MODULI OF HOLOMORPHIC ISOMETRIC EMBEDDINGS OF $\mathbb{C}\mathrm{P}^1$ INTO QUADRICS

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1 Geometry of Grassmannians

W : $\mathbb{R}$ or $\mathbb{C}$ vector space of dimension N

$\mathrm{Gr}_p(W)$: Grassmannian of p-planes in W

Exact sequence of bundles

$$0 \longrightarrow S \xrightarrow{i_S} \underline{W} \xrightarrow{\pi_Q} Q \longrightarrow 0$$

$\underline{W} \to \mathrm{Gr}_p(W)$: $\mathrm{Gr}_p(W) \times W \to \mathrm{Gr}_p(W)$

$S \to \mathrm{Gr}_p(W)$: Tautological bundle over $\mathrm{Gr}_p(W)$.

$Q \to \mathrm{Gr}_p(W)$: Universal quotient bundle.

2 Induced fibre metrics

Fix an inner product ($\mathbb{R}$) or a Hermitian product ($\mathbb{C}$) on W.

$$0 \longrightarrow S \underset{\pi_S}{\overset{i_S}{\rightleftarrows}} \underline{W} \underset{i_Q}{\overset{\pi_Q}{\rightleftarrows}} Q \longrightarrow 0$$

$S \to \mathrm{Gr}_p(W)$: fibre metric g_S.

$Q \to \mathrm{Gr}_p(W)$: fibre metric g_Q.

3 Connections and Second Fundamental Forms

$$s \in \Gamma(S) \Rightarrow i_S(s) \in \Gamma(\underline{W}) \Rightarrow di_S(s) \in \Omega^1(\underline{W})$$
$$di_S(s) = \pi_S di_S(s) + \pi_Q di_S(s) = \nabla^S s + Hs$$

Connection on $S \to \mathrm{Gr}_p(W)$

$$\nabla^S = \pi_S di_S \in \Omega^1(\mathrm{Hom}(S, S))$$

2nd fundamental form of $S \to \mathrm{Gr}_p(W)$

$$H = \pi_Q di_S \in \Omega^1(\mathrm{Hom}(S, Q))$$

$\ldots \nabla^Q,\ K \ldots$

4 Pull-backs

(M, g) Riemannian manifold (RM)

$(\mathrm{Gr}_p(W), g_{Gr})$ Grassmannian as RM.

$$f : M \longrightarrow \mathrm{Gr}_p(W)$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U & \xrightarrow{i_U} & \underline{W}' & \xrightarrow{\pi_V} & V & \longrightarrow & 0 \\
 & & {\scriptstyle f^*}\uparrow & & \downarrow{\scriptstyle f} & & {\scriptstyle f^*}\uparrow & & \\
0 & \longrightarrow & S & \xrightarrow[i_S]{} & \underline{W}'' & \xrightarrow[\pi_Q]{} & Q & \longrightarrow & 0
\end{array}$$

$\underline{W}' \to M$ $\qquad M \times W \to M$

$\underline{W}'' \to \mathrm{Gr}_p(W)$ $\qquad \mathrm{Gr}_p(W) \times W \to \mathrm{Gr}_p(W)$

5 Mean curvature operator

Definition

The bundle homomorphism $A \in \Gamma(\mathrm{Hom}\, V)$ defined as

$$A := \sum_{i=1}^{n} H^U_{e_i} \circ K^V_{e_i},$$

where $e_1, \cdots, e_n$ is an orthonormal basis of $T_x M$ is called the *mean curvature operator of* f.

(EH) Einstein–Hermitian condition

$$A = -\mu\, \mathrm{id}_V, \qquad \mu \in \mathbb{R}_+$$

6 Evaluation & globally generated vector bundles

$V \to M$ — VB

$W \subset \Gamma(V)$ — finite-dimensional vector space

$\underline{W}$ — $M \times W \to M$

Evaluation homomorphism

$$\mathrm{ev} : \underline{W} \longrightarrow V$$

$$\mathrm{ev}_x(t) = t(x) \in V_x, \ t \in W, x \in M$$

Definition

The vector bundle $V \to M$ is said to be *globally generated by* W if $\mathrm{ev} : \underline{W} \to V$ is surjective.

