X'$ has an action induced by Y.

Twist data



- M manifold;
- $X \in \mathfrak{X}_M$, circle action;
- $F \in \Omega^2_M$ closed, X-invariant, with integral periods;
- $a \in C^\infty_M$ with $da = -X \lrcorner F$.

$\mathcal{H}$-related tensors.



- Horizontal distribution: $\mathcal{H} = ker\ \theta \subset TP$
- α tensor on M is $\mathcal{H}$–related to α_W on W if

$$\pi_M^* \alpha = \pi_W^* \alpha_W \quad \text{on } \mathcal{H} = ker\ \theta$$

Write $\alpha \sim_{\mathcal{H}} \alpha_W$.

Lemma

(Swann, 2010) Each X-invariant p-form $\alpha \in \Omega_M^p$ is $\mathcal{H}$-related to a unique p-form $\alpha_W \in \Omega_W^p$ given by

$$\pi_W^* \alpha_W = \pi_M^* \alpha - \theta \wedge \pi^*(a^{-1} X \lrcorner \alpha)$$

Twist computations

Lemma

(S'10) If $\alpha \in \Omega^p_M$ is a X-invariant p-form on M with exterior differential $d\alpha$, and $\alpha \sim_{\mathcal{H}} \alpha_W$ then

$$d\alpha_W \sim_{\mathcal{H}} d\alpha - \frac{1}{a} F \wedge (X \lrcorner \alpha) .$$

- $$\alpha_W \sim_{\mathcal{H}} \alpha, \qquad d\alpha_W \sim_{\mathcal{H}} d_W \alpha$$

  where

  $$d_W := d - \frac{1}{a} F \wedge X \lrcorner$$

  is the *twisted exterior differential.*

- $$(\Lambda_W, d) \sim_{\mathcal{H}} (\Lambda^X_M, d_W)$$

Twist and complex structures

In general, twisting do not preserve integrability.

Proposition

(S'10) For an invariant complex structure I on M that is $\mathcal{H}$-related to an almost complex structure I_W on W we have that I_W is integrable iff $F \in \Omega^{1,1}_I(M)$.

$$F(IA, IB) = F(A, B), \qquad \forall A, B \in TM.$$

Basic example

$$S^{2n+1} \times T^2$$

$$\mathbf{C}\mathrm{P}^n \times T^2 \xleftarrow{\pi_M} S^{2n+1} \times T^2 \xrightarrow{\pi_W} S^{2n+1} \times S^1$$

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  1. $F \in \Omega_I^{1,1}(\mathbf{C}P^n)$, comlex;
  2. $b_2(W) = 0$, non-Kaehler.

(Second) motivating problem

- HOW CAN WE GET HYPERKAEHLER, QUATERNIONIC KAEHLER TWISTS?

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- Which TWIST DATA determines W to be HK, QK?

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- In order to get $d\alpha_W = 0$ we may need to start not from α but from some deformation α^N.
- We will exploit the symmetries of M to get the deformation α^N (defining the same structure on M.)

Symmetries of the HK structure

- We say that $X \in \mathfrak{X}_M$ is a symmetry of a HK manifold (M, g, I, J, K) iff
  1. X is an isometry of the HK metric: $L_X(g) = 0$.
  2. X preserves the linear span $\langle I, J, K \rangle \in EndTM$, ie.,

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- We will say that the symmetry X is rotating if

$$L_X(I) = 0, \qquad L_X(J) = K.$$

Elementary deformations

- Define one-forms $(\alpha_0, \alpha_A) \; : \; A = \mathrm{I}, \mathrm{J}, \mathrm{K}:$

$$\alpha_0 = g(X, \cdot), \qquad \alpha_A = -g(AX, \cdot), \; A = \mathrm{I}, \mathrm{J}, \mathrm{K}.$$

  Then,

$$g_\alpha := \alpha_0^2 + \sum_A \alpha_A^2$$

  is definite semi-positive and proportional to the restriction $g|\mathbf{H}X$ where $\mathbf{H}X = \langle X, IX, JX, KX \rangle$.

