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On the relativistic heat equation and an asymptotic regime of it

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In [1] the authors prove existence and uniqueness of entropy solutions for the Cauchy problem for the quasi-linear parabolic equation $u_t = \operatorname{div} \mathbf{a}(u, Du)$, where $\mathbf{a}(z,\xi) = \nabla_\xi f(z,\xi)$, and f is a convex function of ξ with linear growth as $\|\xi\| \to \infty$, satisfying other additional assumptions. In particular, this class includes a relativistic heat equation and a flux limited diffusion equation used in the theory of radiation hydrodynamics. The relativistic heat equation has been derived by Brenier [2] using Monge-Kantorovich's mass transport theory, and the Cauchy problem solved in [1] for this equation is the following

$$(RHE) \quad \begin{cases} u_t = \nu \operatorname{div}\left(\frac{|u|Du}{\sqrt{u^2+a^2|Du|^2}}\right) & \text{in } (0,\infty)\times\mathbb{R}^N \\ \\ u(t,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

with $0 \leq u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $a = \frac{\nu}{c}$, c being a bound of the propagation speed, and ν being a constant representing a kinematic viscosity. The equation in (RHE) interpolates ([2]) between the usual heat equation (when $c \to \infty$) and the diffusion equation in transparent media (when $\nu \to \infty$) with constant speed of propagation c, that is

$$u_t = c \operatorname{div}\left(u \frac{Du}{|Du|}\right). \tag{1}$$

We study the Cauchy problem associated with equation (1). We define the notion of entropy solution and we prove existence and uniqueness of entropy solutions when $0 \leq u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Existence will be proved by means of Crandall-Ligget's scheme and uniqueness by means of Kruzhkov's technique of doubling variables.

We also construct some explicit solutions of the Cuachy problem for (1) exhibiting fronts propagating at light speed c. Finally, we prove that the same conclusion holds for the (RHE), by constructing a family of compactly supported supersolutions whose front moves at light speed and proving the corresponding comparison principle. More precisely, we show that if $u(t)$ is the entropy solution of (RHE) and $\operatorname{supp}(u_0)$ is compact, then

$$\operatorname{supp}(u(t)) \subset \{x \in \mathbb{R}^N \ : \ d(x, \operatorname{supp}(u_0) \leq ct\} \quad \text{for all } \ t \geq 0.$$

References

[1] F. Andreu, V. Caselles, and J.M. Mazón, *The Cauchy Problem for a Strong Degenerate Quasilinear Equation.* Submited.

[2] Y. Brenier, *Extended Monge-Kantorovich Theory,* in Optimal Transportation and Applications: Lectures given at the C.I.M.E. Summer School help in Martina Franca, L.A. Caffarelli and S. Salsa (eds.), Lecture Notes in Math. 1813, Springer-Verlag, 2003, pp. 91-122.

Quantum-kinetik Fokker-Planck equations:
global-in-time solutions and dispersive effects

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Quantum-Fokker-Planck equations play an important role for the modeling of nano-semiconductor devices, e.g. They combine the description of the electron transport in a selfconsistent electric field (Hartree potential) with diffusive effects of the electron-phonon interaction.

For the mathematical analysis (existence of global-in-time classical solutions) this model lends itself to two different approaches: density matrix operators (however, only for the whole space) and the kinetic Wigner formulation. We shall emphasize the latter approach, where the main analytical difficulty stems from defining the particle density. This shall be solved by exploiting dispersive effects of the free transport operator, and it is inspired by strategies from the classical Vlasov-Fokker-Planck equation.

SOME EXISTENCE RESULTS ON QUASI-LINEAR ELLIPTIC PROBLEMS HAVING NATURAL GROWTH TERMS

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We consider nonlinear boundary value problems whose simplest model is the following:

$$(0.1)\qquad \begin{cases} -\Delta u = \gamma|\nabla u|^2 + \frac{A}{|x|^2} & \text{in } \Omega \qquad (\gamma,\, A \in I\!R) \\ \qquad\qquad u = 0 & \text{on } \partial\Omega. \end{cases}$$

where Ω is a bounded open set in $I\!R^N$, $N > 2$.

