L^1 Existence and Uniqueness Results for Quasi-linear Elliptic Equations with Nonlinear Boundary Conditions

Résultats d'Éxistence et d'Unicité dans L^1 pour des Equations Elliptiques Quasi-linéaires avec des Conditions au Bord non Linéaires

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Abstract

In this paper we study the questions of existence and uniqueness of weak and entropy solutions for equations of type $-\text{div }\mathbf{a}(x, Du) + \gamma(u) \ni \phi$, posed in an open bounded subset Ω of $\mathbb{R}^N$, with nonlinear boundary conditions of the form $\mathbf{a}(x, Du) \cdot \eta + \beta(u) \ni \psi$. The nonlinear elliptic operator div $\mathbf{a}(x, Du)$ modeled on the p-Laplacian operator $\Delta_p(u) = \text{div } (|Du|^{p-2}Du)$, with $p > 1$, γ and β maximal monotone graphs in $\mathbb{R}^2$ such that $0 \in \gamma(0)$ and $0 \in \beta(0)$, and the data $\phi \in L^1(\Omega)$ and $\psi \in L^1(\partial\Omega)$.

Résumé

Dans ce papier nous étudions les questions d'existence et d'unicité de solution faibles et entropiques pour des équations elliptiques de la forme $-\text{div }\mathbf{a}(x, Du) + \gamma(u) \ni \phi$, dans un domaine borné $\Omega \subset \mathbb{R}$, avec des conditions au bord générale de la forme $\mathbf{a}(x, Du) \cdot \eta + \beta(u) \ni \psi$. L'opérateur div $\mathbf{a}(x, Du)$ généralise l'operateur p-Laplacien $\Delta_p(u) = \text{div } (|Du|^{p-2}Du)$, avec $p > 1$, γ et β sont des graphes maximaux monotonnes dans $\mathbb{R}^2$ tels que $0 \in \gamma(0) \cap \beta(0)$, et les données ϕ et ψ sont des fonctions L^1.

Mathematics Subject Classification (2000): 35J60, 35D02

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $1 < p < N$, and let $\mathbf{a} : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function satisfying

(H_1) there exists $\lambda > 0$ such that $\mathbf{a}(x, \xi) \cdot \xi \geq \lambda |\xi|^p$ for *a.e.* $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$,

(H_2) there exists $\sigma > 0$ and $g \in L^{p'}(\Omega)$ such that $|\mathbf{a}(x, \xi)| \leq \sigma(g(x) + |\xi|^{p-1})$ for *a.e.* $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$, where $p' = \frac{p}{p-1}$,

(H_3) $(\mathbf{a}(x,\xi) - \mathbf{a}(x,\eta)) \cdot (\xi - \eta) > 0$ for *a.e.* $x \in \Omega$ and for all $\xi, \eta \in \mathbb{R}^N$, $\xi \neq \eta$.

The hypotheses $(H_1 - H_3)$ are classical in the study of nonlinear operators in divergent form (cf. [23] or [5]). The model example of function $\mathbf{a}$ satisfying these hypotheses is $\mathbf{a}(x,\xi) = |\xi|^{p-2}\xi$. The corresponding operator is the p-Laplacian operator $\Delta_p(u) = \mathrm{div}(|Du|^{p-2}Du)$.

We are interested in the elliptic problem

$$(S^{\gamma,\beta}_{\phi,\psi}) \quad \begin{cases} -\mathrm{div}\ \mathbf{a}(x,Du) + \gamma(u) \ni \phi & \text{in } \Omega \\ \\ \mathbf{a}(x,Du)\cdot\eta + \beta(u) \ni \psi & \text{on } \partial\Omega, \end{cases}$$

where η is the unit outward normal on $\partial\Omega$, $\psi \in L^1(\partial\Omega)$ and $\phi \in L^1(\Omega)$. The nonlinearities γ and β are maximal monotone graphs in $\mathbb{R}^2$ (see, e.g. [12]) such that $0 \in \gamma(0)$ and $0 \in \beta(0)$. In particular, they may be multivalued and this allows to include the Dirichlet condition (taking β to be the monotone graph D defined by $D(0) = \mathbb{R}$) and the Neumann condition (taking β to be the monotone graph N defined by $N(r) = 0$ for all $r \in \mathbb{R}$) as well as many other nonlinear fluxes on the boundary that occur in some problems in Mechanic and Physics (see, e.g., [16] or [11]). Note also that, since γ may be multivalued, problems of type $(S^{\gamma,\beta}_{\phi,\psi})$ appears in various phenomena with changes of state like multiphase Stefan problem (cf [14]) and in the weak formulation of the mathematical model of the so called Hele Shaw problem (cf. [15] and [17]).

Particular instances of problem $(S^{\gamma,\beta}_{\phi,\psi})$ have been studied in [9], [5], [3] and [1]. Let us describe their results in some detail. The work of Bénilan, Crandall and Sacks [9] was pioneer in this kind of problems. They study problem $(S^{\gamma,\beta}_{\phi,0})$ for any γ and β maximal monotone graphs in $\mathbb{R}^2$ such that $0 \in \gamma(0)$ and $0 \in \beta(0)$, for the Laplacian operator, i.e., for $\mathbf{a}(x,\xi) = \xi$, and prove, between other results, that, for any $\phi \in L^1(\Omega)$,

$$\inf\{\mathrm{Ran}(\gamma)\}\mathrm{meas}(\Omega) + \inf\{\mathrm{Ran}(\beta)\}\mathrm{meas}(\partial\Omega) < \int_\Omega \phi$$
$$< \sup\{\mathrm{Ran}(\gamma)\}\mathrm{meas}(\Omega) + \sup\{\mathrm{Ran}(\beta)\}\mathrm{meas}(\partial\Omega),$$

there exists a unique, up to a constant for u, named weak solution, $[u,z,w] \in W^{1,1}(\Omega)\times L^1(\Omega)\times L^1(\partial\Omega)$, $z(x) \in \gamma(u(x))$ *a.e.* in Ω, $w(x) \in \beta(u(x))$ *a.e.* in $\partial\Omega$, such that

$$\int_\Omega Du\cdot Dv + \int_\Omega zv + \int_{\partial\Omega} wv = \int_\Omega \phi v,$$

for all $v \in W^{1,\infty}(\Omega)$. For nonhomogeneous boundary condition, i.e. $\psi \not\equiv 0$, one can see [18] for $\psi \in \mathrm{Ran}(\beta)$, and [19, 20] for some particular situations of β and γ.

Another important work in the L^1-Theory for p-Laplacian type equations is [5], where problem

$$(D^\gamma_\phi) \quad \begin{cases} -\mathrm{div}\ \mathbf{a}(x,Du) + \gamma(u) \ni \phi & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is studied for any γ maximal monotone graph in $\mathbb{R}^2$ such that $0 \in \gamma(0)$. It is proved that, for any $\phi \in L^1(\Omega)$, there exists a unique, named entropy solution, $[u,z] \in \mathcal{T}^{1,p}_0(\Omega)\times L^1(\Omega)$, $z(x) \in \gamma(u(x))$ *a.e.* in Ω, such that

$$\int_\Omega \mathbf{a}(.,Du)\cdot DT_k(u-v) + \int_\Omega zT_k(u-v) \le \int_\Omega \phi T_k(u-v) \quad \forall k > 0,$$

for all $v \in L^\infty(\Omega)\cap W^{1,p}(\Omega)$, $v(x) = 0$ *a.e.* in $\partial\Omega$ (see Section 2 for the definition of $\mathcal{T}^{1,p}_0(\Omega)$).

Following [5], problems $(S^{id,\beta}_{\phi,0})$ and $(S^{id,\beta}_{\phi,\psi})$, where $id(r)=r$ for all $r\in\mathbb{R}$, are studied in [3] and [1], for any β maximal monotone graph in $\mathbb{R}^2$ with closed domain such that $0\in\beta(0)$. It is proved that, for any $\phi\in L^1(\Omega)$ and $\psi\in L^1(\partial\Omega)$, there exists a unique $u\in\mathcal{T}^{1,p}_{tr}(\Omega)$, and there exists $w\in L^1(\partial\Omega)$, $w(x)\in\beta(u(x))$ *a.e.* in $\partial\Omega$, such that

$$\int_\Omega \mathbf{a}(.,Du)\cdot DT_k(u-v)+\int_\Omega uT_k(u-v)+\int_{\partial\Omega} wT_k(u-v)$$

$$\leq \int_{\partial\Omega}\psi T_k(u-v)+\int_\Omega \phi T_k(u-v)\quad \forall k>0,$$

for all $v\in L^\infty(\Omega)\cap W^{1,p}(\Omega)$, $v(x)\in\beta(u(x))$ *a.e.* in $\partial\Omega$.

Our aim is to prove existence and uniqueness of weak and entropy solutions for the general elliptic problem $(S^{\gamma,\beta}_{\phi,\psi})$. The main interest in our work is that we are dealing with general nonlinear operator $-\mathrm{div}\,\mathbf{a}(x,Du)$ with nonhomogeneous boundary condition and general nonlinearities β and γ. As in [9], a range condition relating the average of ϕ and ψ to the range of β and γ is necessary for existence of weak and entropy solution. However, in contrast to the smooth homogeneous case, $\mathbf{a}$ smooth and $\psi=0$, for the nonhomogeneous case this range condition is not sufficient for the existence of weak solution. Indeed, in general, the intersection of the domains of β and γ seems to create some obstruction phenomena for the existence of these solutions. In general, even if $D(\beta)=\mathbb{R}$, it does not exist weak solution, as the following example shows. Let γ be such that $D(\gamma)=[0,1]$, $\beta=\mathbb{R}\times\{0\}$, and let $\phi\in L^1(\Omega)$, $\phi\leq 0$ *a.e.* in Ω, and $\psi\in L^1(\partial\Omega)$, $\psi\leq 0$ *a.e.* in $\partial\Omega$. If there exists $[u,z,w]$ weak solution of problem $(S^{\gamma,\beta}_{\phi,\psi})$, then $z\in\gamma(u)$, therefore $0\leq u\leq 1$ *a.e.* in Ω, $w=0$, and it holds that for any $v\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$,

$$\int_\Omega \mathbf{a}(x,Du)Dv+\int_\Omega zv=\int_{\partial\Omega}\psi v+\int_\Omega \phi v.$$

Taking $v=u$, as $u\geq 0$, we get

$$0\leq \int_\Omega \mathbf{a}(x,Du)Du+\int_\Omega zu=\int_{\partial\Omega}\psi u+\int_\Omega \phi u\leq 0.$$

Therefore, we obtain that $\int_\Omega |Du|^p=0$, so u is constant and

$$\int_\Omega zv=\int_{\partial\Omega}\psi v+\int_\Omega \phi v,$$

for any $v\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$, and in particular, for any $v\in W^{1,p}_0(\Omega)\cap L^\infty(\Omega)$. Consequently, $\phi=z$ *a.e.* in Ω, and ψ must be 0 *a.e.* in $\partial\Omega$.

The main applications we have in mind is the study of doubly nonlinear evolution problems of elliptic-parabolic type and degenerate parabolic problems of Stefan or Hele-Shaw type, with nonhomogeneous boundary conditions and/or dynamical boundary conditions (see [2]). Notice that in all these applications one has $D(\gamma)=\mathbb{R}$, which is sufficiently covered in this paper.

The results we obtain have an interpretation in terms of accretive operators. Indeed, we can define the (possibly multivalued) operator $\mathcal{B}^{\gamma,\beta}$ in $X:=L^1(\Omega)\times L^1(\partial\Omega)$ as

$$\mathcal{B}^{\gamma,\beta}:=\Big\{((v,w),(\hat v,\hat w))\in X\times X\ :\ \exists u\in\mathcal{T}^{1,p}_{tr}(\Omega),\ \text{with } [u,v,w] \text{ an entropy solution of } (S^{\gamma,\beta}_{v+\hat v,w+\hat w})\Big\}.$$

Then, under certain assumptions, $\mathcal{B}^{\gamma,\beta}$ is an m-T-accretive operator in X. Therefore, by the theory of Evolution Equations Governed by Accretive Operators (see, [4], [8] or [13]), for any $(v_0,w_0)\in\overline{D(\mathcal{B}^{\gamma,\beta})}^X$ and any $(f,g)\in L^1(0,T;L^1(\Omega))\times L^1(0,T;L^1(\partial\Omega))$, there exists a unique mild-solution of the problem

$$V'+\mathcal{B}^{\gamma,\beta}(V)\ni(f,g),\qquad V(0)=(v_0,w_0),$$

which rewrites, as an abstract Cauchy problem in X, the following degenerate elliptic-parabolic problem with nonlinear dynamical boundary conditions

$$DP(\gamma,\beta) \quad \begin{cases} v_t - \operatorname{div} \mathbf{a}(x, Du) = f, \ v \in \gamma(u), \quad \text{in } \Omega \times (0,T) \\ \\ w_t + \mathbf{a}(x, Du) \cdot \eta = g, \ w \in \beta(u), \quad \text{on } \partial\Omega \times (0,T) \\ \\ v(0) = v_0 \ \text{ in } \Omega, \ w(0) = w_0 \ \text{ in } \partial\Omega. \end{cases}$$

In principle, it is not clear how these mild solutions have to be interpreted respect to the problem $DP(\gamma,\beta)$. In a next paper ([2]) we characterize these mild solutions.

Let us briefly summarize the contents of the paper. In Section 2 we fix the notation and give some preliminaries. In Section 3 we give the definitions of the different concepts of solution we use. The next section is dedicated to establish the uniqueness results. Finally, in Section 5 we prove the existence results. First, we study the existence of solutions of approximated problems, next we prove the existence of weak solutions for data in $L^{p'}$ and finally the existence of entropy solutions for data in L^1.

2 Preliminaires

For $1 \leq p < +\infty$, $L^p(\Omega)$ and $W^{1,p}(\Omega)$ denote respectively the standard Lebesgue space and Sobolev space, and $W_0^{1,p}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$. For $u \in W^{1,p}(\Omega)$, we denote by u or $\tau(u)$ the trace of u on $\partial\Omega$ in the usual sense and by $W^{\frac{1}{p'},p}(\partial\Omega)$ the set $\tau(W^{1,p}(\Omega))$. Recall that $\operatorname{Ker}(\tau) = W_0^{1,p}(\Omega)$.

In [5], the authors introduce the set

$$\mathcal{T}^{1,p}(\Omega) = \{u : \Omega \longrightarrow \mathbb{R} \text{ measurable such that } T_k(u) \in W^{1,p}(\Omega) \quad \forall k > 0\},$$

where $T_k(s) = \sup(-k, \inf(s,k))$. They also prove that given $u \in \mathcal{T}^{1,p}(\Omega)$, there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that

$$DT_k(u) = v\chi_{\{|v|<k\}} \quad \forall k > 0.$$

This function v will be denoted by Du. It is clear that if $u \in W^{1,p}(\Omega)$, then $v \in L^p(\Omega)$ and $v = Du$ in the usual sense.