- An elementary deformation g^N of a HK mertric g wrt a symmetry X is a new metric of the form

$$g^N = fg + hg_\alpha$$

  where $f, h \in C^\infty_M$.

Main theorem

Theorem

Let (M, g, I, J, K) be a hyperKaehler manifold with non-null rotating symmetry X and Kaehler moment map μ. If $\dim M \geq 8$, then up to homothety, the only twists of elementary deformations $g^N = fg + hg_\alpha$ of g that are quaternion-Kaehler have

$$g^N = \frac{1}{(\mu - c)^2} g_\alpha - \frac{1}{\mu - c} g$$

for some constant c. The corresponding twist data is given by

$$F = k(d\alpha_0 + \omega_I), \qquad a = k(g(X, X) - \mu + c).$$

Meaning

- The unicity statement is particularly important, since it shows that previous constructions using different methods by Haydys, Hitchin, Alekseevsky et al, agree.
- The constant k changes the curvature form: ie., affects the topology of the twist.
- Constant c affects the local properties of the QK metric.

Sketch of proof

- From g^N and (I, J, K) construct Ω^N, and impose that an arbitrary twist Ω_W of Ω^N is QK.
- Decompose these equations in type components relative to $\mathbf{H}X$ and its orthogonal complement.
- This computation leads eventually to

$$f = f(\mu), \quad h = h(\mu), \quad h = f'$$

where μ is the moment map for X.
- First consider the relation $-XF$ and determine the twist function a.
- Investigate the condition $dF = 0$ which lads to a ODE to determine f.

The hyperbolic plane

- Step I.– Define indefinite PSK structures on open subsets S of the hyperbolic plane $\mathbf{R}\mathrm{H}(2)$.

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- Local basis of one forms $\{a, b\} \in \Omega^1_S$:

$$da = 0, \quad db = -\lambda a \wedge b$$

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$$g_S = a^2 + b^2$$

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3. Kaehler 2-form:

$$\omega_S = a \wedge b.$$

Local cone structure

- The PSK manifold S is the Kaehler quotient of a conic SK manifold $C \equiv (C, g, \omega, \nabla, X)$

$$\begin{array}{ccc} C_0 & \xrightarrow{\ i\ } & C \\ \downarrow{\scriptstyle \pi} & & \\ S & & \end{array} \qquad\qquad S = C//_c X$$

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- Locally $C = \mathbf{R}_{>0} \times C_0$.
- C_0 is the level set $\mu^{-1}(c)$ for the moment map of X.
- $C_0 \to S$ is a bundle with connection 1-form φ :

$$d\varphi = 2\pi^* \omega_S = i^* \omega$$

Metric and Kaehler form on C

- Write t for the standard coordinate on $\mathbf{R}_{>0}$ and $\hat{\psi} = dt$.
- Write $\hat{a} = t\pi^*a$, $\hat{b} = t\pi^*b$, $\hat{\varphi} = t\varphi$

Lemma

The $(2,2)$ pseudo–Riemannian metric and the Kaehler form of $C = \mathbf{R}_{>0} \times C_0$ are

$$g_C = \hat{a}^2 + \hat{b}^2 - \hat{\varphi}^2 - \hat{\psi}^2 = -dt^2 + t^2 g_{C_0}$$

$$\omega_C = \hat{a} \wedge \hat{b} - \hat{\varphi} \wedge \hat{\psi}.$$

The conic isometry satisfies

$$IX = t\frac{\partial}{\partial t}$$

LC connection

- Coframe $s^*\theta$:

$$(\hat{a}, \hat{b}, \hat{\varphi}, \hat{\psi})$$

- Exterior differential $d(s^*\theta)$:

$$d\hat{a} = \frac{1}{t} dt \wedge \hat{a} \qquad d\hat{b} = \frac{1}{t}(dt \wedge \hat{b} - \lambda \hat{a} \wedge \hat{b})$$

$$d\hat{\varphi} = \frac{1}{t}(dt \wedge \hat{\varphi} + 2\hat{a} \wedge \hat{b}) \qquad d\hat{\psi} = 0$$

