ELEMENTARY DEFORMATIONS AND THE HYPERKÄHLER / QUATERNIONIC KÄHLER CORRESPONDENCE

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ICMAT, Madrid

April 25, 2017

1 Planning

Introduction and motivation. The c-map

Differential-geometric construction of the c-map

Twist and HK/QK correspondence

Planning 1/3

Introduction and motivation. The c-map

3 Berger's List

Theorem

Let M be a Riemannian, oriented, simply-connected n-dimensional manifold, which is not locally a product, nor symmetric. Then its holonomy group belongs to the following list:

M. Berger (1955)

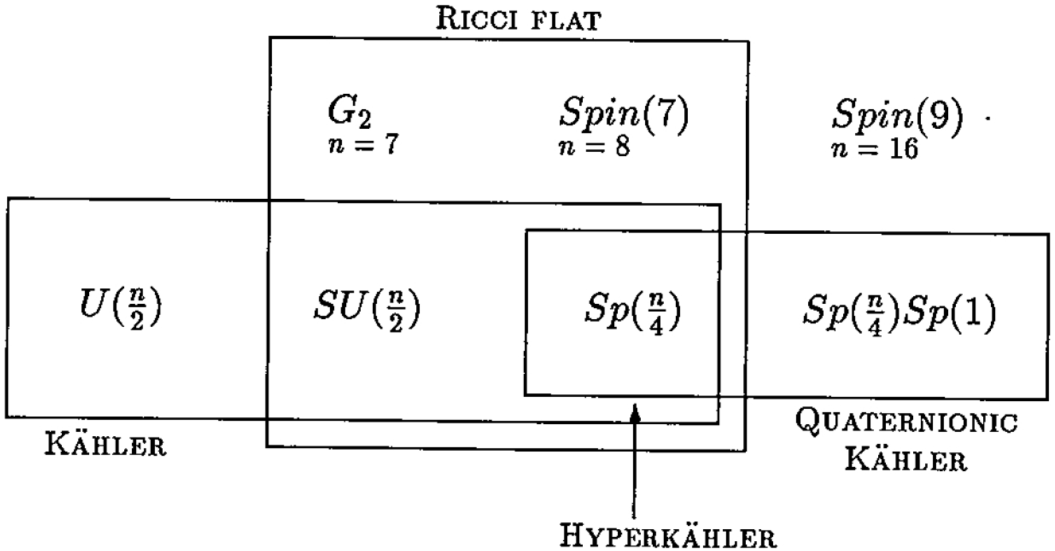


image credit: S.M. Salamon (1989)

4 Kähler, and Quaternionic Kähler Manifolds

$$(M^{2m}, g) + \{\mathrm{Hol} \subseteq \mathrm{U}(m)\} \Rightarrow \mathrm{K}$$

$$\mathrm{U}(m) = \mathrm{SO}(2m) \cap \mathrm{Sp}(m, \mathbb{R}) \subset \mathrm{GL}(m, \mathbb{C})$$

$$(M^{4k}, g) + \{\mathrm{Hol} \subseteq \mathrm{Sp}(k)\mathrm{Sp}(1)\} \Rightarrow \mathrm{QK}$$

$$\mathrm{Sp}(k)\mathrm{Sp}(1) \not\subset \mathrm{U}(m) \Rightarrow \mathrm{QK} \not\subset \mathrm{K}$$

$$\mathrm{Hol} \subsetneq \mathrm{Sp}(k)\mathrm{Sp}(1) \Rightarrow \begin{cases} -\mathrm{Hol} \subseteq \mathrm{Sp}(k) \subset \mathrm{Sp}(k)\mathrm{Sp}(1) \Rightarrow \mathrm{HK} \\ \quad \mathrm{Sp}(k) \subset \mathrm{U}(m) \Rightarrow \mathrm{HK} \subset \mathrm{K} \\ -M \text{ symmetric space} \end{cases}$$

5 Why QK?

Curvature

QK $\Rightarrow$ Einstein $\qquad$ QK $+ \{s = 0\} \Rightarrow$ HK

Wolf spaces

$$\mathbb{H}\mathrm{P}^n, \quad \mathrm{Gr}_2(\mathbb{C}^{n+2}), \quad \mathrm{Gr}_4(\mathbb{R}^{n+4})$$

$$\frac{\mathrm{G}_2}{\mathrm{SO}(4)}, \ \frac{\mathrm{F}_4}{\mathrm{Sp}(3)\mathrm{Sp}(1)}, \ \frac{\mathrm{E}_6}{\mathrm{SU}(6)\mathrm{Sp}(1)}, \ \frac{\mathrm{E}_7}{\mathrm{Spin}(12)\mathrm{Sp}(1)}, \ \frac{\mathrm{E}_8}{\mathrm{E}_7\mathrm{Sp}(1)}$$

(A) LeBrun-Salamon Conjecture: $\{\mathrm{QK} + s > 0\} \Rightarrow$ **Wolf**

Alekseevsky spaces

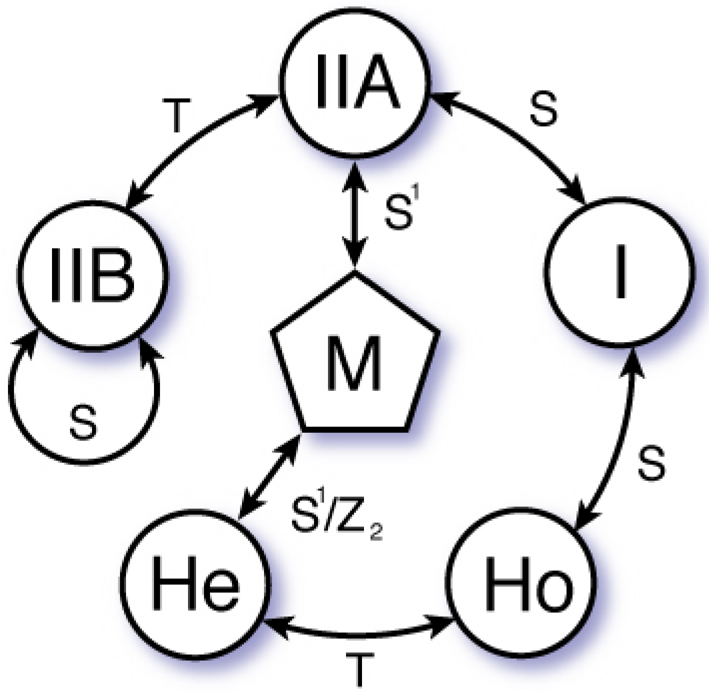
$\exists$ Homogeneous, non-symmetric QK, with $s < 0$.