7 Map to a Grassmannian induced by a VB

$V \to M$ VB globally generated by W $(\dim W = N)$

Induced map by $(V \to M, W)$

$$f : M \longrightarrow \mathrm{Gr}_p(W)$$

$$f(x) := \ker \mathrm{ev}_x$$

where

$$p = N - \mathrm{rnk}\ V$$

Lemma

$V \to M$ can be naturally identified with $f^*Q \to M$

(G) Gauge condition

$$\nabla^V \overset{\mathrm{G}}{\equiv} f^*\nabla^Q$$

8 Standard maps

(M, g) compact RM

$(V \to M,\ h_V,\ \nabla)$ VB + fibre metric + connection

Eigenspaces of the Laplacian (acting on sections)

$$\Gamma(V) = \bigoplus_{\mu} W_\mu, \quad W_\mu := \{t \in \Gamma(V) \,|\, \Delta t = \mu t\}\,.$$

Standard induced map by W_μ

Suppose $V \to M$ globally generated by W_μ

$$f_0 : M \longrightarrow \mathrm{Gr}_p(W_\mu)$$

$$f_0(x) = \ker\ \mathrm{ev}_x|W_\mu$$

Image equivalence of maps

$f_1, f_2 : M \to \mathrm{Gr}_p(\mathbb{K}^m)$ mappings

Image equivalence of maps

f_1 is *image equivalent* to f_2, if $\exists$ isometry ϕ of $\mathrm{Gr}_p(\mathbb{K}^m)$ such that the diagram commutes

$$\begin{array}{ccc} & & \mathrm{Gr}_p(\mathbb{K}^m) \\ & \overset{f_1}{\nearrow} & \downarrow \phi \\ M & & \\ & \underset{f_2}{\searrow} & \\ & & \mathrm{Gr}_p(\mathbb{K}^m) \end{array} \qquad f_2 = \phi \circ f_1$$

10 Gauge equivalence of maps

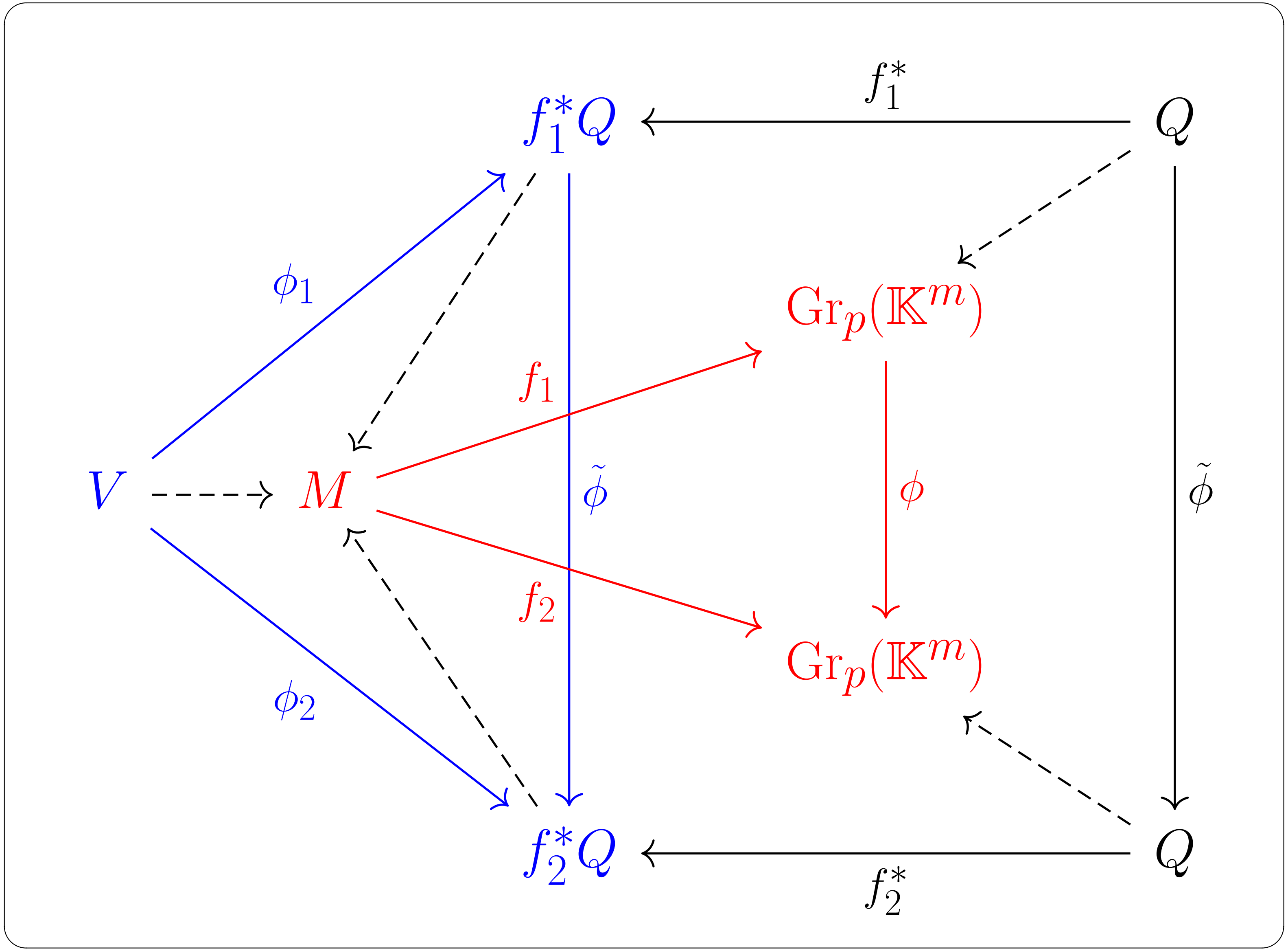
$V \to M$	VB
$f_i : M \to \mathrm{Gr}_p(\mathbb{K}^m)$	mappings
$\phi_i : V \to f_i^* Q$	VB isomorphisms