- LC connection $s^*\omega_{LC}$

$$s^*\omega_{LC} = \frac{1}{t}\begin{pmatrix} 0 & \hat{\varphi} + \lambda\hat{b} & \hat{b} & \hat{a} \\ -\hat{\varphi} - \lambda\hat{b} & 0 & -\hat{a} & \hat{b} \\ \hat{b} & -\hat{a} & 0 & \hat{\varphi} \\ \hat{a} & \hat{b} & -\hat{\varphi} & 0 \end{pmatrix}$$

Curvature 2-form

- Curvature 2-form $s^*\Omega_{LC}$:

$$s^*\Omega_{LC} = \frac{4-\lambda^2}{t^2}\begin{pmatrix} 0 & \hat{a}\wedge\hat{b} & 0 & 0 \\ -\hat{a}\wedge\hat{b} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Lemma

The pseudo–Riemannian metric g_C is flat iff $\lambda^2 = 4$.

Conic special Kaehler condition

- Symplectic connection in terms of LC connection:

$$s^*\omega_{\nabla} = s^*\omega_{LC} + \eta.$$

- Conditions on η.
  1. $\eta \wedge s^*\theta$ (from the special condition);
  2. $\mathbf{i}\,\eta = -\eta\,\mathbf{i}$ (anti-complex condition);c
  3. ${}^t\eta\,\mathbf{s} = -\mathbf{s}\,\eta$ (special connection is symplectic)
  4. $X \lrcorner\, \eta = IX \lrcorner\, \eta = 0$ (η is $\{3,0\}$ totally symmetric).

Proposition

The cone (C, g_C, ω_C) over $S \subset \mathbf{R}\mathrm{H}(2)$ is conic special Kaehler iff $\lambda^2 = \frac{4}{3}$ or $\lambda^4 = 4$.

Hyperkaehler structure on $H = T^*C$ from the RIGID c-map

Step II.– HK metric on T^*C.

- Hyperkaehler metric:

$$g_H = \left(\hat{a}^2 + \hat{b}^2 - \hat{\varphi}^2 - \hat{\psi}^2\right) + \left(\hat{A}^2 + \hat{B}^2 - \hat{\Phi}^2 - \hat{\Psi}^2\right)$$

for $(\hat{A}, \hat{B}, \hat{\Phi}, \hat{\Psi}) = s^*\alpha = dx - xs^*\omega_{LC}$.

- Hyperkaehler structure:

$$\omega_I = \hat{a} \wedge \hat{b} - \hat{\varphi} \wedge \hat{\psi} - \hat{A} \wedge \hat{B} + \Phi \wedge \hat{\Psi}$$
$$\omega_J = \hat{A} \wedge \hat{a} + \hat{B} \wedge \hat{b} + \hat{\Phi} \wedge \hat{\varphi} + \hat{\Psi} \wedge \hat{\psi}$$
$$\omega_K = \hat{A} \wedge \hat{b} - \hat{B} \wedge \hat{a} + \hat{\Phi} \wedge \hat{\psi} - \hat{\Psi} \wedge \hat{\varphi}.$$

Elementary deformation of g_H

Step III.– Elementary deformations, twist data, etc.

- Deformation elements $\alpha_0, \ldots \alpha_K$:

$$\alpha_I = I\widetilde{X} \lrcorner g_H = -t\hat{\psi}, \qquad \alpha_0 = -I\alpha_I = -t\hat{\varphi}$$
$$\alpha_J = I\widetilde{X} \lrcorner g_H = -t\hat{\Phi}, \qquad \alpha_K = I\alpha_J = -t\hat{\Psi}$$

- Moment map:

$$\mu = \frac{1}{2}||\widetilde{X}||^2 = -\frac{t^2}{2}$$

- Deformed metric on $H = T^*C$:

$$\begin{aligned} g_N &= -\frac{1}{\mu}g_H + \frac{1}{\mu^2}(\alpha_0^2 + \alpha_I^2 + \alpha_J^2 + \alpha_K^2) \\ &= \frac{2}{t^2}(\hat{a}^2 + \hat{b}^2 + \hat{\varphi}^2 + \hat{\psi}^2 + \hat{A}^2 + \hat{B}^2 + \hat{\Phi}^2 + \hat{\Psi}^2) \end{aligned}$$