(B) Find explicit examples of non-compact QK manifolds

6 Enter the physics

> *"The mathematical problems that have been solved or techniques that have arisen out of physics in the past have been the lifeblood of mathematics."*
>
> Sir Michael F. Atiyah *Collected Works Vol. 1 (1988), 19, p.13*

M-Theory:



7 c-map

c-map

$$K_1 \times QK_2 \times \mathbb{C}H(1) \longrightarrow K_2 \times QK_1 \times \mathbb{C}H(1)$$

$$K^{2m} \xrightarrow{\text{c-map}} QK^{4(m+1)}$$

Cecotti, Ferrara & Girardello (1989), Ferrara & Sabharwal (1990)

Rigid c-map

$$K^{2m} \xrightarrow{\text{rigid c-map}} HK^{4m}$$

Alekseevsky (1975), Cecotti, Ferrara & Girardello (1989), Ferrara & Sabharwal (1990), de Witt & Van Proeyen (1992), Cortes (1996).

Field content of $N = 2,\ D = 4$ SIMPLE SUGRA

$$a = 0, \ldots, 3, \qquad \hat{\mu} = 0, \ldots 3, \qquad \Lambda = 1, 2$$

Gravity supermultiplet

$$(V^a_{\hat{\mu}},\ \psi^{\Lambda},\ A^0_{\hat{\mu}})$$

m Vector supermultiplets

$$(A^i_{\hat{\mu}},\ \lambda^{i\,\Lambda},\ \phi^i), \qquad i = 1, \ldots, m$$

ϕ^i

m $\mathbb{C}$-scalars $\cong$ **2m** $\mathbb{R}$-scalars

Coordinates on a $2m$-dimensional, PSK Σ-model

9 $4D \to 3D$ Kaluza–Klein compactification, I

$$\hat{\mu} = 0, \dots 3 \longmapsto (0, \mu), \;\; \mu = 1, 2, 3$$

Metric

$$\eta_{ab} V^a_{\hat{\mu}} V^b_{\hat{\nu}} = g_{\hat{\mu}\hat{\nu}} \overset{\text{KK}}{=} \left(\begin{array}{c|c} e^{2\sigma} & e^{2\sigma} A_\nu \\ \hline e^{2\sigma} A_\mu & e^{i\sigma} g_{\mu\nu} + e^{2\sigma} A_\mu A_\nu \end{array} \right)$$

4-vectors

$$A^0_{\hat{\mu}} \overset{\text{KK}}{=} (A^0_0, A^0_\mu) \equiv (\zeta_0, A^0_\mu)$$
$$A^i_{\hat{\mu}} \overset{\text{KK}}{=} (A^i_0, A^i_\mu) \equiv (\zeta_i, A^i_\mu)$$

$\sigma, \; \zeta_0, \; \zeta_i$ — $\mathbf{m+2}$ extra scalar fields

$A_\mu, \; A^0_\mu, \; A^i_\mu$ — $\mathbf{m+2}$ extra $3D$-vector vields.

$4D \to 3D$ Kaluza–Klein compactification, II

Dualization of $3D$-vector fields

$$(A_\mu,\ A^0_\mu,\ A^i_\mu) \xrightarrow{Duality} (a, \widetilde{\zeta}_0,\ \widetilde{\zeta}_i)$$

$a, \widetilde{\zeta}_0,\ \widetilde{\zeta}_i$ $\mathbf{m+2}$ extra scalar fields.

11 **SUGRA** $N = 2, D = 4$

Kaluza-Klein 4D→3D

Gravity supermultiplet

$4D$	#s	$3D$	#s	*	#s
$V_{\hat{\mu}}^{i}$ ψ^{Λ} $A_{\hat{\mu}}^{0}$		$V_{\mu}^{i},\ A_{\mu},\ e^{2\sigma}$ ψ^{Λ} $A_{0}^{0},\ A_{\mu}^{0}$	**1** **1**	$V_{\mu}^{i},\ a,\ \phi$ ψ^{Λ} $\zeta^{0},\ \widetilde{\zeta}^{0}$	**2** **2**

m Vector supermultiplet

$4D$	#s	$3D$	#s	*	#s
$A_{\hat{\mu}}^{i}$ λ_{Λ}^{i} ϕ^{i}	 $2\mathbf{m}, \mathbb{R}$	$A_{0}^{i},\ A_{\mu}^{i}$ λ_{Λ}^{i} ϕ^{i}	**m** **2m**	$\zeta^{i},\ \widetilde{\zeta}^{i}$ λ_{Λ}^{i} ϕ^{i}	**2m** **2m**

Planning 2/3

Introduction and motivation: The c-map.

Differential-Geometric construction of the c-map

Quaternionic Kähler Moduli Spaces

Nigel Hitchin

Abstract We describe in differential-geometric language a class of naturally occurring quaternionic Kähler moduli spaces due originally to the physicists Ferrara and Sabharwal. This class yields an example in real dimension $4n$ for every projective special Kähler manifold of real dimension $2n-2$ and can be applied in particular to the case of the moduli space of complex structures on a Calabi–Yau threefold.