As in [3], $\mathcal{T}_{tr}^{1,p}(\Omega)$ denotes the set of functions u in $\mathcal{T}^{1,p}(\Omega)$ satisfying the following conditions, there exists a sequence u_n in $W^{1,p}(\Omega)$ such that

(a) u_n converges to u *a.e.* in Ω,

(b) $DT_k(u_n)$ converges to $DT_k(u)$ in $L^1(\Omega)$ for all $k > 0$,

(c) there exists a measurable function v on $\partial\Omega$, such that u_n converges to v *a.e.* in $\partial\Omega$.

The function v is the trace of u in the generalized sense introduced in [3]. In the sequel, the trace of $u \in \mathcal{T}_{tr}^{1,p}(\Omega)$ on $\partial\Omega$ will be denoted by $tr(u)$ or u. Let us recall that in the case where $u \in W^{1,p}(\Omega)$, $tr(u)$ coincides with the trace of u, $\tau(u)$, in the usual sense, and the space $\mathcal{T}_0^{1,p}(\Omega)$, introduced in [5] to study (D_ϕ^γ), is equal to $\operatorname{Ker}(tr)$. Moreover, for every $u \in \mathcal{T}_{tr}^{1,p}(\Omega)$ and every $k > 0$, $\tau(T_k(u)) = T_k(tr(u))$, and, if $\phi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, then $u - \phi \in \mathcal{T}_{tr}^{1,p}(\Omega)$ and $tr(u-\phi) = tr(u) - \tau(\phi)$.

We denote

$$V^{1,p}(\Omega) := \left\{ \phi \in L^1(\Omega) : \exists M > 0 \text{ such that } \int_\Omega |\phi v| \leq M \|v\|_{W^{1,p}(\Omega)} \ \forall v \in W^{1,p}(\Omega) \right\}$$

and

$$V^{1,p}(\partial\Omega) := \Big\{ \psi \in L^1(\partial\Omega) : \exists M > 0 \text{ such that } \int_{\partial\Omega} |\psi v| \leq M \|v\|_{W^{1,p}(\Omega)} \ \forall v \in W^{1,p}(\Omega) \Big\}.$$

$V^{1,p}(\Omega)$ is a Banach space endowed with the norm

$$\|\phi\|_{V^{1,p}(\Omega)} := \inf\{M > 0 : \int_{\Omega} |\phi v| \leq M \|v\|_{W^{1,p}(\Omega)} \ \forall v \in W^{1,p}(\Omega)\},$$

and $V^{1,p}(\partial\Omega)$ is a Banach space endowed with the norm

$$\|\psi\|_{V^{1,p}(\partial\Omega)} := \inf\{M > 0 : \int_{\partial\Omega} |\psi v| \leq M \|v\|_{W^{1,p}(\Omega)} \ \forall v \in W^{1,p}(\Omega)\}.$$

Observe that, Sobolev embeddings and Trace theorems imply, for $1 \leq p < N$,

$$L^{p'}(\Omega) \subset L^{(Np/(N-p))'}(\Omega) \subset V^{1,p}(\Omega)$$

and

$$L^{p'}(\partial\Omega) \subset L^{((N-1)p/(N-p))'}(\partial\Omega) \subset V^{1,p}(\partial\Omega).$$

Also,

$$V^{1,p}(\Omega) = L^1(\Omega) \ \text{ and } \ V^{1,p}(\partial\Omega) = L^1(\partial\Omega) \ \text{ when } \ p > N,$$
$$L^q(\Omega) \subset V^{1,N}(\Omega) \ \text{ and } \ L^q(\partial\Omega) \subset V^{1,N}(\partial\Omega) \ \text{ for any } \ q > 1.$$

We say that $\mathbf{a}$ is *smooth* (see [3]) when, for any $\phi \in L^\infty(\Omega)$ such that there exists a bounded weak solution u of the homogeneous Dirichlet problem

$$(D) \qquad \begin{cases} -\operatorname{div} \mathbf{a}(x, Du) = \phi & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

there exists $g \in L^1(\partial\Omega)$ such that u is also a weak solution of the Neumann problem

$$(N) \qquad \begin{cases} -\operatorname{div} \mathbf{a}(x, Du) = \phi & \text{in } \Omega \\ \mathbf{a}(x, Du) \cdot \eta = g & \text{on } \partial\Omega. \end{cases}$$

Functions $\mathbf{a}$ corresponding to linear operators with smooth coefficients and p-Laplacian type operators are smooth (see [11] and [22]). The smoothness of the Laplacian operator is even stronger than this, in fact, there is a bounded linear mapping $T : L^1(\Omega) \to L^1(\partial\Omega)$, such that the weak solution of (D) for $\phi \in L^1(\Omega)$ is also a weak solution of (N) for $g = T(\phi)$ (see [9]).

For a maximal monotone graph η in $\mathbb{R} \times \mathbb{R}$ and $r \in \mathbb{N}$ we denote by η_r the *Yosida approximation* of η, given by $\eta_r = r(I - (I + \frac{1}{r}\eta)^{-1})$. The function η_r is maximal monotone and Lipschitz. We recall the definition of the *main section* η^0 of η

$$\eta^0(s) := \begin{cases} \text{the element of minimal absolute value of } \eta(s) \ \text{ si } \eta(s) \neq \emptyset \\ +\infty \qquad \text{if } [s, +\infty) \cap D(\eta) = \emptyset \\ -\infty \qquad \text{if } (-\infty, s] \cap D(\eta) = \emptyset. \end{cases}$$

If $s \in D(\eta)$, $|\eta_r(s)| \leq |\eta^0(s)|$ and $\eta_r(s) \to \eta^0(s)$ as $r \to +\infty$, and if $s \notin D(\eta)$, $|\eta_r(s)| \to +\infty$ as $r \to +\infty$.

We will denote by P_0 the following set of functions,

$$P_0 = \{p \in C^\infty(\mathbb{R}) : 0 \le p' \le 1, \mathrm{supp}(p') \text{ is compact, and } 0 \notin \mathrm{supp}(p)\}.$$

In [7] the following relation for $u, v \in L^1(\Omega)$ is defined,

$$u \ll v \ \text{ if}$$

$$\int_\Omega (u-k)^+ \le \int_\Omega (v-k)^+ \text{ and } \int_\Omega (u+k)^- \le \int_\Omega (v+k)^- \text{ for any } k > 0,$$

and the following facts are proved.

Proposition 2.1 *Let Ω be a bounded domain in $\mathbb{R}^N$.*
(i) For any $u, v \in L^1(\Omega)$, if $\int_\Omega up(u) \le \int_\Omega vp(u)$ for all $p \in P_0$, then $u \ll v$.
(ii) If $u, v \in L^1(\Omega)$ and $u \ll v$, then $\|u\|_q \le \|v\|_q$ for any $q \in [1, +\infty]$.
(iii) If $v \in L^1(\Omega)$, then $\{u \in L^1(\Omega) : u \ll v\}$ is a weakly compact subset of $L^1(\Omega)$.

3 Weak solutions and entropy solutions

In this section we give the different concepts of solutions we use. The first one is the concept of weak solution.

Definition 3.1 Let $\phi \in L^1(\Omega)$ and $\psi \in L^1(\partial\Omega)$. A triple of functions $[u, z, w] \in W^{1,p}(\Omega) \times L^1(\Omega) \times L^1(\partial\Omega)$ is a *weak solution* of problem $(S^{\gamma,\beta}_{\phi,\psi})$ if $z(x) \in \gamma(u(x))$ *a.e.* in Ω, $w(x) \in \beta(u(x))$ *a.e.* in $\partial\Omega$, and

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv + \int_\Omega zv + \int_{\partial\Omega} wv = \int_{\partial\Omega} \psi v + \int_\Omega \phi v, \tag{1}$$

for all $v \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$.

In general, as it is remarked in [5], for $1 < p \le 2 - \frac{1}{N}$, there exists $f \in L^1(\Omega)$ such that the problem

$$u \in W^{1,1}_{loc}(\Omega), \ u - \Delta_p(u) = f \quad \text{in} \quad \mathcal{D}'(\Omega),$$

has no solution. In [5], to overcome this difficulty and to get uniqueness, it was introduced a new concept of solution, named entropy solution. As in [3] or [1], following these ideas, we introduce the following concept of solution.

Definition 3.2 Let $\phi \in L^1(\Omega)$ and $\psi \in L^1(\partial\Omega)$. A triple of functions $[u, z, w] \in \mathcal{T}^{1,p}_{tr}(\Omega) \times L^1(\Omega) \times L^1(\partial\Omega)$ is an *entropy solution* of problem $(S^{\gamma,\beta}_{\phi,\psi})$ if $z(x) \in \gamma(u(x))$ *a.e.* in Ω, $w(x) \in \beta(u(x))$ *a.e.* in $\partial\Omega$ and

$$\begin{aligned} &\int_\Omega \mathbf{a}(x, Du) \cdot DT_k(u - v) + \int_\Omega zT_k(u - v) + \int_{\partial\Omega} wT_k(u - v) \\ &\le \int_{\partial\Omega} \psi T_k(u - v) + \int_\Omega \phi T_k(u - v) \quad \forall k > 0, \end{aligned} \tag{2}$$

for all $v \in L^\infty(\Omega) \cap W^{1,p}(\Omega)$.

Obviously, every weak solution is an entropy solution and an entropy solution with $u \in W^{1,p}(\Omega)$ is a weak solution.

Remark 3.3 If we take $v = T_h(u) \pm 1$ as test functions in (2) and let h go to $+\infty$, we get that

$$\int_\Omega z + \int_{\partial\Omega} w = \int_{\partial\Omega} \psi + \int_\Omega \phi.$$

Then necessarily ϕ and ψ must satisfy

$$\mathcal{R}^-_{\gamma,\beta} \leq \int_{\partial\Omega} \psi + \int_\Omega \phi \leq \mathcal{R}^+_{\gamma,\beta},$$

where

$$\mathcal{R}^+_{\gamma,\beta} := \sup\{\mathrm{Ran}(\gamma)\}\mathrm{meas}(\Omega) + \sup\{\mathrm{Ran}(\beta)\}\mathrm{meas}(\partial\Omega)$$

and

$$\mathcal{R}^-_{\gamma,\beta} := \inf\{\mathrm{Ran}(\gamma)\}\mathrm{meas}(\Omega) + \inf\{\mathrm{Ran}(\beta)\}\mathrm{meas}(\partial\Omega).$$

We will write $\mathcal{R}_{\gamma,\beta} :=]\mathcal{R}^-_{\gamma,\beta}, \mathcal{R}^+_{\gamma,\beta}[$ when $\mathcal{R}^-_{\gamma,\beta} < \mathcal{R}^+_{\gamma,\beta}$.

Remark 3.4 Let $\phi \in V^{1,p}(\Omega)$ and $\psi \in V^{1,p}(\partial\Omega)$. Then, if $[u, z, w]$ is a weak solution of problem $(S^{\gamma,\beta}_{\phi,\psi})$, it is easy to see that

$$\int_\Omega \mathbf{a}(x, Du) \cdot Du + \int_\Omega zu + \int_{\partial\Omega} wu = \int_{\partial\Omega} \psi u + \int_\Omega \phi u.$$

Moreover, if $D(\gamma) \neq \{0\}$ and $D(\beta) \neq \{0\}$, it follows that $z \in V^{1,p}(\Omega)$, $w \in V^{1,p}(\partial\Omega)$ and

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv + \int_\Omega zv + \int_{\partial\Omega} wv = \int_{\partial\Omega} \psi v + \int_\Omega \phi v,$$

for any $v \in W^{1,p}(\Omega)$.

In fact, let $v \in W^{1,p}(\Omega)$ and take $T_k(|v|)\frac{1}{r}T_r(u)$ as test function in (1). Then, letting r go to 0, there exists $M_1 > 0$ such that

$$\int_{\{x\in\Omega: u(x)\neq 0\}} |z| T_k(|v|) + \int_{\{x\in\partial\Omega: u(x)\neq 0\}} |w| T_k(|v|) \leq M_1 \|v\|_{W^{1,p}(\Omega)}.$$

Letting now k go to $+\infty$, applying Fatou's Lemma, we get

$$\int_{\{x\in\Omega: u(x)\neq 0\}} |z||v| + \int_{\{x\in\partial\Omega: u(x)\neq 0\}} |w||v| \leq M_1 \|v\|_{W^{1,p}(\Omega)}.$$

If $\beta(0)$ is bounded, there exists $M_2 > 0$ such that

$$\int_{\{x\in\partial\Omega: u(x)=0\}} |w||v| \leq M_2 \|v\|_{W^{1,p}(\Omega)}.$$

In the case $\beta(0)$ is unbounded from above (a similar argument can be done in the case of being unbounded from below) let us take $T_k(|v|)S_r(u)$ as test function in (1), where $S_r(s) := \frac{s+r}{r}\chi_{[-r,0]}(s) + \chi_{[0,+\infty[}(s)$, then, letting r go to 0, there exists $M_2 > 0$ such that

$$\int_{\{x\in\partial\Omega: u(x)=0\}} w T_k(|v|) \leq M_2 \|v\|_{W^{1,p}(\Omega)},$$

and consequently, since $\beta(0)$ must be bounded from below (because $D(\beta) \neq \{0\}$), there exists $M_3 > 0$ such that

$$\int_{\{x\in\partial\Omega:u(x)=0\}} |w| T_k(|v|) \leq M_3 \|v\|_{W^{1,p}(\Omega)}.$$

Letting now k go to $+\infty$, applying Fatou's Lemma, we get

$$\int_{\{x\in\partial\Omega:u(x)=0\}} |w||v| \leq M_4 \|v\|_{W^{1,p}(\Omega)}.$$

Similarly, there exists $M_5 > 0$ such that

$$\int_{\{x\in\Omega:u(x)=0\}} |z||v| \leq M_5 \|v\|_{W^{1,p}(\Omega)}.$$

4 Uniqueness results

This section deals with uniqueness results for entropy solutions and therefore for weak solutions. We firstly need the following result.