It is well known that the minimization in $W_0^{1,2}(\Omega)$ (Ω is a bounded domain in $I\!R^N$) of simple functionals like

$$I(v) = \frac{1}{2}\int_\Omega a(x,v)|\nabla v|^2 - \int_\Omega f(x)v(x),$$

where a is a bounded, smooth function and $f \in L^2(\Omega)$, leads to the following Euler-Lagrange equation

$$(0.2)\qquad \begin{cases} -\mathrm{div}(a(x,u)\nabla u) + \frac{1}{2}a'(x,u)|\nabla u|^2 = f & \text{in } \Omega \\ \qquad\qquad u = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus Calculus of Variations (and also Stochastic Control) is a motivation to the study of quasilinear Dirichlet problems having lower order terms with quadratic growth with respect to the gradient, even if the equation is not the Euler-Lagrange equation of integral functionals.

In this paper we give some contributions to the existence results for quasi-linear elliptic problems having natural growth terms.

We are interested in existence and nonexistence of weak solutions of

$$(0.3)\qquad \begin{cases} -\mathrm{div}(M(x,u)\nabla u) = b(x,u,\nabla u) + f(x) & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega. \end{cases}$$

where Ω is a bounded open subset of $I\!R^N$, $0 \in \Omega$, manly if $|f| \le \frac{A}{|x|^2}$ (recall that $\frac{A}{|x|^2}$ does not belong to $L^{\frac{N}{2}}(\Omega)$).

We assume that M is a Carathéodory matrix and b is a Carathéodory functions (that is, measurable with respect to x for every $s \in I\!R$, and continuous with respect to s for almost every $x \in \Omega$) which satisfy, for some positive constants α, β, γ, a.e. in $x \in \Omega$, $\forall s \in I\!R$, $\forall \xi \in I\!R^N$

$$M(x,s)\xi \cdot \xi \geq \alpha |\xi|^2 \tag{0.4}$$

$$|M(x,s)| \leq \beta \tag{0.5}$$

$$|b(x,s,\xi)| \leq \gamma |\nabla u|^2 \tag{0.6}$$

In papers (see L. Boccardo, F. Murat, J.P. Puel: L^∞-estimate for nonlinear elliptic partial differential equations and application to an existence result; SIAM J. Math. Anal. 23 (1992), 326-333 and the references therein), we proved existence of bounded weak solutions in $W_0^{1,p}$ for (0.3) under suitable assumptions on the data (in particular on the summability of f). We have developped a method which essentially allows to prove the existence of a solution once one can provide an L^∞ estimate for the solutions of a family of approximate equations.

The main goal of the present paper is to prove existence results if the data are not regular enough in order to have bounded solutions, but no sign condition is assumed; in the paper A. Bensoussan, L. Boccardo, F.Murat: On a nonlinear partial differential equation having natural growth terms and unbounded solution; Ann. Inst. H. Poincaré Anal. non lin. 5 (1988), 347-364 (see also the references therein) the assumptions (0.6) and $b(x,s,\xi)s \leq 0$ easily allow to prove a priori estimate in $W_0^{1,p}$ and also in L^∞, if $f \in L^m$, $m > \frac{N}{2}$, so that, thanks to the above remark, $f \in L^m$, with $m > \frac{N}{2}$, implies easily the existence. Moreover, the assumptions used in the papers L. Boccardo, T. Gallouet: Strongly nonlinear elliptic equations having natural growth terms and L^1 data; Nonlinear Anal. TMA 19 (1992), 573-579 and L. Boccardo, T. Gallouët, L. Orsina: Existence and nonexistence of solutions for some nonlinear elliptic equations; J. Anal. Math. 73 (1997), 203–223 (mainly $b(x,s,\xi)s \leq 0$) get a priori estimate in $W_0^{1,p}$ even f belongs only to L^1 (but not if f is a measure).

For the sake of simplicity, our framework is the Sobolev space $W_0^{1,2}$, instead of $W_0^{1,p}$, and $-\mathrm{div}(M(x,v)\nabla v))$, instead of $-\mathrm{div}(a(x,v,\nabla v))$ as principal part of the differential operator.