1 Introduction

The study of quaternionic Kähler orbifolds of positive scalar curvature is equivalent to the study of 3-Sasakian manifolds, and the procedure of quaternionic Kähler reduction gives many examples of these, starting from a finite-dimensional quaternionic projective space (see [3]). But whereas hyperkähler reduction has been used

13 Special Kähler manifolds, I

Special Kähler manifolds, SK

$$\mathrm{SK} = (\mathrm{K},\ \nabla^s)$$

1. $\nabla^s \omega = 0$,
2. $\mathrm{R}(\nabla^s) = \mathrm{T}(\nabla^s) = 0$,
3. $\nabla^s_X IY = -\nabla^s_Y IX$ ("special condition")

Conic special Kähler manifolds, CSK

$$\mathrm{CSK} = (\mathrm{SK}, X) \qquad X \in \mathfrak{X}M$$

1. $g(X, X) \neq 0$;
2. $\nabla^s X = -I = \nabla^g X$.

Special Kähler manifolds, II

* Moment mapping

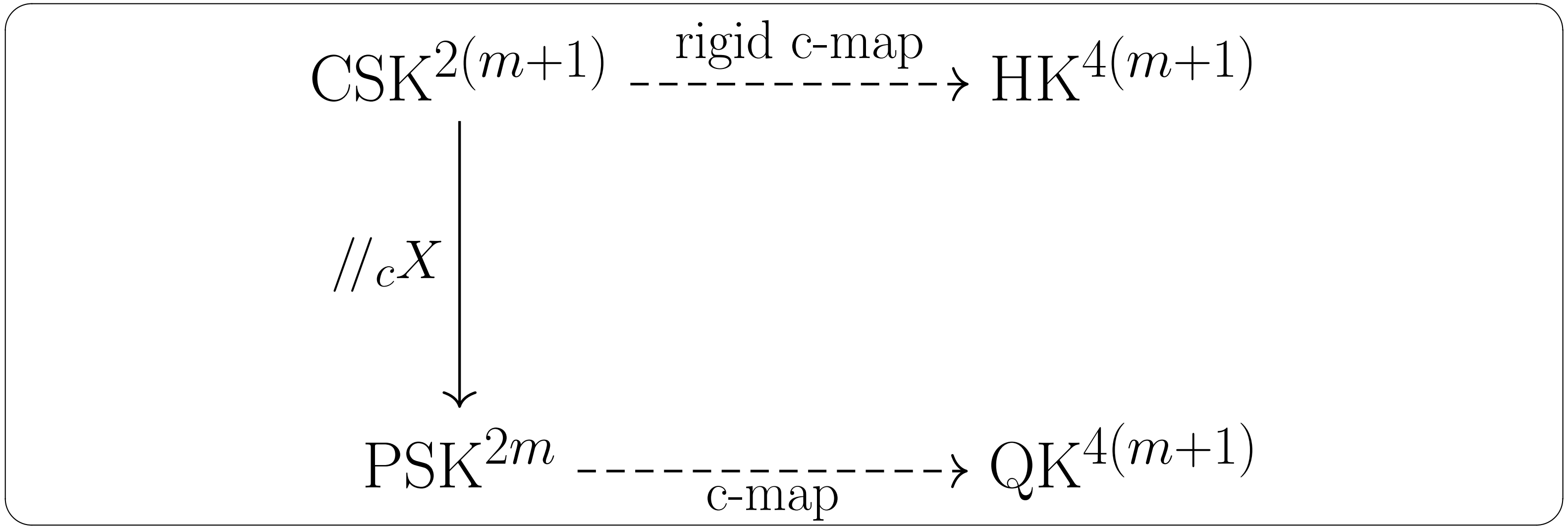
$$\mu : M \to \mathbb{R} : p \mapsto \frac{1}{2}||X_p||^2$$

Projective special Kähler manifolds, PSK

$$\mathrm{PSK} = \mathrm{CSK}//_c X = \mu^{-1}(c)/X \qquad c \in \mathbf{R}$$

$$\begin{array}{ccc} \mathrm{CSK}_0 = \mu^{-1}(c) & \longrightarrow & \mathrm{CSK} \\ \downarrow & & \\ \mathrm{CSK}//_c X = \mathrm{PSK} & & \end{array}$$

15 General picture arising from physics



16 Flat model: HK vector spaces, I

$$V = \mathbb{C}^{p,q} \cong (\mathbb{R}^{2m},\ \mathbf{G},\ \mathbf{i}), \quad m = p+q$$

$$\mathbf{G} = diag\left(\mathrm{id}_{2p}, -\mathrm{id}_{2q}\right),\ \mathbf{i} = diag\left(\mathbf{i}_2, \ldots, \mathbf{i}_2\right),\ \mathbf{i}_2 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$$

Canonical forms on $V,\ V^*$

$$T_xV \cong V = \mathbf{R}^{2m}$$

$$\theta = (dx_1, dy_1, \ldots, dx_m, dy_m)^T \in \Omega^1(V, \mathbf{R}^{2m})$$

$$T_xV^* \cong V^* = (\mathbf{R}^{2m})^*$$

$$\alpha = (du_1, dv_1, \ldots, du_m, dv_m) \in \Omega^1(V^*, (\mathbf{R}^{2m})^*)$$

Kähler forms on $V,\ V^*$

$$\omega = -\frac{1}{2}\theta^T \wedge \mathbf{s}\theta, \qquad \omega^* = -\frac{1}{2}\alpha \wedge \mathbf{s}\alpha^T \qquad (\mathbf{s} = \mathbf{G}\mathbf{i})$$