Gauge equivalence of maps

Two couples (f_i, ϕ_i),$(i = 1, 2)$ are called *gauge equivalent*, if $\exists$ isometry ϕ of $\mathrm{Gr}_p(\mathbb{K}^m)$ such that

$$f_2 = \phi \circ f_1, \qquad \phi_2 = \tilde{\phi} \circ \phi_1$$

where $\tilde{\phi}$ is the bundle automorphism of $Q \to \mathrm{Gr}_p(\mathbb{K}^m)$ covering ϕ.



f_1^*
f_1^*Q
Q
ϕ_1
$\mathrm{Gr}_p(\mathbb{K}^m)$
f_1
V
M
$\tilde{\phi}$
ϕ
$\tilde{\phi}$
f_2
$\mathrm{Gr}_p(\mathbb{K}^m)$
ϕ_2
f_2^*Q
Q
f_2^*

11 Generalisation of the do Carmo – Wallach theory (Holomorphic case)

$M = G/K$	Cpct. irr. Hermitian symm. space
$V \to M$	Complex(*) homogeneous line bundle
∇	canonical connection
$f : M \to \mathrm{Gr}_n(\mathbb{R}^{n+2})$	Full holomorphic map
$\nabla \overset{\mathrm{G}}{\equiv} f^*\nabla^Q$	Gauge condition
$A = -\mu \, \mathrm{id}_V, \; \mu \in \mathbb{R}_+$	Einstein-Hermitian condition

12 Properties of the space of holomorphic sections

Theorem 1A

The space of holomorphic sections $H^0(V) \subset \Gamma(V)$ is eigenspace of Laplacian with eigenvalue μ.

$W := H^0(V)$
Regard W as real $W_{\mathbb{R}}$ + L^2-inner product.

Theorem 1B

$$\iota : \mathbb{R}^{n+2} \longrightarrow W_{\mathbb{R}}$$

$\mathbb{R}^{n+2} \subset W_{\mathbb{R}}$, and $V \to M$ is globally generated by $\mathbb{R}^{n+2}$.

13 The symmetric operator T

$\exists T \in S(W_{\mathbb{R}}) \in \mathrm{End}(W_{\mathbb{R}})$ positive semi-definite

Theorem 2A

$\mathbb{R}^{n+2} = (\ker\ T)^{\perp}$, and $T|_{\mathbb{R}^{n+2}}$ is positive definite.