Lemma 4.1 *Let $[u, z, w]$ be an entropy solution of problem $(S^{\gamma,\beta}_{\phi,\psi})$. Then, for all $h > 0$,*

$$\lambda \int_{\{h<|u|<h+k\}} |Du|^p \leq k \int_{\partial\Omega\cap\{|u|\geq h\}} |\psi| + k \int_{\Omega\cap\{|u|\geq h\}} |\phi|.$$

Proof. Taking $T_h(u)$ as test function in (2), we have

$$\int_\Omega \mathbf{a}(x, Du) \cdot DT_k(u - T_h(u)) + \int_\Omega zT_k(u - T_h(u)) + \int_{\partial\Omega} wT_k(u - T_h(u))$$

$$\leq \int_{\partial\Omega} \psi T_k(u - T_h(u)) + \int_\Omega \phi T_k(u - T_h(u)).$$

Now, using (H_1) and the positivity of the second and third terms, it follows that

$$\lambda \int_{\{h<|u|<h+k\}} |Du|^p \leq k \int_{\partial\Omega\cap\{|u|\geq h\}} |\psi| + k \int_{\Omega\cap\{|u|\geq h\}} |\phi|.$$

□

Theorem 4.2 *Let $\phi \in L^1(\Omega)$ and $\psi \in L^1(\partial\Omega)$, and let $[u_1, z_1, w_1]$ and $[u_2, z_2, w_2]$ be entropy solutions of problem $(S^{\gamma,\beta}_{\phi,\psi})$. Then, there exists a constant $c \in \mathbb{R}$ such that*

$$u_1 - u_2 = c \quad \text{a.e. in } \Omega,$$

$$z_1 - z_2 = 0 \quad \text{a.e. in } \Omega.$$

$$w_1 - w_2 = 0 \quad \text{a.e. in } \partial\Omega.$$

Moreover, if $c \neq 0$, there exists a constant k such that $z_1 = z_2 = k$.

Since every weak solution of problem $(S^{\gamma,\beta}_{\phi,\psi})$ is an entropy solution. The same result is true for weak solutions.

Proof. Let $[u_1, z_1, w_1]$ and $[u_2, z_2, w_2]$ be entropy solutions of problem $(S^{\gamma,\beta}_{\phi,\psi})$. For every $h > 0$, we have that

$$\int_\Omega \mathbf{a}(x, Du_1) \cdot DT_k(u_1 - T_h(u_2)) + \int_\Omega z_1 T_k(u_1 - T_h(u_2))$$

$$+ \int_{\partial\Omega} w_1 T_k(u_1 - T_h(u_2)) \leq \int_{\partial\Omega} \psi T_k(u_1 - T_h(u_2)) + \int_\Omega \phi T_k(u_1 - T_h(u_2))$$

and

$$\int_\Omega \mathbf{a}(x, Du_2) \cdot DT_k(u_2 - T_h(u_1)) + \int_\Omega z_2 T_k(u_2 - T_h(u_1))$$

$$+ \int_{\partial\Omega} w_2 T_k(u_2 - T_h(u_1)) \leq \int_{\partial\Omega} \psi T_k(u_2 - T_h(u_1)) + \int_\Omega \phi T_k(u_2 - T_h(u_1))$$

Adding both inequalities and taking limits when h goes to ∞, on account of the monotonicity of γ and β, if

$$I_{h,k} := \int_\Omega \mathbf{a}(x, Du_1) \cdot DT_k(u_1 - T_h(u_2)) + \int_\Omega \mathbf{a}(x, Du_2) \cdot DT_k(u_2 - T_h(u_1)),$$

we get

$$\limsup_{h\to\infty} I_{h,k} \leq -\int_\Omega (z_1 - z_2) T_k(u_1 - u_2) - \int_{\partial\Omega} (w_1 - w_2) T_k(u_1 - u_2) \leq 0. \tag{3}$$

Let us see that

$$\liminf_{h\to\infty} I_{h,k} \geq 0 \quad \text{for any } k. \tag{4}$$

To prove this, we split

$$I_{h,k} = I^1_{h,k} + I^2_{h,k} + I^3_{h,k} + I^4_{h,k},$$

where

$$I^1_{h,k} := \int_{\{|u_1|<h,\ |u_2|<h\}} (\mathbf{a}(x, Du_1) - \mathbf{a}(x, Du_2)) \cdot DT_k(u_1 - u_2),$$

$$I^2_{h,k} := \int_{\{|u_1|<h,\ |u_2|\geq h\}} \mathbf{a}(x, Du_1) \cdot DT_k(u_1 - h\,\mathrm{sign}(u_2)) + \int_{\{|u_1|<h,\ |u_2|\geq h\}} \mathbf{a}(x, Du_2) \cdot DT_k(u_2 - u_1)$$

$$\geq \int_{\{|u_1|<h,\ |u_2|\geq h\}} \mathbf{a}(x, Du_2) \cdot DT_k(u_2 - u_1),$$

$$I^3_{h,k} := \int_{\{|u_1|\geq h,\ |u_2|<h\}} \mathbf{a}(x, Du_1) \cdot DT_k(u_1 - u_2) + \int_{\{|u_1|\geq h,\ |u_2|<h\}} \mathbf{a}(x, Du_2) \cdot DT_k(u_2 - h\,\mathrm{sign}(u_1))$$

$$\geq \int_{\{|u_1|\geq h,\ |u_2|<h\}} \mathbf{a}(x, Du_1) \cdot DT_k(u_1 - u_2)$$

and

$$I^4_{h,k} := \int_{\{|u_1|\geq h,\ |u_2|\geq h\}} \mathbf{a}(x, Du_1) \cdot DT_k(u_1 - h\,\mathrm{sign}(u_2))$$

$$+ \int_{\{|u_1|\geq h,\ |u_2|\geq h\}} \mathbf{a}(x, Du_2) \cdot DT_k(u_2 - h\,\mathrm{sign}(u_1)) \geq 0.$$

Combining the above estimates we get

$$I_{h,k} \geq I^1_{h,k} + L^1_{h,k} + L^2_{h,k}, \tag{5}$$

where

$$L^1_{h,k} := \int_{\{|u_1|<h,\ |u_2|\geq h\}} \mathbf{a}(x, Du_2)\cdot DT_k(u_2-u_1),$$

$$L^2_{h,k} := \int_{\{|u_1|\geq h,\ |u_2|<h\}} \mathbf{a}(x, Du_1)\cdot DT_k(u_1-u_2)$$

and $I^1_{h,k}$ is non negative and non decreasing in h. Now, if we set

$$C(h,k) := \{h < |u_1| < k+h\} \cap \{h-k < |u_2| < h\},$$

we have

$$\begin{aligned}|L^2_{h,k}| &\leq \int_{\{|u_1-u_2|<k,\ |u_1|\geq h,\ |u_2|<h\}} |\mathbf{a}(x, Du_1)\cdot (Du_1 - Du_2)| \\ &\leq \int_{C(h,k)} |a(x, Du_1)\cdot Du_1| + \int_{C(h,k)} |\mathbf{a}(x, Du_1)\cdot Du_2|.\end{aligned}$$

Then, by Hölder's inequality, we get

$$|L^2_{h,k}| \leq \left(\int_{C(h,k)} |\mathbf{a}(x, Du_1)|^{p'}\right)^{1/p'} \left(\left(\int_{C(h,k)} |Du_1|^p\right)^{1/p} + \left(\int_{C(h,k)} |Du_2|^p\right)^{1/p}\right).$$

Now, by (H_2),

$$\begin{aligned}\left(\int_{C(h,k)} |\mathbf{a}(x, Du_1)|^{p'}\right)^{1/p'} &\leq \left(\int_{C(h,k)} \sigma^{p'}\left(g(x) + |Du_1|^{p-1}\right)^{p'}\right)^{1/p'} \\ &\leq \sigma 2^{\frac{1}{p}}\left(\|g\|^{p'}_{p'} + \int_{\{h<|u_1|<k+h\}} |Du_1|^p\right)^{1/p'}.\end{aligned}$$

On the other hand, by Lemma 4.1, we obtain

$$\int_{\{h<|u_1|<k+h\}} |Du_1|^p \leq \frac{k}{\lambda}\left(\int_{\{|u_1|\geq h\}} |\psi| + \int_{\{|u_1|\geq h\}} |\phi|\right)$$

and

$$\int_{\{h-k<|u_2|<h\}} |Du_2|^p \leq \frac{k}{\lambda}\left(\int_{\{|u_2|\geq h-k\}} |\psi| + \int_{\{|u_2|\geq h-k\}} |\phi|\right).$$

Then, since $\phi \in L^1(\Omega)$, $\psi \in L^1(\partial\Omega)$ and having in mind that

$$\lim_{r\to+\infty} \text{meas}\{x \in \Omega : |u_i(x)| \geq r\} = 0$$

and

$$\lim_{r\to+\infty} \text{meas}\{x \in \partial\Omega : |u_i(x)| \geq r\} = 0,$$

since $u_i \in \mathcal{T}^{1,p}_{tr}(\Omega)$, we obtain that

$$\lim_{h\to\infty} L^2_{h,k} = 0.$$

Similarly, $\lim_{h\to\infty} L^1_{h,k} = 0$. Therefore by (5), (4) holds. Now, from (4), (3) and (5), we have that

$$\lim_{h\to+\infty} \int_{\{|u_1|<h,\ |u_2|<h\}} (\mathbf{a}(x, Du_1) - \mathbf{a}(x, Du_2))\cdot DT_k(u_1-u_2) = 0.$$

Therefore, for any $h > 0$, $DT_h(u_1) = DT_h(u_2)$ *a.e.* in Ω. Consequently, there exists a constant c such that

$$u_1 - u_2 = c \ \ a.e. \ \text{ in } \Omega.$$

Moreover, by (3) and (4), we have

$$\int_\Omega (z_1 - z_2) T_k(u_1 - u_2) + \int_{\partial\Omega} (w_1 - w_2) T_k(u_1 - u_2) = 0 \quad \forall\, k > 0, \tag{6}$$

from where it follows that

$$(w_1 - w_2)\chi_{\{u_1 - u_2 \neq 0\}} = 0 \ \ a.e. \ \text{ in } \partial\Omega,$$

and

$$(z_1 - z_2)\chi_{\{u_1 - u_2 \neq 0\}} = 0 \ \ a.e. \ \text{ in } \Omega.$$

Then, if $c \neq 0$ it follows that $w_1 = w_2$, and $z_1 = z_2$.

In order to see that $z_1 = z_2$ in the case $c = 0$, we take $T_h(u_1) - \varphi$ and $T_h(u_1) + \varphi$, $\varphi \in D(\Omega)$, as test functions in (2) for the solution $[u_1, z_1, w_1]$ and $[u_1, z_2, w_2]$, respectively, adding these inequalities and letting h go to $+\infty$, if $k > \|\varphi\|_\infty$, we get

$$\lim_{h\to\infty} J_{h,k} + \int_\Omega (z_1 - z_2)\varphi \leq 0,$$

where

$$\begin{aligned} J_{h,k} &= \int_\Omega \mathbf{a}(x, Du_1) \cdot [DT_k(u_1 - T_h(u_1) + \varphi) + DT_k(u_1 - T_h(u_1) - \varphi)] \\ &= \int_{\{|u_1| > h\}} \mathbf{a}(x, Du_1) \cdot [DT_k(u_1 - T_h(u_1) + \varphi) + DT_k(u_1 - T_h(u_1) - \varphi)]. \end{aligned}$$

Then, using Hölder's inequality and Lemma 4.1, we obtain that

$$\lim_{h\to\infty} J_{h,k} = 0.$$

Hence

$$\int_\Omega z_1\varphi \leq \int_\Omega z_2\varphi.$$

Similarly,

$$\int_\Omega z_2\varphi \leq \int_\Omega z_1\varphi.$$

Therefore $z_1 = z_2$.

If $c \neq 0$, following the arguments of Lemma 3.5 of [6], we have that $z_1 = z_2$ is constant. In fact, let $j(r) = \int_0^r \gamma^0(s)ds$, therefore, $\gamma = \partial j$, the subdifferential of j. Now, $z_1(x) \in \gamma(u_1(x)) \cap \gamma(u_1(x) + c)$ *a.e.* $x \in \Omega$, consequently, $j(u_1(x) + c) - j(u_1(x)) = cz_1(x)$ *a.e.* in Ω. Moreover, if $\gamma(\mathbb{R})$ is bounded, j is Lipschitz continuous, $j(T_k(u_1) + c), j(T_k(u_1)) \in W^{1,p}(\Omega)$ and $\nabla\left(j(T_k(u_1) + c) - j(T_k(u_1))\right) = 0$ *a.e.* in Ω. The above identity is obvious when $|u_1| \geq k$, and in the case $|u_1| < k$, we have $\nabla\left(j(u_1 + c) - j(u_1)\right) = 0$. Therefore $j(T_k(u_1) + c) - j(T_k(u_1))$ is constant (this constant, in fact, does not depend on k) and consequently cz_1 is constant. As $c \neq 0$, z_1 is constant. In the case γ is not bounded, we work, again as in Lemma 3.5 of [6], truncating γ.

Finally, in order to see that $w_1 = w_2$, we use the fact that we can take as test function in (2), for the corresponding $(S^{\gamma,\beta}_{\phi,\psi})$, $v = T_h(u_i) \pm \varphi$, for any $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then, since $u_1 = u_2 + c$ and $z_1 = z_2$, we get

$$\int_{\partial\Omega} w_1\varphi = \int_{\partial\Omega} w_2\varphi.$$

Therefore $w_1 = w_2$. □

5 Existence results

In this section we give the existence results. Let us begin with some approximation results which allow to get the existence results.

5.1 Approximated problems

For $m, n \in \mathbb{N}$, we approximate γ and β by $\gamma_{m,n}(r) = \gamma(r) + \frac{1}{m}r^+ - \frac{1}{n}r^-$ and $\beta_{m,n}(r) = \beta(r) + \frac{1}{m}r^+ - \frac{1}{n}r^-$ respectively, so we first consider the problem

$$(S_{\phi,\psi}^{\gamma_{m,n},\beta_{m,n}}) \quad \begin{cases} -\text{div } \mathbf{a}(x, Du) + \gamma_{m,n}(u) \ni \phi & \text{in } \Omega \\ \\ \mathbf{a}(x, Du) \cdot \eta + \beta_{m,n}(u) \ni \psi & \text{on } \partial\Omega. \end{cases}$$

For $(S_{\phi,\psi}^{\gamma_{m,n},\beta_{m,n}})$, we have the following existence and uniqueness results.