17 Flat model: HK vector spaces, II

$$T^*V = V + V^*$$

HK structure

$$\begin{aligned} \omega_J &= \quad du_i \wedge dx_i + dv_i \wedge dy_i = \quad \alpha \wedge \theta \\ \omega_K &= \quad du_i \wedge dy_i - dv_i \wedge dx_i = \quad -\alpha \wedge \mathbf{i}\theta \\ \omega_I &= \quad \omega - \omega^* = \quad \frac{1}{2}(\alpha \wedge \mathbf{s}\alpha^T - \theta^T \wedge \mathbf{s}\theta) \end{aligned}$$

$$g_{\mathrm{HK}} = \varepsilon_i(dx_i^2 + dy_i^2 + du_i^2 + dv_i^2) \qquad (\varepsilon_i = \mathbf{G}_{ii})$$

$$Idx_i = dy_i, \quad Idu_i = -dv_i$$

$$Jdx_i = -du_i, \quad Jdv_i = dy_i$$

$$Kdx_i = dv_i, \quad Kdu_i = dy_i$$

HK structure on the cotangent bundle

$$T^*M = \mathrm{GL}(M) \times_{\mathrm{GL}(2m,\mathbb{R})} (\mathbb{R}^{2m})^*$$

$$\theta \in \Omega^1(\mathrm{GL}(M), \mathbb{R}^{2m}), \qquad \omega_\nabla \in \Omega^1(\mathrm{GL}(M), \mathrm{End}(\mathbb{R}^{2m}))$$

$$\alpha = dx - x\omega_\nabla \in \Omega^1(\mathrm{GL}(M) \times (\mathbb{R}^{2m})^*, (\mathbb{R}^{2m})^*)$$

Lemma

$\omega_J = \alpha \wedge \theta$ iff ω_∇ is torsion-free

* (M, I) : I acs

$$T^*M \cong \Lambda^{1,0}M = \mathrm{GL}(\mathbb{C}, M) \times_{\mathrm{GL}(m,\mathbb{C})} (\mathbb{C}^m)^*$$

$$\omega = \omega_J + i\omega_K \in \Lambda^{2,0}M$$

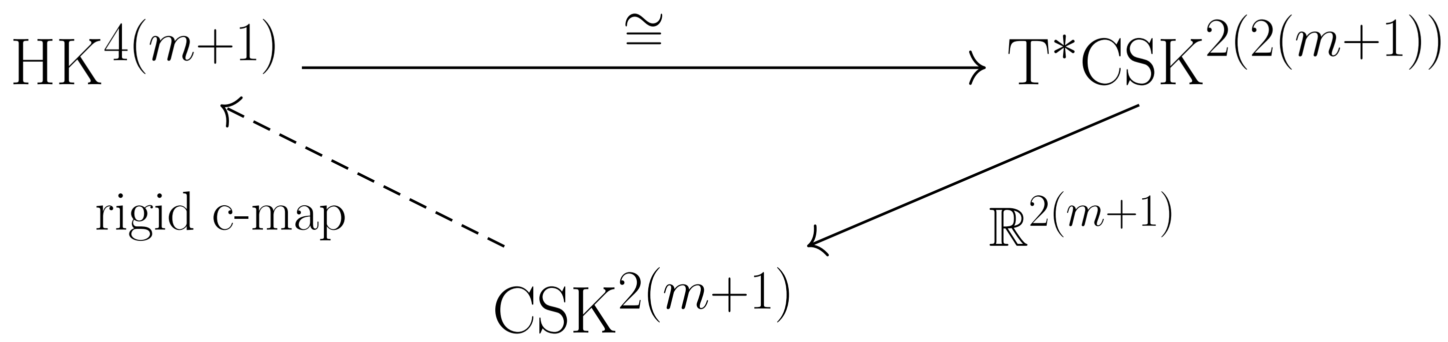
Lemma

Suppose ω_∇ is torsion-free. $\omega_K = -\alpha \wedge \mathbf{i}\theta$ iff the acs I on M is integrable and (∇, I) satisfies the 'special condition'.

19 Meaning of the SK condition

Proposition

The two-forms $\omega_I = \frac{1}{2}(\alpha \wedge \mathbf{s}\alpha^T - \theta^T \wedge \mathbf{s}\theta)$, ω_J, ω_K on T^*M give a HK structure compatible with the standard complex symplectic structure iff (M, I, g, ∇) is SK.

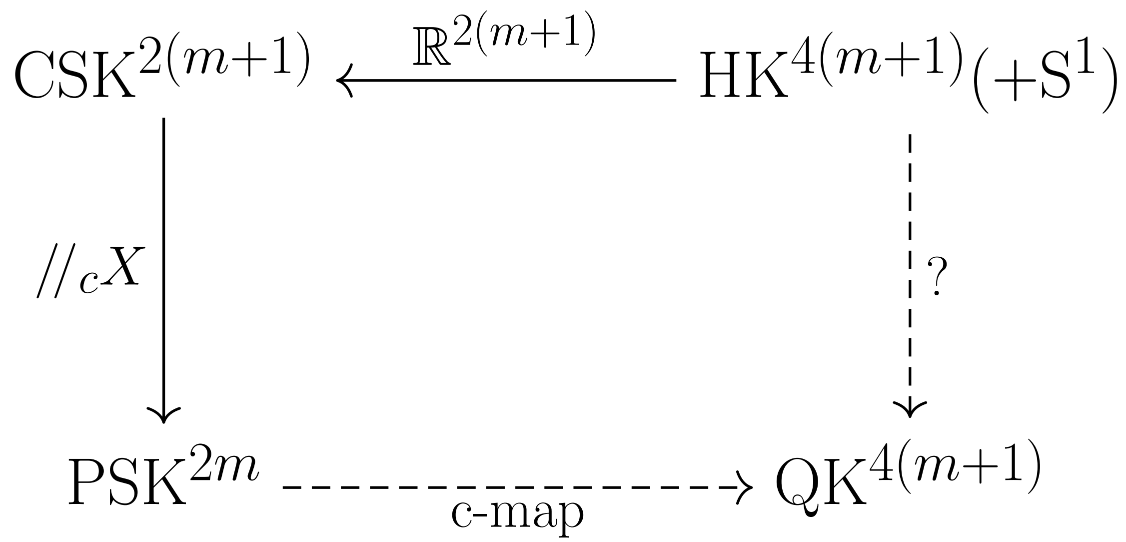


Effect of the conic hypothesis

Proposition

Let $\widetilde{X}$ be the horizontal lift of the conic isometry X to $T^*(\mathrm{CSK})$ Then $\widetilde{X}$ is an isometry of $T^*(\mathrm{CSK})$ and

$$L_{\widetilde{X}}\omega_I = 0, \qquad L_{\widetilde{X}}\omega_J = \omega_K, \qquad L_{\widetilde{X}}\omega_K = -\omega_J$$



Planning 3/3

Introduction and Motivation. The c-map.