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Intermediate asymptotics beyond homogeneity and self-similarity:
long time behavior for $u_t = \Delta\phi(u)$

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We investigate the long time asymptotics in $L^1_+(\mathbf{R})$ for solutions of general nonlinear diffusion equations $u_t = \Delta\phi(u)$. We describe, for the first time, the intermediate asymptotics for a very large class of non-homogeneous nonlinearities ϕ for which long time asymptotics cannot be characterized by self-similar solutions. Scaling the solutions by their own second moment (temperature in the kinetic theory language) we obtain a universal asymptotic profile characterized by fixed points of certain maps in probability measures spaces endowed with the euclidean Wasserstein distance d_2.

In the particular case of $\phi(u) \sim u^m$ at first order when $u \sim 0$, we also obtain an optimal rate of convergence in L^1 towards the asymptotic profile identified, in this case, as the Barenblatt self–similar solution corresponding to the exponent m. This second result holds for a larger class of nonlinearities compared to results in the existing literature and is achieved by a variation of the entropy dissipation method in which the nonlinear filtration equation is considered as a perturbation of the porous medium equation.

This is a joint work in collaboration with Marco Di Francesco and Giussepe Toscani.

"Uniqueness for the mean curvature equation - the case of radial graphs"

Emmanuel Chasseigne
Université de Tours

Abstract. We consider the Mean Curvature Equation for Graphs:

$$\frac{\partial u}{\partial t} - \Delta u + \frac{\langle D^2 u Du, Du\rangle}{1+|Du|^2} = 0, \tag{1}$$

where $N \geqslant 1$, $(x,t) \in R^N \times (0,\infty)$. Ecker and Huisken proved that for any initial data $u_0 \in C^0(R^N)$, there exists a regular solution $u(x,t)$. We are interested here in the uniqueness of such solutions, and prove the following result:

Theorem - *For any continuous and radial initial data, there exists a unique viscosity solution of* (1). *Moreover, the solution is radial and regular.*

This is the first uniqueness result obtained in several space dimensions, without growth restrictions on the initial data or the solution at infinity. To get this, we use various techniques borrowed from viscosity, diffusion and geometry.

An optimization result for elliptic equations with drift

Francois Hamel
Universite Paul Cezanne Aix-Marseille III

We consider the problem of minimizing the first eigenvalue of an elliptic operator of the type $-\Delta + v \cdot \nabla$ with bounded drifts v, in bounded domains with Dirichlet boundary conditions. The minimization or maximization problems over the fields v in a given domain lead to some nonlinear equations. The minimization problem over the domain with given measure and over the vector field with a given bound has a unique solution and the minimizing domain is a ball. This result generalizes the usual result of Faber and Krahn for the Laplace operator. This talk is based on a joint paper with N. Nadirashvili (CNRS, Marseille) and E. Russ (Universite Paul Cezanne Aix-Marseille III)

Optimal convergence rates for the fastest conservative nonlinear diffusions

Robert J. McCann
University of Toronto

In many diffusive settings, initial disturbances will gradually disappear and all but their crudest features — such as size and location — will eventually be forgotten. Quantifying the rate at which this information is lost is sometimes a question of central interest. Joint work with Yong Jung Kim (University of California at Riverside and KAIST) addresses this issue for the fastest conservative nonlinearities in a model problem known as the fast diffusion equation

$$u_t = \Delta(u^m), \qquad (n-2)_+/n < m \leq n/(n+2), \quad u, t \geq 0, \quad x \in \mathbf{R}^n,$$

which governs the decay of any integrable, compactly supported initial density towards a characteristically spreading self-similar profile. For other values of the parameter m, this equation has been used to model heat transport, population spreading, fluid seepage, curvature flows, and avalanches in sandpiles. For the fastest conservative nonlinearities, we develop a potential theoretic comparison technique which establishes the sharp conjectured power law rate of decay $1/t$ uniformly in relative error, and in weaker norms such as $L^1(\mathbf{R}^n)$.