Proposition 5.1 *Assume $D(\gamma) = D(\beta) = \mathbb{R}$. Let $m, n \in \mathbb{N}$, $m \le n$. Then, the following hold.*

(i) For $\phi \in L^\infty(\Omega)$ and $\psi \in L^\infty(\partial\Omega)$, there exist $u = u_{\phi,\psi,m,n} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $z = z_{\phi,\psi,m,n} \in L^\infty(\Omega)$, $z(x) \in \gamma(u(x))$ a.e. in Ω, and $w = w_{\phi,\psi,m,n} \in L^\infty(\partial\Omega)$, $w(x) \in \beta(u(x))$ a.e. in $\partial\Omega$, such that $[u, z, w]$ is a weak solution of $(S_{\phi,\psi}^{\gamma_{m,n},\beta_{m,n}})$.
Moreover, if $M := \|\phi\|_\infty + \|\psi\|_\infty$,

$$-nM \le u \le nM,$$

$$-\gamma^0(-nM) \le z \le \gamma^0(nM),$$

and there exists $c(\Omega, N, p) > 0$ such that

$$\|Du\|_{L^p(\Omega)}^{p-1} \le \frac{c(\Omega, N, p)}{\lambda} \left(\|\phi\|_{V^{1,p}(\Omega)} + \|\psi\|_{V^{1,p}(\partial\Omega)}\right).$$

(ii) If $m_1 \le m_2 \le n_2 \le n_1$, $\phi_1, \phi_2 \in L^\infty(\Omega)$, $\psi_1, \psi_2 \in L^\infty(\partial\Omega)$ then

$$\int_\Omega (z_{\phi_1,\psi_1,m_1,n_1} - z_{\phi_2,\psi_2,m_2,n_2})^+ + \int_{\partial\Omega} (w_{\phi_1,\psi_1,m_1,n_1} - w_{\phi_2,\psi_2,m_2,n_2})^+$$

$$\le \int_{\partial\Omega} (\psi_1 - \psi_2)^+ + \int_\Omega (\phi_1 - \phi_2)^+.$$

Proof. Observe that $\frac{1}{m}s^+ - \frac{1}{n}s^- = \frac{1}{m}s + \left(\frac{1}{m} - \frac{1}{n}\right)s^- = \left(\frac{1}{m} - \frac{1}{n}\right)s^+ + \frac{1}{n}s$.

Let us take

$$c_r > \sup\{nM, \gamma_r(nM), -\gamma_r(-nM), \beta_r(nM), -\beta_r(-nM)\},$$

where γ_r and β_r are the Yosida approximations of γ and β, respectively. For $r \in \mathbb{N}$, it is easy to see that the operator $B_r : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$ defined by

$$\langle B_r u, v\rangle = \int_\Omega \mathbf{a}(x, Du) \cdot Dv + \int_\Omega T_{c_r}(\gamma_r(u))v + \frac{1}{r}\int_\Omega |u|^{p-2}uv$$

$$+\frac{1}{m}\int_\Omega T_{c_r}(u^+)v - \frac{1}{n}\int_\Omega T_{c_r}(u^-)v + \int_{\partial\Omega} T_{c_r}(\beta_r(u))v$$

$$+\frac{1}{m}\int_{\partial\Omega} T_{c_r}(u^+)v - \frac{1}{n}\int_{\partial\Omega} T_{c_r}(u^-)v - \int_{\partial\Omega} \psi v - \int_\Omega \phi v,$$

is bounded, coercive, monotone and hemicontinuous. Let $K = W^{1,p}(\Omega)$. Then, by a classical result of Browder ([21]), there exists $u_r = u_{\phi,\psi,m,n,r} \in K$, such that

$$\begin{aligned} \int_\Omega \mathbf{a}(x, Du_r) \cdot Dv + \int_\Omega T_{c_r}(\gamma_r(u_r))v + \frac{1}{r}\int_\Omega |u_r|^{p-2}u_r v \\ + \frac{1}{m}\int_\Omega T_{c_r}((u_r)^+)v - \frac{1}{n}\int_\Omega T_{c_r}((u_r)^-)v \\ + \int_{\partial\Omega} T_{c_r}(\beta_r(u_r))v + \frac{1}{m}\int_{\partial\Omega} T_{c_r}((u_r)^+)v - \frac{1}{n}\int_{\partial\Omega} T_{c_r}((u_r)^-)v \\ = \int_{\partial\Omega} \psi v + \int_\Omega \phi v, \end{aligned} \tag{7}$$

for all $v \in K$.

Taking $v = T_k((u_r - mM)^+)$ in (7), misleading non negative terms, dividing by k, and taking limits as k goes to 0, we get

$$\frac{1}{m}\int_\Omega T_{c_r}(u_r)\mathrm{sign}^+(u_r - mM) + \frac{1}{m}\int_{\partial\Omega} T_{c_r}(u_r)\mathrm{sign}^+(u_r - mM)$$

$$\leq \int_{\partial\Omega} \psi\, \mathrm{sign}^+(u_r - mM) + \int_\Omega \phi\, \mathrm{sign}^+(u_r - mM).$$

Consequently

$$\int_\Omega (T_{c_r}(u_r) - mM)\mathrm{sign}^+(u_r - mM) + \int_{\partial\Omega} (T_{c_r}(u_r) - mM)\mathrm{sign}^+(u_r - mM)$$

$$\leq \int_{\partial\Omega} (m\psi - mM)\mathrm{sign}^+(u_r - mM) + \int_\Omega (m\phi - mM)\mathrm{sign}^+(u_r - mM) \leq 0.$$

Therefore, since $m \leq n$,

$$u_r(x) \leq nM \quad a.e. \text{ in } \Omega.$$

Similarly, taking $v = T_k((u_r + nM)^-)$ in (7), we get

$$u_r(x) \geq -nM \quad a.e. \text{ in } \Omega.$$

Consequently,

$$\|u_r\|_\infty \leq nM, \tag{8}$$

and (7) yields

$$\begin{aligned} \int_\Omega \mathbf{a}(x, Du_r) \cdot Dv + \int_\Omega \gamma_r(u_r)v + \frac{1}{r}\int_\Omega |u_r|^{p-2}u_r v + \frac{1}{m}\int_\Omega u_r^+ v - \frac{1}{n}\int_\Omega u_r^- v \\ + \int_{\partial\Omega} \beta_r(u_r)v + \frac{1}{m}\int_{\partial\Omega} u_r^+ v - \frac{1}{n}\int_{\partial\Omega} u_r^- v = \int_{\partial\Omega} \psi v + \int_\Omega \phi v, \end{aligned} \tag{9}$$

for all $v \in W^{1,p}(\Omega)$.

Taking $v = T_k((u_r)^+)$ in (9), disregarding some positive terms, dividing by k and letting k go to ∞ we get that

$$\frac{1}{m}\int_\Omega u_r^+ + \int_\Omega \gamma_r(u_r)^+ + \int_{\partial\Omega}\beta_r(u_r)^+ \le \int_\Omega \phi^+ + \int_{\partial\Omega}\psi^+, \tag{10}$$

and, similarly, taking $T_k((u_r)^-)$ we get

$$\frac{1}{n}\int_\Omega u_r^- + \int_\Omega \gamma_r(u_r)^- + \int_{\partial\Omega}\beta_r(u_r)^- \le \int_\Omega \phi^- + \int_{\partial\Omega}\psi^-. \tag{11}$$

Taking $v = u_r - \frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega} u_r$ as test function in (9) and having in mind that

$$\int_{\partial\Omega}\beta_r(u_r)\left(u_r - \frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega}u_r\right)$$
$$= \int_{\partial\Omega}\left(\beta_r(u_r) - \beta_r\left(\frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega}u_r\right)\right)\left(u_r - \frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega}u_r\right) \ge 0;$$

$$\int_\Omega \gamma_r(u_r)\left(u_r - \frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega}u_r\right) = \int_\Omega\left(\gamma_r(u_r) - \gamma_r\left(\frac{1}{\text{meas}(\Omega)}\int_\Omega u_r\right)\right)\left(u_r - \frac{1}{\text{meas}(\Omega)}\int_\Omega u_r\right)$$
$$-\int_\Omega \gamma_r(u_r)\left(\frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega}u_r - \frac{1}{\text{meas}(\Omega)}\int_\Omega u_r\right)$$
$$\ge -\int_\Omega \gamma_r(u_r)\left(\frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega}u_r - \frac{1}{\text{meas}(\Omega)}\int_\Omega u_r\right)$$

and working similarly with the other terms, we get

$$\lambda\int_\Omega |Du_r|^p \le \int_\Omega \phi\left(u_r - \frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega}u_r\right) + \int_{\partial\Omega}\psi\left(u_r - \frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega}u_r\right)$$
$$-\int_\Omega \gamma_r(u_r)\left(\frac{1}{\text{meas}(\Omega)}\int_\Omega u_r - \frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega}u_r\right)$$
$$-\frac{1}{m}\int_\Omega u_r^+\left(\frac{1}{\text{meas}(\Omega)}\int_\Omega u_r - \frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega}u_r\right)$$
$$+\frac{1}{n}\int_\Omega u_r^-\left(\frac{1}{\text{meas}(\Omega)}\int_\Omega u_r - \frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega}u_r\right).$$

Now, by Poincaré's inequality and the trace Theorem, there exists $c_1 = c_1(\Omega, N, p) > 0$ such that

$$\int_\Omega \phi\left(u_r - \frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega}u_r\right) \le c_1\|\phi\|_{V^{1,p}(\Omega)}\|Du_r\|_{L^p(\Omega)},$$

and

$$\int_{\partial\Omega}\psi\left(u_r - \frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega}u_r\right) \le c_1\|\psi\|_{V^{1,p}(\partial\Omega)}\|Du_r\|_{L^p(\Omega)}.$$

On the other hand, by (10) and (11),

$$-\int_\Omega \gamma_r(u_r)\left(\frac{1}{\text{meas}(\Omega)}\int_\Omega u_r - \frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega}u_r\right) - \frac{1}{m}\int_\Omega u_r^+\left(\frac{1}{\text{meas}(\Omega)}\int_\Omega u_r - \frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega}u_r\right)$$

$$+\frac{1}{n}\int_\Omega u_r^- \left(\frac{1}{\text{meas}(\Omega)}\int_\Omega u_r - \frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega} u_r\right)$$
$$\leq 2\left(\int_{\partial\Omega}|\psi| + \int_\Omega |\phi|\right)\left|\frac{1}{\text{meas}(\Omega)}\int_\Omega u_r - \frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega} u_r\right|.$$

Moreover, applying again the generalized Poincaré inequality, there exists $c_2 = c_2(\Omega, N, p) > 0$ such that

$$\left|\frac{1}{\text{meas}(\Omega)}\int_\Omega u_r - \frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega} u_r\right| \leq$$
$$\frac{1}{\text{meas}(\Omega)^{\frac{1}{p}}}\left(\left\|u_r - \frac{1}{\text{meas}(\Omega)}\int_\Omega u_r\right\|_{L^p(\Omega)} + \left\|u_r - \frac{1}{\text{meas}(\partial\Omega)}\int_{\partial\Omega} u_r\right\|_{L^p(\Omega)}\right)$$
$$\leq c_2\|Du_r\|_{L^p(\Omega)}.$$

Therefore, there exists $c_3 = c_3(\Omega, N, p) > 0$, such that

$$\|Du_r\|^{p-1}_{L^p(\Omega)} \leq \frac{c_3}{\lambda}\left(\|\phi\|_{V^{1,p}(\Omega)} + \|\psi\|_{V^{1,p}(\partial\Omega)}\right). \tag{12}$$

As a consequence of (8) and (12) we can suppose that there exists a subsequence, still denoted u_r, such that

$$u_r \quad \text{converges weakly in } W^{1,p}(\Omega) \text{ to } u \in W^{1,p}(\Omega),$$
$$u_r \quad \text{converges in } L^q(\Omega) \text{ and } a.e. \text{ in } \Omega \text{ to } u, \text{ for any } q \geq 1,$$
$$u_r \quad \text{converges in } L^p(\partial\Omega) \text{ and } a.e. \text{ to } u,$$

with

$$-nM \leq u \leq nM. \tag{13}$$

Taking into account (13), we get that $|\gamma_r(u_r)|$ is uniformly bounded. Consequently, we can assume that $\gamma_r(u_r) \to z \in L^\infty(\Omega)$ weakly*, moreover

$$-\gamma^0(-nM) \leq z \leq \gamma^0(nM).$$

Since $u_r \to u$ in $L^1(\Omega)$, applying [9, Lemma G], it follows that $z(x) \in \gamma(u(x))$ $a.e.$ on Ω.

On the other hand, since $\beta_r(u_r)$ is also uniformly bounded, we can assume that $\beta_r(u_r) \to w \in L^\infty(\partial\Omega)$ weakly*. Again, applying [9, Lemma G], it follows that $w(x) \in \beta(u(x))$ $a.e.$ in $\partial\Omega$.

Let us see now that $\{Du_r\}$ converges in measure to Du. We follow the technique used in [10] (see also [3]). Since Du_r converges to Du weakly in $L^p(\Omega)$, it is enough to show that $\{Du_r\}$ is a Cauchy sequence in measure. Let t and $\epsilon > 0$. For some $A > 1$, we set

$$C(x, A, t) := \inf\{(\mathbf{a}(x,\xi) - \mathbf{a}(x,\eta))\cdot(\xi - \eta) \;:\; |\xi| \leq A,\ |\eta| \leq A,\ |\xi - \eta| \geq t\ \}.$$

Having in mind that the function $\psi \to a(x,\psi)$ is continuous for almost all $x \in \Omega$ and the set $\{(\xi,\eta) : |\xi| \leq A,\ |\eta| \leq A,\ |\xi - \eta| \geq t\ \}$ is compact, the infimum in the definition of $C(x, A, t)$ is a minimum. Hence, by (H_3), it follows that

$$C(x, A, t) > 0 \quad \text{for almost all } \ x \in \Omega. \tag{14}$$

Now, for $r, s \in \mathbb{N}$ and any $k > 0$, the following inclusion holds

$$\begin{gathered}\{|Du_r - Du_s| > t\} \\ \subset \{|Du_r| \geq A\} \cup \{|Du_s| \geq A\} \cup \{|u_r - u_s| \geq k^2\} \cup \{C(x, A, t) \leq k\} \cup G,\end{gathered} \tag{15}$$

where

$$G = \{|u_r - u_s| \le k^2,\ C(x,A,t) \ge k,\ |Du_r| \le A,\ |Du_s| \le A,\ |Du_r - Du_s| > t\}.$$

Since the sequence Du_r is bounded in $L^p(\Omega)$ we can choose A large enough in order to have

$$\text{meas}(\{|Du_r| \ge A\} \ \cup \ \{|Du_s| \ge A\}) \le \frac{\epsilon}{4} \quad \text{for all} \quad r,s \in \mathbb{N}. \tag{16}$$