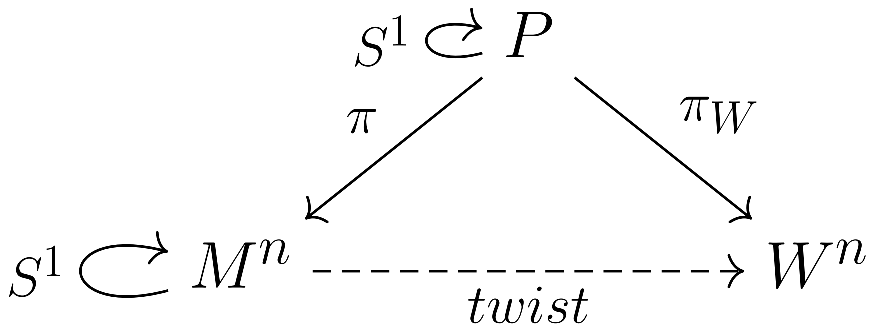
Differential-Geometric construction of the c-map.

Twist and HK/QK correspondence

The Twist construction (sketch)

The twsit construction associates to a manifold M with a S^1-action, a new space W of the same dimension, with a distinguished vector field.

This construction fits into a double fibration



so W is M twisted by the S^1-bundle P.

A. Swann, (2007, 2010)

Twists & HK/QK correspondence

1992, 2001

Instanton twists (Hypercomplex, Quaternionic, HKT)

D.Joyce (1992), G.Grantcharov & Y.S. Poon (2001)

2007, 2010

General twists (T-duality, HKT, KT, SKT, ...)

A.F. Swann (2007, 2010)

2008

HK/QK correspondence

$\{\mathrm{HK} + \text{symmetry fixing one } \omega\} \Leftrightarrow \{\mathrm{QK} + \text{circle action}\}$

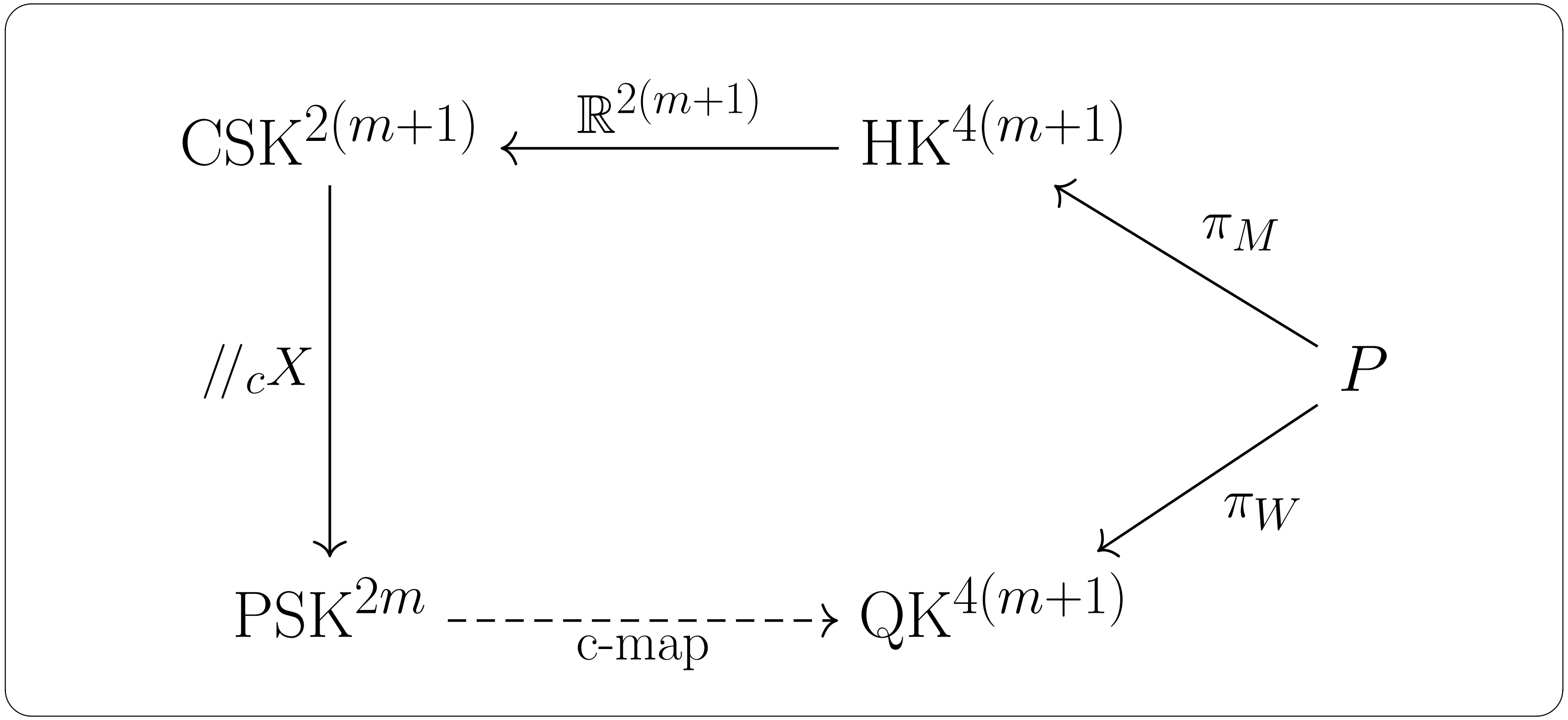
A. Haydys (2008)