Theorem 2B

$$(T^2 - \mathrm{id}_W, \mathrm{GS}(V_0, V_0))_S = 0$$
$$(T^2, \mathrm{GS}(\mathfrak{m}V_0, V_0))_S = 0$$

Theorem 2C

T provides holomorphic embedding

$$\mathrm{Gr}_n(\mathbb{R}^{n+2}) \longrightarrow \mathrm{Gr}_{n'}(W) \qquad n' = n + \dim\ker T$$

and a bundle isomorphism $\phi : V \to f^*Q$

14 Moduli theorems

Theorem 3A

$f : M \to \mathrm{Gr}_n(\mathbb{R}^{n+2})$ can be expressed as

$$f([g]) = (\iota^* \mathrm{T} \iota)^{-1} \left(f_0([g]) \cap (\ker\ T)^{\perp} \right)$$

where $f_0([g])$ is the standard map.

Theorem 3B

$$[f]_{gauge} \stackrel{1:1}{\longleftrightarrow} T$$

where $[f]_{gauge}$ is represented by $\iota^* \mathrm{T} \iota$ and ϕ.

15 Holomorphic isometric embeddings of degree k

Holo.emb.

$$f : \mathbb{C}\mathrm{P}^1 \hookrightarrow \mathrm{Gr}_n(\mathbb{R}^{n+2}) \subset \mathbb{C}\mathrm{P}^{n+1}$$

Iso. of $\deg\ k$

$$f^*\omega_Q = k\omega_0, \qquad k \in \mathbb{N}$$

Lemma

$f : \mathbb{C}\mathrm{P}^1 \to \mathrm{Gr}_n(\mathbb{R}^{n+2})$ holo.emb. is k-holo.iso.emb. iff

$$\begin{array}{ccccc} \mathcal{O}(k) & \xleftrightarrow{\text{(G)}} & f^*Q & \xleftarrow{f^*} & Q \\ & \searrow & \swarrow & & \downarrow \\ & & \mathbb{C}\mathrm{P}^1 & \xrightarrow[f]{} & \mathrm{Gr}_n(\mathbb{R}^{n+2}) \end{array}$$

Lemma

$f : \mathbb{C}\mathrm{P}^1 \to \mathrm{Gr}_n(\mathbb{R}^{n+2})$ holo.emb. $+$ iso. degree k, then

$$\text{(EH)} \quad A = -\mu \, \mathrm{id}, \qquad \mu \in \mathbb{R}_{>0}$$

16 $\mathbb{C}$ vs $\mathbb{R}$ representations

$\mathcal{O}(k) \to \mathbb{C}\mathrm{P}^1$ $\qquad$ $\mathrm{SU}(2) \times_{\mathrm{U}(1)} V_0 \to \mathrm{SU}(2)/\mathrm{U}(1)$

Holomorphic sections + Borel–Weil Thm.

$$W := H^0(\mathcal{O}(k)) = S^k\mathbb{C}^2 \qquad \mathrm{SU}(2) - \mathrm{IRREP}$$

$$\dim_{\mathbb{C}} W = k+1$$

$\mathrm{SU}(2) \cong \mathrm{Spin}(3)$

$$S_0^l\mathbb{R}^3 (\dim = 2l+1) \qquad \mathrm{SO}(3) - \mathrm{IRREP}$$

$k = 2l$

$$W_{\mathbb{R}} = (S^{2l}\mathbb{C}^2)_{\mathbb{R}} = \mathbb{R}^{4l+2} = 2S_0^{2l}\mathbb{R}^3$$

$k = 2l+1$

$$W_{\mathbb{R}} = (S^{2l+1}\mathbb{C}^2)_{\mathbb{R}} = \mathbb{R}^{4l+4}$$

17 Spectral formulae

$$\mathrm{H}(W) \subset \mathrm{S}(W_{\mathbb{R}}) \subset \mathrm{End}(W_{\mathbb{R}})$$

Lemma

$W_{\mathbb{R}} = (S^{2l}\mathbb{C}^2)_{\mathbb{R}} = \mathbb{R}^{4l+2}$

$$S^2\mathbb{R}^{4l+2} = 3\bigoplus_{r=0}^{l} S_0^{2l-2r}\mathbb{R}^3 \oplus \bigoplus_{r=0}^{l-1} S_0^{(2l-1)-2r}\mathbb{R}^3$$