By (14), we can choose k small enough in order to have

$$\text{meas}(\{C(x,A,t) \le k\}) \le \frac{\epsilon}{4}. \tag{17}$$

On the other hand, if we use $T_k(u_r - u_s)$ and $T_k(u_r - u_s)$ as test functions in (9) for u_r and u_s respectively, we obtain

$$\begin{gathered}\int_\Omega \mathbf{a}(x,Du_r)\cdot DT_k(u_r-u_s) + \int_\Omega \gamma_r(u_r)T_k(u_r-u_s)\\ +\frac{1}{r}\int_\Omega |u_r|^{p-2}u_rT_k(u_r-u_s) + \frac{1}{m}\int_\Omega u_r^+T_k(u_r-u_s) - \frac{1}{n}\int_\Omega u_r^-T_k(u_r-u_s)\\ +\int_{\partial\Omega}\beta_r(u_r)T_k(u_r-u_s) + \frac{1}{m}\int_{\partial\Omega} u_r^+T_k(u_r-u_s) - \frac{1}{n}\int_{\partial\Omega} u_r^-T_k(u_r-u_s)\\ = \int_{\partial\Omega}\psi T_k(u_r-u_s) + \int_\Omega \phi T_k(u_r-u_s),\end{gathered} \tag{18}$$

and

$$\begin{gathered}-\int_\Omega \mathbf{a}(x,Du_s)\cdot DT_k(u_r-u_s) - \int_\Omega \gamma_s(u_s)T_k(u_r-u_s)\\ -\frac{1}{s}\int_\Omega |u_s|^{p-2}u_sT_k(u_r-u_s) - \frac{1}{m}\int_\Omega u_s^+T_k(u_r-u_s) + \frac{1}{n}\int_\Omega u_s^-T_k(u_r-u_s)\\ -\int_{\partial\Omega}\beta_s(u_s)T_k(u_r-u_s) - \frac{1}{m}\int_{\partial\Omega} u_s^+T_k(u_r-u_s) + \frac{1}{n}\int_{\partial\Omega} u_s^-T_k(u_r-u_s)\\ = -\int_{\partial\Omega}\psi T_k(u_r-u_s) - \int_\Omega \phi T_k(u_r-u_s).\end{gathered} \tag{19}$$

Adding (18) and (19) and disregarding some positive terms, we get

$$\begin{gathered}\int_\Omega (\mathbf{a}(x,Du_r)-\mathbf{a}(x,Du_s))\cdot DT_k(u_r-u_s) \le -\int_\Omega (\gamma_r(u_r)-\gamma_s(u_s))\,T_k(u_r-u_s)\\ -\int_\Omega\left(\frac{1}{r}|u_r|^{p-2}u_r - \frac{1}{s}|u_s|^{p-2}u_s\right)T_k(u_r-u_s) - \int_{\partial\Omega}(\beta_r(u_r)-\beta_s(u_s))\,T_k(u_r-u_s).\end{gathered}$$

Consequently, there exists a constant $\hat{M}$ independent of r and s such that

$$\int_\Omega (\mathbf{a}(x,Du_r)-\mathbf{a}(x,Du_s))\cdot DT_k(u_r-u_s) \le k\hat{M}.$$

Hence

$$
\begin{aligned}
&\operatorname{meas}(G) \\
&\leq \operatorname{meas}(\{|u_r - u_s| \leq k^2,\ (\mathbf{a}(x, Du_r) - \mathbf{a}(x, Du_s)) \cdot D(u_r - u_s) \geq k\}) \\
&\qquad \leq \frac{1}{k} \int_{\{|u_r - u_s| < k^2\}} (\mathbf{a}(x, Du_r) - \mathbf{a}(x, Du_s)) \cdot D(u_r - u_s) \\
&\qquad = \frac{1}{k} \int_\Omega (\mathbf{a}(x, Du_r) - \mathbf{a}(x, Du_s)) \cdot DT_{k^2}(u_r - u_s) \leq \frac{1}{k}\ k^2 \hat{M} \leq \frac{\epsilon}{4}
\end{aligned}
\tag{20}
$$

for k small enough.

Since A and k have been already chosen, if r_0 is large enough we have for $r, s \geq r_0$ the estimate $\operatorname{meas}(\{|u_r - u_s| \geq k^2\}) \leq \frac{\epsilon}{4}$. From here, using (15), (16), (17) and (20), we can conclude that

$$
\operatorname{meas}(\{|Du_r - Du_s| \geq t\}) \leq \epsilon \quad \text{for} \quad r, s \geq r_0.
$$

From here, up to extraction of a subsequence, we also have $\mathbf{a}(., Du_r)$ converges in measure and $a.e.$ to $\mathbf{a}(., Du)$. Now, by (H_2) and (12),

$$
\mathbf{a}(., Du_r) \text{ converges weakly in } L^{p'}(\Omega)^N \text{ to } \mathbf{a}(., Du).
$$

Finally, letting $r \to +\infty$ in (9), we prove (i).

In order to prove (ii), we write $u_{1,r} = u_{\phi_1,\psi_1,m_1,n_1,r}$ and $u_{2,r} = u_{\phi_2,\psi_2,m_2,n_2,r}$. Taking $T_k((u_{1,r} - u_{2,r})^+)$, with r large enough, as test function in (9) for $u_{1,r}$, $m = m_1$ and $n = n_1$, we get

$$
\begin{aligned}
&\int_\Omega \mathbf{a}(x, Du_{1,r}) \cdot DT_k((u_{1,r} - u_{2,r})^+) + \int_\Omega \gamma_r(u_{1,r}) T_k((u_{1,r} - u_{2,r})^+) \\
&+\frac{1}{r} \int_\Omega |u_{1,r}|^{p-2} u_{1,r} T_k((u_{1,r} - u_{2,r})^+) + \frac{1}{m_1} \int_\Omega u_{1,r}^+ T_k((u_{1,r} - u_{2,r})^+) - \frac{1}{n_1} \int_\Omega u_{1,r}^- T_k((u_{1,r} - u_{2,r})^+) \\
&+ \int_{\partial\Omega} \beta_r(u_{1,r}) T_k((u_{1,r} - u_{2,r})^+) + \frac{1}{m_1} \int_{\partial\Omega} u_{1,r}^+ T_k((u_{1,r} - u_{2,r})^+) - \frac{1}{n_1} \int_{\partial\Omega} u_{1,r}^- T_k((u_{1,r} - u_{2,r})^+) \\
&= \int_{\partial\Omega} \psi_1 T_k((u_{1,r} - u_{2,r})^+) + \int_\Omega \phi_1 T_k((u_{1,r} - u_{2,r})^+),
\end{aligned}
$$

and taking $T_k(u_{1,r} - u_{2,r})^+$ as test function in (9) for $u_{2,r}$, $m = m_2$ and $n = n_2$, we get

$$
\begin{aligned}
&- \int_\Omega \mathbf{a}(x, Du_{2,r}) \cdot DT_k((u_{1,r} - u_{2,r})^+) - \int_\Omega \gamma_r(u_{2,r}) T_k((u_{1,r} - u_{2,r})^+) \\
&-\frac{1}{r} \int_\Omega |u_{2,r}|^{p-2} u_{2,r} T_k((u_{1,r} - u_{2,r})^+) - \frac{1}{m_2} \int_\Omega u_{2,r}^+ T_k((u_{1,r} - u_{2,r})^+) + \frac{1}{n_2} \int_\Omega u_{2,r}^- T_k((u_{1,r} - u_{2,r})^+) \\
&- \int_{\partial\Omega} \beta_r(u_{2,r}) T_k((u_{1,r} - u_{2,r})^+) - \frac{1}{m_2} \int_{\partial\Omega} u_{2,r}^+ T_k((u_{1,r} - u_{2,r})^+) + \frac{1}{n_2} \int_{\partial\Omega} u_{2,r}^- T_k((u_{1,r} - u_{2,r})^+) \\
&= - \int_{\partial\Omega} \psi_2 T_k((u_{1,r} - u_{2,r})^+) - \int_\Omega \phi_2 T_k((u_{1,r} - u_{2,r})^+).
\end{aligned}
$$

Adding these two inequalities, misleading some non negative terms, dividing by k, and letting $k \to 0$, we get

$$\begin{aligned}\int_\Omega (\gamma_r(u_{1,r}) - \gamma_r(u_{2,r}))^+ + \int_{\partial\Omega} (\beta_r(u_{1,r}) - \beta_r(u_{2,r}))^+ \\ \leq \int_{\partial\Omega} (\psi_1 - \psi_2)^+ + \int_\Omega (\phi_1 - \phi_2)^+.\end{aligned} \tag{21}$$

Therefore, taking into account the above convergence, (ii) is obtained. □

In the case $\psi = 0$, we have the following result.

Proposition 5.2 *Assume $D(\beta) = \mathbb{R}$. Let $m, n \in \mathbb{N}$, $m \leq n$. Then, the following hold.*

(i) For $\phi \in L^\infty(\Omega)$, there exist $u = u_{\phi,m,n} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $z = z_{\phi,m,n} \in L^\infty(\Omega)$, $z(x) \in \gamma(u(x))$ a.e. in Ω, and $w = w_{\phi,m,n} \in L^\infty(\partial\Omega)$, $w(x) \in \beta(u(x))$ a.e. in $\partial\Omega$, such that $[u, z, w]$ is a weak solution of problem $(S_{\phi,0}^{\gamma_{m,n},\beta_{m,n}})$, and $z << \phi$.

(ii) If $m_1 \leq m_2 \leq n_2 \leq n_1$, $\phi_1, \phi_2 \in L^\infty(\Omega)$, then

$$\int_\Omega (z_{\phi_1,m_1,n_1} - z_{\phi_2,m_2,n_2})^+ + \int_{\partial\Omega} (w_{\phi_1,m_1,n_1} - w_{\phi_2,m_2,n_2})^+ \leq \int_\Omega (\phi_1 - \phi_2)^+.$$

Proof. Following the proof of Proposition 5.1 there exists $u_r = u_{\phi,m,n,r} \in K = W^{1,p}(\Omega)$, such that

$$\|u_r\|_\infty \leq n\|\phi\|_\infty,$$

and

$$\begin{aligned}\int_\Omega \mathbf{a}(x, Du_r) \cdot Dv + \frac{1}{r} \int_\Omega |u_r|^{p-2} u_r v \\ + \int_\Omega \gamma_r(u_r)v + \frac{1}{m} \int_\Omega u_r^+ (u_r - v) - \frac{1}{n} \int_\Omega u_r^- v \\ + \int_{\partial\Omega} \beta_r(u_r)v + \frac{1}{m} \int_{\partial\Omega} u_r^+ v - \frac{1}{n} \int_{\partial\Omega} u_r^- v = \int_\Omega \phi v,\end{aligned} \tag{22}$$

for all $v \in K$.

We can finish the proof as in Propositions 5.1 if we prove that $\gamma_r(u_r)$ is weakly convergent in $L^1(\Omega)$. Taking $v = p(\gamma_r(u_r))$, $p \in P_0$, as test function in (22) we have that, after misleading non negative terms,

$$\int_\Omega \gamma_r(u_r) p(\gamma_r(u_r)) \leq \int_\Omega \phi p(\gamma_r(u_r)),$$

which implies, $\gamma_r(u_r) << \phi$. In particular, see Proposition 2.1, $\|\gamma_r(u_r)\|_\infty \leq \|\phi\|_\infty$ and $\gamma_r(u_r) \to z \in L^\infty(\Omega)$ weakly in $L^1(\Omega)$, with $z << \phi$. □

Remark 5.3 Observe that if $D(\beta) = \{0\}$ and γ is anyone, rewriting the proof of Proposition 5.2, with $K = W_0^{1,p}(\Omega)$, we find $u = u_{\phi,m,n} \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $z = z_{\phi,m,n} \in L^\infty(\Omega)$, $z(x) \in \gamma(u(x))$ *a.e.* in Ω, such that

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv + \int_\Omega zv + \frac{1}{m} \int_\Omega u^+ v - \frac{1}{n} \int_\Omega u^- v = \int_\Omega \phi v,$$

for all $v \in W_0^{1,p}(\Omega)$. Moreover, if $m_1 \leq m_2 \leq n_2 \leq n_1$, $\phi_1, \phi_2 \in L^\infty(\Omega)$, then

$$\int_\Omega (z_{\phi_1,\psi_1,m_1,n_1} - z_{\phi_2,\psi_2,m_2,n_2})^+ \leq \int_\Omega (\phi_1 - \phi_2)^+.$$

Proposition 5.4 *Assume $D(\gamma)=\mathbb{R}$ and* **a** *smooth. Let $m,n\in\mathbb{N}$, $m\leq n$. Then, the following hold.*

(i) For $\phi\in L^\infty(\Omega)$ and $\psi\in L^\infty(\partial\Omega)$, there exist $u=u_{\phi,\psi,m,n}\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$, $z=z_{\phi,\psi,m,n}\in L^\infty(\Omega)$, $z(x)\in\gamma(u(x))$ a.e. in Ω, and $w=w_{\phi,\psi,m,n}\in L^1(\partial\Omega)$, $w(x)\in\beta(u(x))$ a.e. in $\partial\Omega$, such that $[u,z,w]$ is a weak solution of $(S_{\phi,\psi}^{\gamma_{m,n},\beta_{m,n}})$.
Moreover, there exists $c(\Omega,p,\lambda)>0$ such that

$$\|Du\|_{L^p(\Omega)}\leq c(\Omega,p,\lambda)\left(\|\phi\|_{V^{1,p}(\Omega)}+\|\psi\|_{V^{1,p}(\partial\Omega)}\right)^{\frac{1}{p-1}}.$$

(ii) If $m_1\leq m_2\leq n_2\leq n_1$, $\phi_1,\phi_2\in L^\infty(\Omega)$, $\psi_1,\psi_2\in L^\infty(\partial\Omega)$ then

$$\int_\Omega (z_{\phi_1,\psi_1,m_1,n_1}-z_{\phi_2,\psi_2,m_2,n_2})^+ + \int_{\partial\Omega}(w_{\phi_1,\psi_1,m_1,n_1}-w_{\phi_2,\psi_2,m_2,n_2})^+$$

$$\leq \int_{\partial\Omega}(\psi_1-\psi_2)^+ + \int_\Omega(\phi_1-\phi_2)^+.$$

Proof. Applying Proposition 5.1 to β_r, the Yosida approximation of β, there exists $u_r=u_{\phi,\psi,m,n,r}\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ and $z_r=z_{\phi,\psi,m,n,r}\in L^\infty(\Omega)$, $z_r\in\gamma(u_r)$ *a.e.* in Ω, such that

$$\begin{aligned}&\int_\Omega \mathbf{a}(x,Du_r)\cdot Dv+\int_\Omega z_r v+\int_{\partial\Omega}\beta_r(u_r)v\\ &+\frac{1}{m}\int_\Omega u_r^+v-\frac{1}{n}\int_\Omega u_r^-v+\frac{1}{m}\int_{\partial\Omega}u_r^+v-\frac{1}{n}\int_{\partial\Omega}u_r^-v\\ &=\int_{\partial\Omega}\psi v+\int_\Omega\phi v,\end{aligned}\tag{23}$$

for all $v\in W^{1,p}(\Omega)$. Moreover, $|u_r|$ is uniformly bounded by nM, $M:=\|\phi\|_\infty+\|\psi\|_\infty$,

$$-\gamma^0(-nM)\leq z_r\leq\gamma^0(nM),$$

and

$$\int_\Omega z_r^\pm+\int_{\partial\Omega}w_r^\pm\leq\int_{\partial\Omega}\psi^\pm+\int_\Omega\phi^\pm.$$

Let now $\hat{u}\in L^\infty(\Omega)$ and $\hat{z}\in\gamma(\hat{u})$, $\hat{z}\in L^\infty(\Omega)$, be such that $\hat{u}$ is solution of the Dirichlet problem (see Remark 5.3)

$$\begin{cases}-\text{div }\mathbf{a}(x,D\hat{u})+\hat{z}+\frac{1}{m}\hat{u}^+-\frac{1}{n}\hat{u}^-=\phi & \text{in }\Omega\\ \\ \hat{u}=0 & \text{on }\partial\Omega.\end{cases}$$

Since **a** is smooth, there exists $\hat{\psi}\in L^1(\partial\Omega)$ such that

$$\int_\Omega\mathbf{a}(x,D\hat{u})\cdot Dv+\int_\Omega\hat{z}v+\frac{1}{m}\int_\Omega\hat{u}^+v-\frac{1}{n}\int_\Omega\hat{u}^-v=\int_{\partial\Omega}\hat{\psi}v+\int_\Omega\phi v,\tag{24}$$

for any $v\in W^{1,p}(\Omega)\cap L^\infty(\Omega)$.