2013

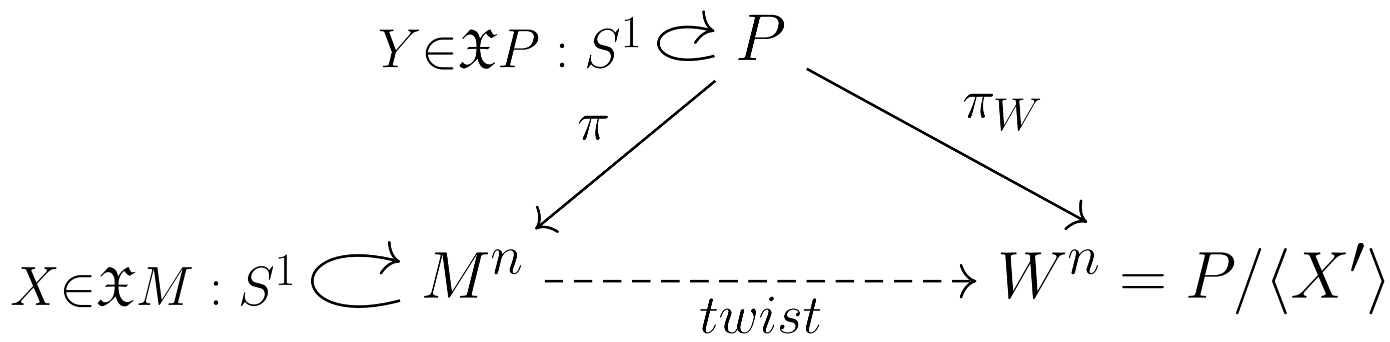
Twistor interpretation

N.J. Hitchin (2013)

Idea


$\mathrm{CSK}^{2(m+1)}$
$\mathbb{R}^{2(m+1)}$
$\mathrm{HK}^{4(m+1)}$
π_M
$//_c X$
P
π_W
PSK^{2m}
$\mathrm{QK}^{4(m+1)}$
c-map

The Twist construction (in detail)



A. Swann, (2007, 2010)

1. $P(M, S^1)$: connection θ, curvature $\pi_M^* F = d\theta$.
2. $L_X F = 0$
3. $X' = X^\theta + aY \in \mathfrak{X}P$: such that $L_{X'}\theta = L_{X'}Y = 0$.
4. $W = P/\langle X' \rangle$ with induced action by Y.

Twist data

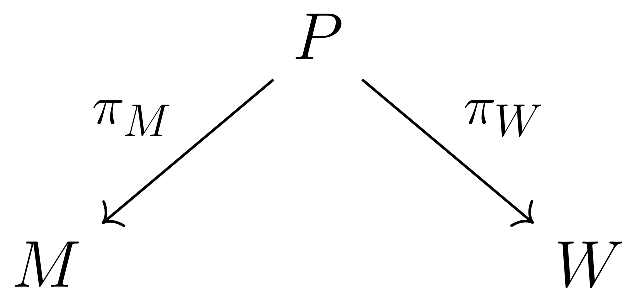
$$(M, X, F, a) \Longrightarrow \quad M \xleftarrow{\pi_M} P \xrightarrow{\pi_W} W$$

Twist data

1) M, a C^∞ manifold.
2) $X \in \mathfrak{X}M$, generating the S^1-action.
3) $F \in \Omega^2 M$, X-invariant, with integral periods.
4) $a \in C^\infty M$ such that

$$\mathrm{d}a = -X \lrcorner F$$

27 $\mathcal{H}$-related tensors



$$T_pP = \mathcal{H}_p + \mathcal{V}_p$$
$$\mathcal{H}_p \cong T_{\pi(p)}M \cong T_{\pi_W(p)}W$$

$\sim_{\mathcal{H}}$

$$\alpha \in \mathbf{T}M, \qquad \alpha_W \in \mathbf{T}W$$
$$\alpha \sim_{\mathcal{H}} \alpha_W \iff (\pi_M^*\alpha)|_{\mathcal{H}} = (\pi_W^*\alpha_W)|_{\mathcal{H}}$$

Lemma

$$\alpha \in \Omega^p M^X \Rightarrow \exists! \alpha_W \in \Omega^p W \ : \ \alpha_W \sim_{\mathcal{H}} \alpha$$
$$\pi_W^*\alpha_W = \pi^*\alpha - \theta \wedge \pi^*(a^{-1} X \lrcorner \alpha)$$

Computing the Twist

$d\alpha_W$

$\alpha \in \Omega^p M^X$,

$$d\alpha_W \sim_{\mathcal{H}} d\alpha - \frac{1}{a} F \wedge (X \lrcorner\, \alpha)\,.$$

Twisted exterior differential, d_W

$$\alpha_W \sim_{\mathcal{H}} \alpha, \qquad d\alpha_W \sim_{\mathcal{H}} d_W \alpha$$

$$d_W := d - \frac{1}{a} F \wedge X \lrcorner$$

29 Twist vs Integrability

Complex case

Let I an invariant complex structure on M, $\mathcal{H}$-related to an almost-complex structure I_W on W, I_W is integrable iff $F \in \Omega^{1,1}_I M$.