$W_{\mathbb{R}} = (S^{2l+1}\mathbb{C}^2)_{\mathbb{R}} = \mathbb{R}^{4l+4}$

$$S^2\mathbb{R}^{4l+4} = 3\bigoplus_{r=0}^{l} S_0^{(2l+1)-2r}\mathbb{R}^3 \oplus \bigoplus_{r=0}^{l-1} S_0^{2l-2r}\mathbb{R}^3$$

Lemma

$$\mathrm{S}(W_{\mathbb{R}}) = \mathrm{H}_+(W) \oplus \mathrm{H}_-(W) \oplus \sigma\mathrm{H}_+(W) \oplus J\sigma\mathrm{H}_+(W)$$

Spaces of Hermitian operators

Proposition

$S^{2l}\mathbb{C}^2$

$$\mathrm{H}_+(S^{2l}\mathbb{C}^2) = \bigoplus_{r=0}^{l} S_0^{2l-2r}\mathbb{R}^3$$

$$\mathrm{H}_-(S^{2l}\mathbb{C}^2) = \bigoplus_{r=0}^{l-1} S_0^{(2l-1)-2r}\mathbb{R}^3$$

$S^{2l+1}\mathbb{C}^2$

$$\mathrm{H}_+(S^{2l+1}\mathbb{C}^2) = \bigoplus_{r=0}^{l} S_0^{(2l+1)-2r}\mathbb{R}^3$$

$$\mathrm{H}_-(S^{2l+1}\mathbb{C}^2) = \bigoplus_{r=0}^{l} S_0^{2l-2r}\mathbb{R}^3$$

Finding T via orthogonality relations

$$S^k\mathbb{C}^2|^{\mathrm{SU}(2)}_{\mathrm{U}(1)} = \mathbb{C}_{-k} \oplus \mathbb{C}_{-k+2} \oplus \cdots = V_0 \oplus \mathfrak{m}V_0 \oplus \dots$$

Lemma

(a) $\mathrm{H}(W) \subset \mathrm{GS}(\mathfrak{m}V_0, V_0)$

(b) $\mathrm{GS}(\mathfrak{m}V_0, V_0) \cap (\sigma H_+(W) \oplus J\sigma H_+(W)) \neq \emptyset$

$$\mathrm{GS}(\mathfrak{m}V_0, V_0) = \mathrm{H}(W) \oplus S^k_0\mathbb{R}^3 \oplus S^k_0\mathbb{R}^3$$

Corollary

$$\mathcal{M}_k \cong (\mathrm{GS}(\mathfrak{m}V_0, V_0) \oplus \mathbb{R}\,\mathbf{id})^\perp \in \mathrm{S}(W_\mathbb{R})$$

$$\mathcal{M}_k \cong 2 \bigoplus_{r=1}^{k\geq 2r} S_0^{k-2r}\mathbb{R}^3, \qquad \dim_\mathbb{R} \mathcal{M}_k = k(k-1)$$

20 Main Results about Moduli Spaces

$f : \mathbf{C}P^1 \to \mathrm{Gr}_n(\mathbb{R}^{n+2})$ Full Holo.Iso.Emb of $\deg\ k$

$\mathcal{M}_k$ Moduli by gauge equivalence.

Theorem 1

$$n \leqq 2k.$$

Theorem 2

If $n = 2k$ (target $\mathrm{Gr}_{2k}(\mathbb{R}^{2k+2})$) then $\mathcal{M}_k$ can be regarded as an open convex body in

$$2 \bigoplus_{r=1}^{k \geq 2r} S_0^{k-2r}\mathbb{R}^3$$

$\overline{\mathcal{M}_k}$ Compactification of $\mathcal{M}_k$

Theorem 3

Boundary points correspond to maps whose images are included in totally geodesic submanifolds

$$\mathrm{Gr}_p(\mathbb{R}^{p+2}) \subset \mathrm{Gr}_{2k}(\mathbb{R}^{2k+2}), \qquad p < 2k.$$

$\mathbf{M}_k$ Moduli space by image equivalence

Theorem 4

If $n = 2k$ (target $\mathrm{Gr}_{2k}(\mathbb{R}^{2k+2})$)

$$\mathbf{M}_k = \mathcal{M}_k / S^1.$$

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