Taking $v=p(\beta_r(u_r-\hat{u}))$, $p\in P_0$, as test function in (23), and $p(\beta_r(u_r-\hat{u}))$ as test function in (24), and adding both equalities we get, after misleading non negative terms, that

$$\int_{\partial\Omega}\beta_r(u_r)p(\beta_r(u_r))\leq\int_{\partial\Omega}(\psi-\hat{\psi})p(\beta_r(u_r)),$$

which implies (see Proposition 2.1) that

$$\beta_r(u_r) \to w \in L^1(\partial\Omega) \text{ weakly in } L^1(\partial\Omega).$$

Now, arguing as in the proof of Proposition 5.1, we obtain (i).

To prove (ii), Proposition 5.1 implies, denoting $u_{i,r} = u_{\phi_i,\psi_i,m_i,n_i,r}$ and $z_{i,r} = z_{\phi_i,\psi_i,m_i,n_i,r}$, $i = 1, 2$,

$$\begin{aligned} &\int_\Omega (z_{1,r} - z_{2,r})^+ + \int_{\partial\Omega} (\beta_r(u_{1,r}) - \beta_r(u_{2,r}))^+ \\ &\qquad \leq \int_{\partial\Omega} (\psi_1 - \psi_2)^+ + \int_\Omega (\phi_1 - \phi_2)^+. \end{aligned} \tag{25}$$

Taking limits in (25) when r goes to $+\infty$, (ii) holds. □

In the case $\psi = 0$, we have the following result.

Proposition 5.5 *Assume* **a** *smooth. Let $m, n \in \mathbb{N}$, $m \leq n$. Then, the following hold.*

(i) For $\phi \in L^\infty(\Omega)$, there exist $u = u_{\phi,m,n} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $z = z_{\phi,m,n} \in L^\infty(\Omega)$, $z(x) \in \gamma(u(x))$ a.e. in Ω, and $w = w_{\phi,m,n} \in L^1(\partial\Omega)$, $w(x) \in \beta(u(x))$ a.e. in $\partial\Omega$, such that $[u, z, w]$ is a weak solution of problem $(S_{\phi,0}^{\gamma_{m,n},\beta_{m,n}})$, with $z << \phi$.

(ii) If $m_1 \leq m_2 \leq n_2 \leq n_1$, $\phi_1, \phi_2 \in L^\infty(\Omega)$, then

$$\int_\Omega (z_{\phi_1,m_1,n_1} - z_{\phi_2,m_2,n_2})^+ + \int_{\partial\Omega} (w_{\phi_1,m_1,n_1} - w_{\phi_2,m_2,n_2})^+ \leq \int_\Omega (\phi_1 - \phi_2)^+.$$

From the above results we can obtain existence of entropy solutions for data in L^1 and also existence of weak solutions when the data are more regular. We start with the existence of weak solutions.

5.2 Existence of weak solutions

Theorem 5.6 *Assume $D(\gamma) = \mathbb{R}$ and $\mathcal{R}^-_{\gamma,\beta} < \mathcal{R}^+_{\gamma,\beta}$. Let $D(\beta) = \mathbb{R}$ or* **a** *smooth.*

(i) For any $\phi \in V^{1,p}(\Omega)$ and $\psi \in V^{1,p}(\partial\Omega)$ with

$$\int_\Omega \phi + \int_{\partial\Omega} \psi \in \mathcal{R}_{\gamma,\beta}, \tag{26}$$

there exists a weak solution $[u, z, w]$ of problem $(S_{\phi,\psi}^{\gamma,\beta})$.

(ii) For any $[u_1, z_1, w_1]$ weak solution of problem $(S_{\phi_1,\psi_1}^{\gamma,\beta})$, $\phi_1 \in V^{1,p}(\Omega)$ and $\psi_1 \in V^{1,p}(\partial\Omega)$ satisfying (26), and any $[u_2, z_2, w_2]$ weak solution of problem $(S_{\phi_2,\psi_2}^{\gamma,\beta})$, $\phi_2 \in V^{1,p}(\Omega)$ and $\psi_2 \in V^{1,p}(\partial\Omega)$ satisfying (26), we have that

$$\int_\Omega (z_1 - z_2)^+ + \int_{\partial\Omega} (w_1 - w_2)^+ \leq \int_{\partial\Omega} (\psi_1 - \psi_2)^+ + \int_\Omega (\phi_1 - \phi_2)^+.$$

Proof. We approximate ϕ and ψ by

$$\phi_{m,n} = \sup\{\inf\{m, \phi\}, -n\}$$

and

$$\psi_{m,n} = \sup\{\inf\{m, \psi\}, -n\},$$

respectively. We have, $\phi_{m,n} \in L^\infty(\Omega)$, $\psi_{m,n} \in L^\infty(\partial\Omega)$, are non decreasing in m, non increasing in n, $\|\phi_{m,n}\|_{L^{p'}(\Omega)} \leq \|\phi\|_{L^{p'}(\Omega)}$ and $\|\psi_{m,n}\|_{L^{p'}(\partial\Omega)} \leq \|\psi\|_{L^{p'}(\partial\Omega)}$. Then, if $m \leq n$, by Propositions 5.1 or 5.4, there exist $u_{m,n} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $z_{m,n} \in L^\infty(\Omega)$, $z_{m,n}(x) \in \gamma(u_{m,n}(x))$ *a.e.* in Ω and $w_{m,n} \in L^1(\partial\Omega)$, $w_{m,n}(x) \in \beta(u_{m,n}(x))$ *a.e.* on $\partial\Omega$, such that

$$\begin{gathered}\int_\Omega \mathbf{a}(x, Du_{m,n}) \cdot Dv + \int_\Omega z_{m,n} v + \int_{\partial\Omega} w_{m,n} v \\ + \frac{1}{m}\int_\Omega u_{m,n}^+ v - \frac{1}{n}\int_\Omega u_{m,n}^- v + \frac{1}{m}\int_{\partial\Omega} u_{m,n}^+ v - \frac{1}{n}\int_{\partial\Omega} u_{m,n}^- v \\ = \int_{\partial\Omega} \psi_{m,n} v + \int_\Omega \phi_{m,n} v,\end{gathered} \tag{27}$$

for any $v \in W^{1,p}(\Omega)$. Moreover,

$$\int_\Omega z_{m,n}^\pm + \int_{\partial\Omega} w_{m,n}^\pm \leq \int_\Omega \phi^\pm + \int_{\partial\Omega} \psi^\pm \tag{28}$$

and

$$\|Du_{m,n}\|_{L^p(\Omega)}^{p-1} \leq \frac{c(\Omega, N, p)}{\lambda}\left(\|\phi\|_{V^{1,p}(\Omega)} + \|\psi\|_{V^{1,p}(\partial\Omega)}\right). \tag{29}$$

Fixed $m \in \mathbb{N}$, by Propositions 5.1 or 5.4 (ii), $\{z_{m,n}\}_{n=m}^\infty$ and $\{w_{m,n}\}_{n=m}^\infty$ are monotone non increasing. Then, by (28) and the Monotone Convergence Theorem, there exists $\hat{z}_m \in L^1(\Omega)$, $\hat{w}_m \in L^1(\partial\Omega)$ and a subsequence $n(m)$, such that

$$\|z_{m,n(m)} - \hat{z}_m\|_1 \leq \frac{1}{m}$$

and

$$\|w_{m,n(m)} - \hat{w}_m\|_1 \leq \frac{1}{m}.$$

Thanks again to Proposition 5.1 or 5.4 (ii), $\hat{z}_m$ and $\hat{w}_m$ are non decreasing in m. Now, by (28), we have that $\int_\Omega |\hat{z}_m|$ and $\int_{\partial\Omega} |\hat{w}_m|$ are bounded. Using again the Monotone Convergence Theorem, there exist $z \in L^1(\Omega)$ and $w \in L^1(\partial\Omega)$ such that

$$\hat{z}_m \text{ converges } a.e. \text{ and in } L^1(\Omega) \text{ to } z$$

and

$$\hat{w}_m \text{ converges } a.e. \text{ and in } L^1(\partial\Omega) \text{ to } w.$$

Consequently,

$$z_m := z_{m,n(m)} \text{ converges to } z \ a.e. \text{ and in } L^1(\Omega) \tag{30}$$

and

$$w_m := w_{m,n(m)} \text{ converges to } w \ a.e. \text{ and in } L^1(\partial\Omega). \tag{31}$$

If we set $u_m := u_{m,n(m)}$, $\phi_m := \phi_{m,n(m)}$ and $\psi_m := \psi_{m,n(m)}$, then we have

$$\begin{gathered}\int_\Omega \mathbf{a}(x, Du_m) \cdot Dv + \int_\Omega z_m v + \int_{\partial\Omega} w_m v \\ + \frac{1}{m}\int_\Omega u_m^+ v - \frac{1}{n(m)}\int_\Omega u_m^- v + \frac{1}{m}\int_{\partial\Omega} u_m^+ v - \frac{1}{n(m)}\int_{\partial\Omega} u_m^- v \\ = \int_{\partial\Omega} \psi_m v + \int_\Omega \phi_m v,\end{gathered} \tag{32}$$

for any $v \in W^{1,p}(\Omega)$.

As a consequence of (29),

$$\left\{ u_m - \frac{1}{\text{meas}(\partial\Omega)} \int_{\partial\Omega} u_m \right\}_m \quad \text{is bounded in } W^{1,p}(\Omega). \tag{33}$$

Let us see that

$$\left\{ \frac{1}{\text{meas}(\partial\Omega)} \int_{\partial\Omega} u_m \ : \ m \in \mathbb{N} \right\} \quad \text{is a bounded sequence.} \tag{34}$$

If (34) does not hold, then, extracting a subsequence if necessary, we can suppose that $\int_{\partial\Omega} u_m$ converges to $+\infty$ (or $-\infty$, respectively). Suppose first that $\int_{\partial\Omega} u_m$ converges to $+\infty$. Hence, by (33) we have

$$u_m \ \text{ converges to } \ +\infty \ \ a.e. \text{ in } \Omega, \ \text{ and } a.e. \text{ in } \partial\Omega.$$

Moreover, since for m large enough

$$u_m^- \leq \left(u_m - \frac{1}{\text{meas}(\partial\Omega)} \int_{\partial\Omega} u_m \right)^- + \left(\frac{1}{\text{meas}(\partial\Omega)} \int_{\partial\Omega} u_m \right)^- = \left(u_m - \frac{1}{\text{meas}(\partial\Omega)} \int_{\partial\Omega} u_m \right)^-,$$

by (33), we get

$$\left\{ \int_{\partial\Omega} u_m^- \right\}_{m\in\mathbb{N}} \quad \text{is bounded}$$

and, similarly,

$$\left\{ \int_{\Omega} u_m^- \right\}_{m\in\mathbb{N}} \quad \text{is bounded.}$$

In the case $\int_{\partial\Omega} u_m$ converges to $-\infty$, we similarly obtain that

$$u_m \ \text{ converges to } \ -\infty \ \ a.e. \text{ in } \Omega, \ \text{ and } a.e. \text{ in } \partial\Omega,$$

and

$$\left\{ \int_{\partial\Omega} u_m^+ \right\}_{m\in\mathbb{N}} \quad \text{and} \quad \left\{ \int_{\Omega} u_m^+ \right\}_{m\in\mathbb{N}} \quad \text{are bounded.}$$

Therefore, we have $z = \sup\{\text{Ran}(\gamma)\}$ ($z = \inf\{\text{Ran}(\gamma)\}$, respectively) and $w = \sup\{\text{Ran}(\beta)\}$ ($w = \inf\{\text{Ran}(\beta)\}$, respectively). Now, taking $v = 1$ as test function in (32), we get

$$\begin{aligned} &\frac{1}{m} \int_{\Omega} u_m^+ - \frac{1}{n(m)} \int_{\Omega} u_m^- + \frac{1}{m} \int_{\partial\Omega} u_m^+ - \frac{1}{n(m)} \int_{\partial\Omega} u_m^- \\ &= \int_{\Omega} \phi_m + \int_{\partial\Omega} \psi_m - \int_{\Omega} z_m - \int_{\partial\Omega} w_m, \end{aligned}$$

and we get a contradiction with (26). Hence, (34) is true. By (33) and (34), we have $\{\|u_m\|_{W^{1,p}(\Omega)}\}_m$ is bounded. Therefore, there exists a subsequence, that we denote equal, such that

$$u_m \to u \text{ weakly in } W^{1,p}(\Omega),$$

$$u_m \to u \text{ in } L^p(\Omega) \text{ and } a.e. \text{ in } \Omega,$$

$$u_m \to u \text{ in } L^p(\partial\Omega) \text{ and } a.e. \text{ in } \partial\Omega.$$

Moreover, arguing as in Proposition 5.1, it is not difficult to see that $\{Du_m\}$ is a Cauchy sequence in measure. Then, up to extraction of a subsequence, Du_m converges to Du a.e. in Ω. Consequently, we obtain that

$$\mathbf{a}(., Du_m) \text{ converges weakly in } L^{p'}(\Omega)^N \text{ and } a.e. \text{ in } \Omega \text{ to } \mathbf{a}(., Du).$$

From these convergences, we finish the proof of existence.