Some nonuniqueness results for equations
with critical growth in the gradient

Ireneo Peral
Universidad Autonoma de Madrid

In this talk we analyze existence, nonexistence, multiplicity and regularity of solutions to problem

$$\left\{ \begin{array}{rcll} -\Delta u & = & \beta(u)|\nabla u|^2 + l f(x) & \text{in } \Omega \\ u & = & 0 & \text{on } \partial\Omega, \end{array} \right. \tag{1}$$

where β is a continuous nondecreasing positive function and f belongs to some suitable Lebesgue spaces.

The problem is, for instance, the elliptic counterpart of the Kardar-Parisi-Zhang model in the physical theory of growth and roughening of surfaces.

The work is a collaboration with B. Abdellaoui and A. Dall'Aglio.

The paper complete could be downloaded from:

http://www.uam.es/personal_pdi/ciencias/ireneo/

The structure of the singular set of a free boundary in potential theory

Henrik Shahgholian
KTH Stockholm

Abstract: We characterize the structure of the singular set in the following free boundary problem

$$(\Delta u - f)u = 0, \qquad \text{in} \quad B = B(0,1),$$

where f is Lipschitz, and $u \in W^{2,p}(B)$, $p > n$. The free boundary $\partial\Omega$, represented by $\partial\{\Delta u = f\}$, appears in certain problems in geophysics and inverse problems in potential theory.

(Joint with Luis Caffarelli)

VISCOELASTIC AND NON-VISCOELASTIC SPRING-MASS MODELS

Joan Solà-Morales

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The oscillation of a mass that is attached to an elastic spring and to a damper has always been modelled by the very simple ODE

$$my'' + qy'(t) + \frac{E}{L}y(t) = 0.$$

In this talk we want to discuss how the dynamics of this simple model can be (or not) the limit in some sense of more complete models, where the spring is considered as a continuous medium. In these cases, the deformations of the spring are solutions of a PDE, of a wave type, and the mass motion and the action of the external damper appear only on the (dynamic) boundary conditions.

Our main conclusion is that this limit can only be reasonably (and approximately) obtained when the internal viscosity (also called Kelvin-Voigt damping, or strong damping) of the elastic spring is taken into account. This means that one has to consider the problem

$$\begin{cases} \rho\, u_{tt}(x,t) = E_1\, u_{xxt}(x,t) + E\, u_{xx}(x,t) \quad 0 < x < L, \\ u(0,t) = 0 \\ m\, u_{tt}(L,t) \;=\; -\left(Eu_x(L,t) + r\, u_t(L,t) + E_1\, u_{xt}(L,t)\right). \end{cases}$$

This problem generates an analytic semigroup and we proved that for some values of the parameters it has two dominant complex eigenvalues that produce a dynamics like the one of the ODE.

But we are going to emphasize also that this is no longer the case in several other situations, with $E_1 = 0$ (non-viscoelastic) or E_1 positive but small, or even some cases with E_1 very large. They exhibit very different dynamics that have nothing to do with the ODE.

The mathematical techniques we use are mainly semigroup theory and spectral analysis. The problem has also an interesting nonlinear version, where nonlinear oscillations can be embedded into the nonlinear PDE with the use of invariant manifolds techniques.

The results presented in this talk come from a joint work with Marta Pellicer (UPC-UAB), and also from her recent PhD. thesis.

References:

M. Pellicer and J. Solà-Morales: "On a Model for a Spring-Mass System", J. Math. Anal. Appl. 294 (2004), no. 2, 687-698.

M. Pellicer: "Anàlisi d'un model de suspensió-amortiment", Tesi Doctoral, Universitat Politècnica de Catalunya, 2004.

Multidimensional high-field limit of the Vlasov-Poisson-Fokker-Planck system

Juan SOLER
Universidad de Granada

In this talk, the high-field limit of the Vlasov-Poisson-Fokker-Planck system for charged particles is rigorously derived. The first result is obtained in any space dimension by using modulated energy techniques. It requires the smoothness of the solutions of the limit problem. In dimension 2, it is possible to handle more general data by using methods developed for a diagonal defect measures theory. The convergence of the concentration of particles is obtained in the space of bounded measures. In both cases, the limit of the sequence of densities of distribution functions is shown to solve a non linear system of partial differential equations which is related to Ohm's law.