Twist data

- Twisting (curvature) 2-form:

$$F = -\hat{a} \wedge \hat{b} + \hat{} \wedge \hat{\psi} - \hat{A} \wedge \hat{B} + \hat{\Phi} \wedge \hat{\Psi}$$

- Twisting function

$$a = \mu = -\frac{t^2}{2}$$

Flat case $\lambda^2 = 4$.

Step IV.– Computing d_W

- The LC and the special connection coincide.

Flat case $\lambda^2 = 4$.

Step IV.– Computing d_W

- The LC and the special connection coincide.
- X-invariant coframe on C

$$\gamma = s^*\theta/t = (\tilde{a}, \tilde{b}, \varphi, \tilde{\psi})$$

(we can compute the twisted differentials immediatelly)

$$d_W \tilde{a} = 0 \qquad d_W \tilde{b} = 2\tilde{b} \wedge \tilde{a}$$

$$d_W \varphi = 2\tilde{a} \wedge \tilde{b} + \frac{2}{t^2} F \qquad d_W \tilde{\psi} = 0$$

Flat case $\lambda^2 = 4$.

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- In the vertical directions

$$\tilde{\delta} = (s^*\alpha)/t = (\tilde{A}, \tilde{B}, \tilde{\Phi}, \tilde{\Psi})$$

is NOT $\tilde{X}$-invariant.

$$\delta = \tilde{\delta} e^{\mathbf{i}\tau}$$

$$\epsilon = \frac{1}{2}(\delta_1 + \delta_4, \delta_2 - \delta_3, -\delta_2 - \delta_3, -\delta_1 + \delta_4)$$

$$d_W \epsilon = \epsilon \wedge \begin{pmatrix} \psi - \tilde{a} & 0 & 2\tilde{b} & 0 \\ 0 & \tilde{\psi} - \tilde{a} & 0 & -2\tilde{b} \\ 0 & 0 & \tilde{\psi} - \tilde{a} & 0 \\ 0 & 0 & 0 & \tilde{\psi} + \tilde{a} \end{pmatrix}$$

$$d_W \varphi = 2(\varphi \wedge \tilde{\psi} + \epsilon_{13} + \epsilon_4 \wedge \epsilon_2)$$

The resulting Lie algebra is isomorphic to the solvable algebra corresponding to the non-compact symmetric space

$$Gr_2^+(\mathbf{C}^{2,2}) = \frac{U(2,2)}{U(2) \times U(2)}$$

non-flat case $\lambda^2 = \frac{4}{3}$

- Write
$$(\hat{A}, \hat{B}, \hat{\Phi}, \hat{\Psi}) = dx - xs^*\omega_{LC} - x\eta.$$
- Then, g_H, g_N and the twist data F, a are the same.
- Twisted differentials for $\gamma = (\tilde{a}, \tilde{b}, \varphi, \tilde{\psi})$:

$$d_W \tilde{a} = 0 \qquad d_W \tilde{b} = -\frac{2}{3}\tilde{a} \wedge \tilde{b} \qquad d_W \tilde{\psi} = 0$$

$$d_W \varphi = 2(\varphi \wedge \tilde{\psi} - \tilde{A} \wedge \tilde{B} + \tilde{\Phi} \wedge \tilde{\Psi})$$

- Adjusting the vertical directionss we arrive to

$$d_W \epsilon = \epsilon \wedge \tilde{\psi} \mathbf{Id_4} + \frac{1}{\sqrt{3}} \epsilon \wedge \tilde{a} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$
$$+ \frac{2}{\sqrt{3}} \epsilon \wedge \tilde{b} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

- We see the structure of solvable algebra associated to

$$\frac{G_2^*}{SO(4)}$$

Thank you.

Bibliography

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- The general theory can be consulted in:
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