The proof of (ii) is a consequence of the existence result, Propositions 5.1 or 5.4 (ii), and the uniqueness result. □

Remark 5.7 For positive data ϕ and ψ, it is not necessary the assumption $D(\gamma) = D(\beta) = \mathbb{R}$, that is, we can improve the above result in the following way. Assume $[0, +\infty[\subset D(\gamma)$ and $\mathcal{R}^+_{\gamma,\beta} > 0$. Let $[0, +\infty[\subset D(\beta)$ or $\mathbf{a}$ smooth. For any $0 \leq \phi \in V^{1,p}\Omega)$ and $0 \leq \psi \in V^{1,p}(\partial\Omega)$ with $\int_\Omega \phi + \int_{\partial\Omega} \psi < \mathcal{R}^+_{\gamma,\beta}$, there exists a weak solution of problem $(S^{\gamma,\beta}_{\phi,\psi})$. A similar result holds for non positive data.

We also have existence and uniqueness of weak solutions if $\mathcal{R}^-_{\gamma,\beta} = \mathcal{R}^+_{\gamma,\beta}$, that is when $\gamma(r) = \beta(r) = 0$ for any $r \in \mathbb{R}$.

Theorem 5.8 *For any $\phi \in V^{1,p}(\Omega)$ and $\psi \in V^{1,p}(\partial\Omega)$ with*

$$\int_\Omega \phi + \int_{\partial\Omega} \psi = 0, \tag{35}$$

there exists a unique (up to a constant) weak solution $u \in W^{1,p}(\Omega)$ of the problem

$$\begin{cases} -\text{div } \mathbf{a}(x, Du) = \phi & \text{in } \Omega \\ \\ \mathbf{a}(x, Du) \cdot \eta = \psi & \text{on } \partial\Omega \end{cases}$$

in the sense that

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv = \int_{\partial\Omega} \psi v + \int_\Omega \phi v,$$

for all $v \in W^{1,p}(\Omega)$.

Proof. Let us approximate ϕ by $\phi_m = T_m(\phi) - \frac{1}{\text{meas}(\Omega)}\alpha_m$ and ψ by $\psi_m = T_m(\psi)$, where $\alpha_m = \int_\Omega T_m(\phi) + \int_{\partial\Omega} T_m(\psi)$. Observe that

$$\lim_{m\to+\infty} \alpha_m = 0 \tag{36}$$

and

$$\int_\Omega \phi_m + \int_{\partial\Omega} \psi_m = 0. \tag{37}$$

By Proposition 5.1, there exist $u_m \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that

$$\int_\Omega \mathbf{a}(x, Du_m) \cdot Dv + \frac{1}{m}\int_\Omega u_m v + \frac{1}{m}\int_{\partial\Omega} u_m v = \int_{\partial\Omega} \psi_m v + \int_\Omega \phi_m v, \tag{38}$$

for any $v \in W^{1,p}(\Omega)$.

Taking $v = u_m$ as test function in (38), using (36) and the Poincaré inequality, it is easy to see that

$$\left\{ u_m - \frac{1}{\text{meas}(\partial\Omega)} \int_{\partial\Omega} u_m \right\}_m \text{ is bounded in } W^{1,p}(\Omega). \tag{39}$$

Let us also see that

$$\left\{ \frac{1}{\text{meas}(\partial\Omega)} \int_{\partial\Omega} u_m \ : \ m \in \mathbb{N} \right\} \quad \text{is a bounded sequence.} \tag{40}$$

If (40) does not hold, then, extracting a subsequence if necessary, we can suppose that $\int_{\partial\Omega} u_m$ converges to $+\infty$ (or $-\infty$, respectively). Suppose first that $\int_{\partial\Omega} u_m$ converges to $+\infty$. Hence, as in the proof of Theorem 5.6, we have

$$\left\{ \int_{\Omega} u_m^- \right\}_{m\in\mathbb{N}} \quad \text{is bounded.}$$

Now, taking $v = m$ in (38) and using (37), it follows that

$$\lim_{m\to+\infty} \int_{\Omega} u_m^- = +\infty,$$

which is a contradiction. Similarly, we get a contradiction in the case $\int_{\partial\Omega} u_m$ converging to $-\infty$. Hence, (40) is true. By (39) and (40), we have $\{\|u_m\|_{W^{1,p}(\Omega)}\}_m$ is bounded, and we can finish as in the proof of Theorem 5.6. $\square$

Remark 5.9 Taking into account the arguments used in Remark 3.4, we get that $[u, z, w]$ in the above results (including also the case $\beta = D$) satisfies

$$\int_{\Omega} |zv| + \int_{\partial\Omega} |wv| \leq \int_{\Omega} |\phi v| + \int_{\partial\Omega} |\psi v| + \sigma \left(\|g\|_{L^{p'}(\Omega)} + \|Du\|_{L^p(\Omega)}^{p-1} \right) \|Dv\|_{L^p(\Omega)}$$

for all $v \in W^{1,p}(\Omega)$, and

$$\|Du\|_{L^p(\Omega)}^{p-1} \leq \frac{c(\Omega, N, p)}{\lambda} \left(\|\phi\|_{V^{1,p}(\Omega)} + \|\psi\|_{V^{1,p}(\partial\Omega)} \right),$$

for some $c(\Omega, N, p) > 0$.

Taking $\beta = D$ and $\gamma(r) = 0$ for all $r \in \mathbb{R}$ in Theorem 5.6 for $\mathbf{a}$ smooth, and taking into account Remark 5.9, we have the following result in the line of Proposition C (iv) of [9].

Corollary 5.10 $\mathbf{a}$ *is smooth if and only if for any $\phi \in V^{1,p}(\Omega)$ there exists $T(\phi) \in V^{1,p}(\partial\Omega)$ such that the weak solution u of*

$$\begin{cases} -\text{div}\, \mathbf{a}(x, Du) = \phi & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

is a weak solution of

$$\begin{cases} -\text{div}\, \mathbf{a}(x, Du) = \phi & \text{in } \Omega \\ \mathbf{a}(x, Du) \cdot \eta = T(\phi) & \text{on } \partial\Omega. \end{cases}$$

Moreover, the map $T : V^{1,p}(\Omega) \to V^{1,p}(\partial\Omega)$ satisfies

$$\int_{\Omega} (T(\phi_1) - T(\phi_2))^+ \leq \int_{\Omega} (\phi_1 - \phi_2)^+,$$

for all $\phi_1, \phi_2 \in V^{1,p}(\Omega)$.

In the case $\psi = 0$ we have the following result, which is similar to the one obtained by Bénilan, Crandall and Sack in [9] for the Laplacian operator and $L^1(\Omega)$-data.

Theorem 5.11 *Assume $D(\beta) = \mathbb{R}$ or* **a** *smooth. Let $\mathcal{R}^-_{\gamma,\beta} < \mathcal{R}^+_{\gamma,\beta}$.*

(i) For any $\phi \in V^{1,p}(\Omega)$ such that $\int_\Omega \phi \in \mathcal{R}_{\gamma,\beta}$, there exists a weak solution $[u, z, w]$ of problem $(S^{\gamma,\beta}_{\phi,0})$, with $z << \phi$.

(ii) For any $[u_1, z_1, w_1]$ weak solution of problem $(S^{\gamma,\beta}_{\phi_1,0})$, $\phi_1 \in V^{1,p}(\Omega)$, $\int_\Omega \phi_1 \in \mathcal{R}_{\gamma,\beta}$, and any $[u_2, z_2, w_2]$ weak solution of problem $(S^{\gamma,\beta}_{\phi_2,0})$, $\phi_2 \in V^{1,p}(\Omega)$, $\int_\Omega \phi_2 \in \mathcal{R}_{\gamma,\beta}$, we have that

$$\int_\Omega (z_1 - z_2)^+ + \int_{\partial\Omega} (w_1 - w_2)^+ \leq \int_\Omega (\phi_1 - \phi_2)^+.$$

Taking into account Remark 5.3, we obtain the following existence theorem for Dirichlet boundary condition.

Theorem 5.12 *Assume $D(\beta) = \{0\}$. For any $\phi \in V^{1,p}(\Omega)$, there exists a unique $[u, z] = [u_{\phi,\psi}, z_{\phi,\psi}] \in W^{1,p}_0(\Omega) \times V^{1,p}(\Omega)$, $z \in \gamma(u)$ a.e. in Ω, such that*

$$\int_\Omega \mathbf{a}(x, Du) \cdot Dv + \int_\Omega zv = \int_\Omega \phi v,$$

for all $v \in W^{1,p}_0(\Omega)$.

Moreover, if $\phi_1, \phi_2 \in V^{1,p}(\Omega)$, then

$$\int_\Omega (z_{\phi_1,\psi_1} - z_{\phi_2,\psi_2})^+ \leq \int_\Omega (\phi_1 - \phi_2)^+. \tag{41}$$

5.3 Existence of entropy solutions

Let us see the existence results of entropy solutions for data in L^1.

Theorem 5.13 *Assume $D(\gamma) = \mathbb{R}$, and $D(\beta) = \mathbb{R}$ or* **a** *smooth. Let also assume that, if $[0, +\infty[\subset D(\beta)$,*

$$\lim_{k\to+\infty} \gamma^0(k) = +\infty \text{ and } \lim_{k\to+\infty} \beta^0(k) = +\infty, \tag{42}$$

and if $]-\infty, 0] \subset D(\beta)$,

$$\lim_{k\to-\infty} \gamma^0(k) = -\infty \text{ and } \lim_{k\to-\infty} \beta^0(k) = -\infty. \tag{43}$$

Then,

(i) for any $\phi \in L^1(\Omega)$ and $\psi \in L^1(\partial\Omega)$, there exists an entropy solution $[u, z, w]$ of problem $(S^{\gamma,\beta}_{\phi,\psi})$.

(ii) For any $[u_1, z_1, w_1]$ entropy solution of problem $(S^{\gamma,\beta}_{\phi_1,\psi_1})$, $\phi_1 \in L^1(\Omega)$, $\psi_1 \in L^1(\partial\Omega)$, and any $[u_2, z_2, w_2]$ entropy solution of problem $(S^{\gamma,\beta}_{\phi_2,\psi_2})$, $\phi_2 \in L^1(\Omega)$, $\psi_2 \in L^1(\partial\Omega)$, we have that

$$\int_\Omega (z_1 - z_2)^+ + \int_{\partial\Omega} (w_1 - w_2)^+ \leq \int_{\partial\Omega} (\psi_1 - \psi_2)^+ + \int_\Omega (\phi_1 - \phi_2)^+.$$

Proof. Observe that, under the assumptions of the theorem, we have $\mathcal{R}_{\gamma,\beta} = \mathbb{R}$.

We divide the proof in several steps.

Step 1. Let us approximate ϕ by $\phi_m := T_m(\phi)$ and ψ by $\psi_m := T_m(\psi)$. Then, by Theorem 5.6, there exist $u_m \in W^{1,p}(\Omega)$, $z_m \in V^{1,p}(\Omega)$, $z_m(x) \in \gamma(u_m(x))$ *a.e.* in Ω, and $w_m \in V^{1,p}(\partial\Omega)$, $w_m(x) \in \beta(u_m(x))$ *a.e.* on $\partial\Omega$, such that

$$\int_\Omega \mathbf{a}(x, Du_m) \cdot Dv + \int_\Omega z_m v + \int_{\partial\Omega} w_m v = \int_{\partial\Omega} \psi_m v + \int_\Omega \phi_m v, \tag{44}$$

for any $v \in W^{1,p}(\Omega)$.

Moreover,

$$\int_\Omega z_m^\pm + \int_{\partial\Omega} w_m^\pm \le \int_{\partial\Omega} \psi_m^\pm + \int_\Omega \phi_m^\pm \tag{45}$$

and

$$\int_\Omega |z_n - z_m| + \int_{\partial\Omega} |w_n - w_m| \le \int_{\partial\Omega} |\psi_n - \psi_m| + \int_\Omega |\phi_n - \phi_m|.$$

Consequently

$$\begin{aligned} z_m \to z \ \text{ in } L^1(\Omega) \\ w_m \to w \ \text{ in } L^1(\partial\Omega). \end{aligned} \tag{46}$$

Taking $v = T_k(u_m)$ in (44), we obtain

$$\lambda \int_\Omega |DT_k(u_m)|^p \le k \left(\|\phi\|_1 + \|\psi\|_1 \right), \quad \forall k \in \mathbb{N}. \tag{47}$$

By (47), we have $\{T_k(u_m)\}$ is bounded in $W^{1,p}(\Omega)$. Then, we can suppose that there exists $\sigma_k \in W^{1,p}(\Omega)$ such that

$$T_k(u_m) \text{ converges to } \sigma_k \text{ weakly in } W^{1,p}(\Omega),$$

$$T_k(u_m) \text{ converges to } \sigma_k \text{ in } L^p(\Omega) \text{ and } a.e. \text{ in } \Omega$$

and

$$T_k(u_m) \text{ converges to } \sigma_k \text{ in } L^p(\partial\Omega) \text{ and } a.e. \text{ in } \partial\Omega.$$

Step 2. Let us see that u_m converges almost every where in Ω.

If $D(\beta)$ is bounded from above by r_1, using the Poincaré inequality and (47),

$$\operatorname{meas}\{x \in \Omega \ : \ \sigma_k^+(x) = k\} \le \int_\Omega \frac{(\sigma_k^+)^{p^*}}{k^{p^*}} \le \liminf_m \int_\Omega \frac{(T_k((u_m)^+))^{p^*}}{k^{p^*}}$$

$$\le \frac{C_1}{k^{p^*}} \liminf_m \left(\int_{\partial\Omega} T_k((u_m)^+) + \left(\int_\Omega |DT_k((u_m)^+)|^p \right)^{1/p} \right)^{p^*}$$

$$\le \frac{C_1}{k^{p^*}} \left(r_1 \operatorname{meas}(\partial\Omega) + \left(\frac{\|\phi\|_1 + \|\psi\|_1}{\lambda} k \right)^{1/p} \right)^{p^*} \quad \forall k > 0,$$

where $p^* = \frac{Np}{N-p}$ and C_1 is independent of k and m.