Kähler case K $\to \mathbb{C}$

$$\mathbb{CP}^n \times T^2 \xleftarrow{\pi_M} S^{2n+1} \times T^2 \xrightarrow{\pi_W} S^{2n+1} \times S^1$$

30 Need for a deformation

Problem

$$d\alpha = 0 \not\to d\alpha_W = 0$$

$$d\alpha_W \sim d_W\alpha = d\alpha - \frac{1}{a}F \wedge (X \lrcorner\, \alpha) = -\frac{1}{a}F \wedge (X \lrcorner\, \alpha) \neq 0$$

31 Simetríes of the structure $\mathrm{HK} = (M, g, I, J, K)$

Rotating symmetry

$X \in \mathfrak{X}M$

1. $L_X(g) = 0.$

2. $L_X(I) = \langle I, J, K \rangle$

3. $(a)\ L_X I = 0, \quad (b)\ L_X J = K, \quad (c)\ L_X K = -J$

Elementary deformations of the HK metric

g_α

$$\mathbb{H}X = \langle X, IX, JX, KX \rangle$$

$$\alpha_0 = g(X, \cdot), \qquad \alpha_A = -g(AX, \cdot) \;\; (A = \mathrm{I}, \mathrm{J}, \mathrm{K})$$

$$g_\alpha := \alpha_0^2 + \sum_A \alpha_A^2 \equiv g|_{\mathbb{H}X}$$

g^N, Elementary deformation of the metric

$$g^N = fg + hg_\alpha, \qquad f, g \in \mathcal{C}^\infty M$$

33 Uniqueness

Theorem

(M^{4k}, g, I, J, K) — HK, $k \geq 2$

$X \in \mathfrak{X}M$ — Rotating symmetry

μ — Moment mapping

∃! Elementary deformation

$$g^N = -\frac{1}{\mu - c} g + \frac{1}{(\mu - c)^2} g_\alpha$$

∃! Twist data

$$F = kG = k(d\alpha_0 + \omega_I), \qquad a = k(g(X, X) - \mu + c).$$

W — QK

O.Macia, A.F.Swann (2015)

Idea of proof

1. From g^N and (I, J, K), construct ω_A^N y Ω^N.
2. Impose arbitrary twist of Ω^N to be QK.
3. Decompose equation with respect to the splitting $TM = \langle \mathbb{H}X \rangle \oplus \langle \mathbb{H}X \rangle^{\perp}$.
4. All this leads to $f = f(\mu),\ h = h(\mu)$ y $h = f'$.
5. Impose $da = -X \lrcorner F$ to determine a.
6. Imposing $dF = 0$ leads to ODE's which determine f.

MUCHAS GRACIAS

Planning 4/3 (!)

Introduction and Motivation. The c-map.

Interlude: Differential-Geometric construction of the c-map.
(pre-twist version)

Twist and HK/QK correspondence

Bonus Track: A worked-out example

$\mathbb{R}$H(2)

- $\mathbb{C}$H(1) : Real, 2-dimensional solvable Lie group with Kähler metric of constant curvature.

- Local basis of one-forms on $S \subset \mathbb{C}\mathrm{H}(1)$: $\{a, b\} \in \Omega^1_S$:

$$da = 0, \quad db = -\lambda a \wedge b$$

- Metric:

$$g_S = a^2 + b^2$$

- Almost complex structure:

$$Ia = b$$

- Kähler two-form:

$$\omega_S = a \wedge b.$$

Local cone structure

- The PSK S is the Kähler quotient of a CSK manifold $C \equiv (C, g, \omega, \nabla, X)$

$$\begin{array}{ccc} C_0 & \xrightarrow{i} & C \\ \Big\downarrow \pi & & \\ S & & \end{array} \qquad\qquad S = C//_c X$$

- Locally $C = \mathbf{R}_{>0} \times C_0$.
- C_0 is the level set $\mu^{-1}(c)$ for the moment map of X.
- $C_0 \to S$ is a bundle with connection 1-form φ :

$$d\varphi = 2\pi^* \omega_S$$

38 Metric and Kähler form on C

- Write t for the standard coordinate on $\mathbf{R}_{>0}$ and $\hat{\psi} = dt$.
- Write $\hat{a} = t\pi^*a,\ \hat{b} = t\pi^*b,\ \hat{\varphi} = t\varphi$

Lemma

The $(2,2)$ pseudo–Riemannian metric and the Kähler form of $C = \mathbf{R}_{>0} \times C_0$ are

$$g_C = \hat{a}^2 + \hat{b}^2 - \hat{\varphi}^2 - \hat{\psi}^2 = -dt^2 + t^2 g_{C_0}$$

$$\omega_C = \hat{a} \wedge \hat{b} - \hat{\varphi} \wedge \hat{\psi}.$$

The conic isometry satisfies

$$IX = t\frac{\partial}{\partial t}$$

39 LC connection

- Coframe $s^*\theta = (\hat{a}, \hat{b}, \hat{\varphi}, \hat{\psi})$

Exterior differential $d(s^*\theta)$

$$d\hat{a} = \frac{1}{t} dt \wedge \hat{a} \qquad d\hat{b} = \frac{1}{t}(dt \wedge \hat{b} - \lambda \hat{a} \wedge \hat{b})$$

$$d\hat{\varphi} = \frac{1}{t}(dt \wedge \hat{\varphi} + 2\hat{a} \wedge \hat{b}) \quad d\hat{\psi} = 0$$

LC connection $s^*\omega_{LC}$

$$s^*\omega_{LC} = \frac{1}{t} \begin{pmatrix} 0 & \hat{\varphi} + \lambda \hat{b} & \hat{b} & \hat{a} \\ -\hat{\varphi} - \lambda \hat{b} & 0 & -\hat{a} & \hat{b} \\ \hat{b} & -\hat{a} & 0 & \hat{\varphi} \\ \hat{a} & \hat{b} & -\hat{\varphi} & 0 \end{pmatrix}$$

Curvature 2-form

Curvature 2-form $s^*\Omega_{LC}$

$$s^*\Omega_{LC} = \frac{4-\lambda^2}{t^2}\begin{pmatrix} 0 & \hat{a}\wedge\hat{b} & 0 & 0 \\ -\hat{a}\wedge\hat{b} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Lemma

The pseudo–Riemannian metric g_C is flat iff $\lambda^2 = 4$.

Conic special Kähler condition

Symplectic connection in terms of LC connection

$$s^*\omega_{\nabla} = s^*\omega_{LC} + \eta.$$

Conditions on η.

1. $\eta \wedge s^*\theta$.

2. $\mathbf{i}\,\eta = -\eta\,\mathbf{i}$.

3. ${}^t\eta\,\mathbf{s} = -\mathbf{s}\,\eta$.

4. $X \lrcorner\, \eta = IX \lrcorner\, \eta = 0$.

Proposition

The cone (C, g_C, ω_C) over $S \subset \mathbf{R}\mathrm{H}(2)$ is CSK iff $\lambda^2 = \frac{4}{3}$ or $\lambda^2 = 4$.

HK structure on T*CSK

HK metric

$$g_{HK} = \left(\hat{a}^2 + \hat{b}^2 - \hat{\varphi}^2 - \hat{\psi}^2\right) + \left(\hat{A}^2 + \hat{B}^2 - \hat{\Phi}^2 - \hat{\Psi}^2\right)$$

HK structure

$$\begin{aligned}
\omega_I &= \hat{a} \wedge \hat{b} - \hat{\varphi} \wedge \hat{\psi} - \hat{A} \wedge \hat{B} + \Phi \wedge \hat{\Psi} \\
\omega_J &= \hat{A} \wedge \hat{a} + \hat{B} \wedge \hat{b} + \hat{\Phi} \wedge \hat{\varphi} + \hat{\Psi} \wedge \hat{\psi} \\
\omega_K &= \hat{A} \wedge \hat{b} - \hat{B} \wedge \hat{a} + \hat{\Phi} \wedge \hat{\psi} - \hat{\Psi} \wedge \hat{\varphi}.
\end{aligned}$$

43 Elementary deformation of g_{HK}

$\alpha_0, \dots \alpha_K$

$$\alpha_I = I\widetilde{X} \lrcorner g_H = -t\hat{\psi}, \qquad \alpha_0 = -I\alpha_I = -t\hat{\varphi}$$
$$\alpha_J = I\widetilde{X} \lrcorner g_H = -t\hat{\Phi}, \qquad \alpha_K = I\alpha_J = -t\hat{\Psi}$$

μ

$$\mu = \frac{1}{2}||\widetilde{X}||^2 = -\frac{t^2}{2}$$

Elementary deformation

$$\begin{aligned} g_N &= -\frac{1}{\mu} g_H + \frac{1}{\mu^2}(\alpha_0^2 + \alpha_I^2 + \alpha_J^2 + \alpha_K^2) \\ &= \frac{2}{t^2}(\hat{a}^2 + \hat{b}^2 + \hat{\varphi}^2 + \hat{\psi}^2 + \hat{A}^2 + \hat{B}^2 + \hat{\Phi}^2 + \hat{\Psi}^2) \end{aligned}$$

Twist data

Twisting two-form

$$F = -\hat{a} \wedge \hat{b} + \hat{\varphi} \wedge \hat{\psi} - \hat{A} \wedge \hat{B} + \hat{\Phi} \wedge \hat{\Psi}$$

Twisting function

$$a = \mu = -\frac{t^2}{2}$$

45 Flat case $\lambda^2 = 4$.

- The LC and the special connection coincide.
- X-invariant coframe on CSK

$$\gamma = s^*\theta/t = (\tilde{a}, \tilde{b}, \varphi, \tilde{\psi})$$

(we can compute the twisted differentials immediatelly)

$$\begin{array}{ll} d_W\tilde{a} = 0 & d_W\tilde{b} = 2\tilde{b} \wedge \tilde{a} \\ d_W\varphi = 2\tilde{a} \wedge \tilde{b} + \frac{2}{t^2}F & d_W\tilde{\psi} = 0 \end{array}$$

- The (vertical) coframe

$$\tilde{\delta} = (s^*\alpha)/t = (\tilde{A}, \tilde{B}, \tilde{\Phi}, \tilde{\Psi})$$

is NOT $\tilde{X}$-invariant.

$$\delta = \tilde{\delta} e^{\mathbf{i}\tau}$$

$$\epsilon = \frac{1}{2}(\delta_1 + \delta_4, \delta_2 - \delta_3, -\delta_2 - \delta_3, -\delta_1 + \delta_4)$$

$$d_W \epsilon = \epsilon \wedge \begin{pmatrix} \psi - \tilde{a} & 0 & 2\tilde{b} & 0 \\ 0 & \tilde{\psi} - \tilde{a} & 0 & -2\tilde{b} \\ 0 & 0 & \tilde{\psi} - \tilde{a} & 0 \\ 0 & 0 & 0 & \tilde{\psi} + \tilde{a} \end{pmatrix}$$

$$d_W \varphi = 2(\varphi \wedge \tilde{\psi} + \epsilon_{13} + \epsilon_4 \wedge \epsilon_2)$$

The resulting Lie algebra is isomorphic to the non-compact symmetric space

$$Gr_2^+(\mathbf{C}^{2,2}) = \frac{U(2,2)}{U(2) \times U(2)}$$

46 Non-flat case $\lambda^2 = \frac{4}{3}$

- Same $g_{HK},\ g_N$ and same twist data F, a.

- Twisted differentials for $\gamma = (\tilde{a}, \tilde{b}, \varphi, \tilde{\psi})$:

$$d_W \tilde{a} = 0 \qquad d_W \tilde{b} = -\frac{2}{3}\tilde{a} \wedge \tilde{b} \qquad d_W \tilde{\psi} = 0$$

$$d_W \varphi = 2(\varphi \wedge \tilde{\psi} - \tilde{A} \wedge \tilde{B} + \tilde{\Phi} \wedge \tilde{\Psi})$$

- Adjusting the vertical coframe we arrive to

$$d_W\epsilon = \epsilon\wedge\tilde{\psi}\mathbf{Id_4} + \frac{1}{\sqrt{3}}\epsilon\wedge\tilde{a}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

$$+\frac{2}{\sqrt{3}}\epsilon\wedge\tilde{b}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We see the structure of solvable algebra associated to

$$\frac{G_2^*}{SO(4)}$$

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