If $D(\beta)$ is unbounded from above, then, we are supposing $\lim_{k\to+\infty}\gamma^0(k) = +\infty$. Therefore, for $k > 0$ large enough (in order to have $\gamma^0(k) > 0$), by (45) we have

$$\text{meas}\{x \in \Omega \ : \ \sigma_k^+(x) = k\} = \int_{\{x\in\Omega \ : \ \sigma_k^+(x)=k\}} \frac{\gamma^0(\sigma_k^+(x))}{\gamma^0(k)}$$

$$\leq \frac{1}{\gamma^0(k)} \liminf_m \int_\Omega \gamma^0(T_k((u_m)^+)) \leq \frac{1}{\gamma^0(k)}(\|\phi\|_1 + \|\psi\|_1).$$

Consequently, in any case, there exists $g(k) > 0$, $\lim_{k\to+\infty} g(k) = 0$, such that

$$\text{meas}\{x \in \Omega \ : \ \sigma_k^+(x) = k\} \leq g(k) \quad \forall k > 0. \tag{48}$$

Similarly, if $D(\beta)$ is bounded from below or assumption (43) holds, we can prove that there exists $g(k)$ as above such that

$$\text{meas}\{x \in \Omega \ : \ \sigma_k^-(x) = k\} \leq g(k) \quad \forall k > 0. \tag{49}$$

Note that we have proved (48) and (49) in any case. Consequently, there exists $g(k) > 0$ with $\lim_{k\to+\infty} g(k) = 0$, such that

$$\text{meas}\{x \in \Omega \ : \ |\sigma_k(x)| = k\} \leq g(k) \quad \forall k > 0.$$

Therefore, if we define $u(x) = \sigma_k(x)$ on $\{x \in \Omega : |\sigma_k(x)| < k\}$, then

$$u_m \text{ converges to } u \ a.e. \text{ in } \Omega, \tag{50}$$

and we have that

$$T_k(u_m) \text{ converges weakly in } W^{1,p}(\Omega) \text{ to } T_k(u),$$

$$T_k(u_m) \text{ converges in } L^p(\Omega) \text{ and } a.e. \text{ in } \Omega \text{ to } T_k(u)$$

and

$$T_k(u_m) \text{ converges in } L^p(\partial\Omega) \text{ and } a.e. \text{ in } \partial\Omega \text{ to } T_k(u).$$

Consequently, $u \in \mathcal{T}^{1,p}(\Omega)$.

Arguing as in Proposition 5.1, it is not difficult to see that $\{Du_m\}$ is a Cauchy sequence in measure. Similarly, we can prove that $DT_k(u_m)$ converges in measure to $DT_k(u)$. Then, up to extraction of a subsequence, Du_m converges to Du a.e. in Ω. Consequently, we obtain that

$$\mathbf{a}(., DT_k(u_m)) \text{ converges weakly in } L^{p'}(\Omega)^N \text{ and } a.e. \text{ in } \Omega \text{ to } \mathbf{a}(., DT_k(u)). \tag{51}$$

Step 3. Let us see now that $u \in \mathcal{T}^{1,p}_{tr}(\Omega)$. On the one hand we have that $u_m \to u$ a.e. in Ω. On the other hand, since $DT_k(u_m)$ is bounded in $L^p(\Omega)$ and $DT_k(u_m) \to DT_k(u)$ in measure, it follows from [5, Lemma 6.1] that $DT_k(u_m) \to DT_k(u)$ in $L^1(\Omega)$. Next, let us see that u_m converges $a.e.$ in $\partial\Omega$. Let suppose first that $D(\beta)$ is bounded from above by r_1, then, by (47), there exists a constant C_3 such that

$$\text{meas}\{x \in \partial\Omega \ : \ \sigma_k^+(x) = k\} \leq \int_{\partial\Omega} \frac{\sigma_k^+}{k}$$

$$\leq \liminf_m \int_{\partial\Omega} \frac{T_k((u_m)^+)}{k} \leq \frac{r_1 \text{meas}(\partial\Omega)}{k} \quad \forall k > 0.$$

If $D(\beta)$ is unbounded from above, then, we are supposing $\lim_{k\to+\infty}\beta^0(k) = +\infty$. Therefore, for $k > 0$ large enough (in order to have $\beta^0(k) > 0$), by (45) we have

$$\text{meas}\{x \in \partial\Omega \ : \ \sigma_k^+(x) = k\} = \int_{\{x\in\partial\Omega \ : \ \sigma_k^+(x)=k\}} \frac{\beta^0(\sigma_k^+(x))}{\beta^0(k)}$$

$$\leq \frac{1}{\beta^0(k)} \liminf_m \int_{\partial\Omega} \beta^0(T_k((u_m)^+)) \leq \frac{1}{\beta^0(k)}(\|\phi\|_1 + \|\psi\|_1).$$

We work similarly if $D(\beta)$ is bounded from below or assumption (43) holds, and, in any case, there exists $\hat{g}(k) > 0$, $\lim_{k\to+\infty}\hat{g}(k) = 0$, such that

$$\text{meas}\{x \in \partial\Omega : |\sigma_k(x)| = k\} \leq \hat{g}(k) \quad \forall k > 0.$$

Hence, if we define $v(x) = T_k(u)(x)$ on $\{x \in \partial\Omega \ : \ |T_k(u)(x)| < k\}$, then

$$u_m \text{ converges to } v \ a.e. \text{ in } \partial\Omega. \tag{52}$$

Consequently, $u \in \mathcal{T}_{tr}^{1,p}(\Omega)$.

Since $z_m(x) \in \gamma(u_m(x))$ *a.e.* in Ω and $w_m(x) \in \beta(u_m(x))$ *a.e.* in $\partial\Omega$, from (46), (50), (52) and from the maximal monotonicity of γ and β, we deduce that $z(x) \in \gamma(u(x))$ *a.e.* in Ω and $w(x) \in \beta(u(x))$ *a.e.* in $\partial\Omega$.

Step 4. Finally, let us prove that $[u, z, w]$ is an entropy solution relative to $D(\beta)$ of $(S_{\phi,\psi}^{\gamma,\beta})$. To do that, we introduce the class $\mathcal{F}$ of functions $S \in C^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfying

$$S(0) = 0,\ 0 \leq S' \leq 1,\ S'(s) = 0 \text{ for } s \text{ large enough},$$

$$S(-s) = -S(s),\ \text{and } S''(s) \leq 0 \text{ for } s \geq 0.$$

Let $v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $v(x) \in D(\beta)$ *a.e.* in $\partial\Omega$, and $S \in \mathcal{F}$. Taking $S(u_m - v)$ as test function in (44), we get

$$\begin{aligned} &\int_\Omega \mathbf{a}(x, Du_m) \cdot DS(u_m - v) + \int_\Omega z_m S(u_m - v) + \int_{\partial\Omega} w_m S(u_m - v) \\ &= \int_{\partial\Omega} \psi_m S(u_m - v) + \int_\Omega \phi_m S(u_m - v). \end{aligned} \tag{53}$$

We can write the first term of (53) as

$$\int_\Omega \mathbf{a}(x, Du_m) \cdot Du_m S'(u_m - v) - \int_\Omega \mathbf{a}(x, Du_m) \cdot Dv S'(u_m - v). \tag{54}$$

Since $u_m \to u$ and $Du_m \to Du$ *a.e.*, Fatou's Lemma yields

$$\int_\Omega \mathbf{a}(x, Du) \cdot Du S'(u - v) \leq \liminf_{m\to\infty} \int_\Omega \mathbf{a}(x, Du_m) \cdot Du_m S'(u_m - v).$$

The second term of (54) is estimated as follows. Let $r := \|v\|_\infty + \|S\|_\infty$. By (51)

$$\mathbf{a}(x, DT_r u_m) \to \mathbf{a}(x, DT_r u) \qquad \text{weakly in } L^{p'}(\Omega). \tag{55}$$

On the other hand,

$$|Dv S'(u_m - v)| \leq |Dv| \in L^p(\Omega).$$

Then, by the Dominated Convergence Theorem, we have

$$DvS'(u_m - v) \to DvS'(u - v) \quad \text{in} \;\; L^p(\Omega)^N. \tag{56}$$

Hence, by (55) and (56), it follows that

$$\lim_{m\to\infty} \int_\Omega \mathbf{a}(x, Du_m) \cdot DvS'(u_m - v) = \int_\Omega \mathbf{a}(x, Du) \cdot DvS'(u - v).$$

Therefore, applying again the Dominated Convergence Theorem in the other terms of (53), we obtain

$$\int_\Omega \mathbf{a}(x, Du) \cdot DS(u - v) + \int_\Omega zS(u - v) + \int_{\partial\Omega} wS(u - v)$$

$$\leq \int_{\partial\Omega} \psi S(u - v) + \int_\Omega \phi S(u - v).$$

From here, to conclude, we only need to apply the technique used in the proof of [5, Lemma 3.2].

The proof of (ii) is a consequence of the existence result, Theorem 5.6 (ii), and the uniqueness result. □

Remark 5.14 In Theorem 5.13, if the data ϕ and ψ are non negative (non positive, respectively), then assumption (43) ((42), respectively) is not necessary. That is, only assuming $[0, +\infty[\subset D(\gamma)$, $[0, +\infty[\subset D(\beta)$ or $\mathbf{a}$ smooth, and assumption (42) if $[0, +\infty[\subset D(\beta)$, for any $0 \leq \phi \in L^1(\Omega)$ and $0 \leq \psi \in L^1(\partial\Omega)$, there exists an entropy solution of problem $(S^{\gamma,\beta}_{\phi,\psi})$. A similar result holds for non positive data.

Taking into account Theorem 5.13 and Corollay 5.10, we have the following result.

Corollary 5.15 $\mathbf{a}$ *is smooth if and only if for any* $\phi \in L^1(\Omega)$ *there exists* $T(\phi) \in L^1(\partial\Omega)$ *such that the entropy solution* u *of*

$$\begin{cases} -\mathrm{div}\ \mathbf{a}(x, Du) = \phi & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

is an entropy solution of

$$\begin{cases} -\mathrm{div}\ \mathbf{a}(x, Du) = \phi & \text{in } \Omega \\ \mathbf{a}(x, Du) \cdot \eta = T(\phi) & \text{on } \partial\Omega. \end{cases}$$

Moreover, the map $T : L^1(\Omega) \to L^1(\partial\Omega)$ *satisfies*

$$\int_\Omega (T(\phi_1) - T(\phi_2))^+ \leq \int_\Omega (\phi_1 - \phi_2)^+,$$

for all $\phi_1, \phi_2 \in L^1(\Omega)$, *and* $T\left(V^{1,p}(\Omega)\right) \subset V^{1,p}(\partial\Omega)$.

In the case $\psi = 0$ we have the following result.

Theorem 5.16 *Assume $D(\beta)=\mathbb{R}$ or* **a** *is smooth. Let also assume that, if $[0,+\infty[\subset D(\gamma)\cap D(\beta)$, the assumption (42) holds, and, if $]-\infty,0]\subset D(\gamma)\cap D(\beta)$ the assumption (43) holds. Then,*

(i) for any $\phi\in L^1(\Omega)$, there exists an entropy solution $[u,z,w]$ of problem $(S^{\gamma,\beta}_{\phi,0})$, with $z<<\phi$.

(ii) For any $[u_1,z_1,w_1]$ entropy solution of problem $(S^{\gamma,\beta}_{\phi_1,0})$, $\phi_1\in L^1(\Omega)$, and any $[u_2,z_2,w_2]$ entropy solution of problem $(S^{\gamma,\beta}_{\phi_2,0})$, $\phi_2\in L^1(\Omega)$, we have that

$$\int_\Omega (z_1-z_2)^+ + \int_{\partial\Omega}(w_1-w_2)^+ \leq \int_\Omega(\phi_1-\phi_2)^+.$$

Remark 5.17 In Theorems 5.13 and 5.16, it is not difficult to see that (42) can be substituted by one of the following assumptions,

(42′) $\exists 0<\alpha\leq 1,\ r_0>0:\ \gamma^0(r)\geq r^\alpha \quad \forall r\geq r_0,$

(42″) $\exists 0<\alpha\leq 1, r_0>0:\ \beta^0(r)\geq r^\alpha \quad \forall r\geq r_0;$

and (43) can be substituted by one of the following assumptions,

(43′) $\exists 0<\alpha\leq 1,\ r_0>0:\ \gamma^0(r)\leq -(-r)^\alpha \quad \forall r\leq -r_0,$

(43″) $\exists 0<\alpha\leq 1, r_0>0:\ \beta^0(r)\leq -(-r)^\alpha \quad \forall r\leq -r_0.$

If $D(\beta)=\{0\}$, taking into account Theorem 5.12, it can be proved the following result given by Bénilan et al. in [5] for Dirichlet boundary condition.

Theorem 5.18 *Assume $D(\beta)=\{0\}$. For any $\phi\in L^1(\Omega)$, there exists a unique entropy solution $[u,z]$ of*

$$\begin{cases} -\mathrm{div}\ \mathbf{a}(x,Du)+\gamma(u)\ni\phi & \text{in } \Omega \\ \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

in the sense given by Bénilan et al. in [5].

Remark 5.19 Observe that in all the above existence results, we have that if $[u_1,z_1,w_1]$ and $[u_2,z_2,w_2]$ are entropy solutions of problems $(S^{\gamma,\beta}_{\phi_1,\psi_1})$ and $(S^{\gamma,\beta}_{\phi_2,\psi_2})$ respectively, with $\phi_1\leq\phi_2$ and $\psi_1\leq\psi_2$, then there exists a constance C such that $u_1\leq u_2+C$.

Some extensions

Following the ideas developed in this work, it is possible to find a larger class of entropy solutions when β is only assumed to have closed domain.

Definition 5.20 Let $\phi\in L^1(\Omega)$ and $\psi\in L^1(\partial\Omega)$. A triple of functions $[u,z,w]\in\mathcal{T}^{1,p}_{tr}(\Omega)\times L^1(\Omega)\times L^1(\partial\Omega)$ is an *entropy solution relative to* $D(\beta)$ of problem $(S^{\gamma,\beta}_{\phi,\psi})$ if $z(x)\in\gamma(u(x))$ *a.e.* in Ω, $w(x)\in\beta(u(x))$ *a.e.* in $\partial\Omega$ and

$$\begin{aligned}\int_\Omega \mathbf{a}(x,Du)\cdot DT_k(u-v)+\int_\Omega zT_k(u-v)+\int_{\partial\Omega}wT_k(u-v)\\ \leq \int_{\partial\Omega}\psi T_k(u-v)+\int_\Omega\phi T_k(u-v)\quad \forall k>0,\end{aligned} \tag{57}$$

for all $v\in L^\infty(\Omega)\cap W^{1,p}(\Omega)$, $v(x)\in D(\beta)$ *a.e.* in $\partial\Omega$.

For this concept of solution we can prove the following result.

Theorem 5.21 *Assume $D(\beta)$ is closed and $D(\beta) \subset D(\gamma)$. Let also assume that if $[0, +\infty[\subset D(\beta)$ the assumption (42) holds, and if $]-\infty, 0] \subset D(\beta)$ the assumption (43) holds. Then,*

(i) for any $\phi \in L^1(\Omega)$ and $\psi \in L^1(\partial\Omega)$ there exists an entropy solution $[u, z, w] = [u_{\phi,\psi}, z_{\phi,\psi}, w_{\phi,\psi}]$ relative to $D(\beta)$ of problem $(S^{\gamma,\beta}_{\phi,\psi})$. Moreover,

$$\beta^0(\inf D(\beta)) \leq w \leq \beta^0(\sup D(\beta))$$

and

$$\int_\Omega z^\pm + \int_{\partial\Omega} w^\pm \leq \int_{\partial\Omega} \psi^\pm + \int_\Omega \phi^\pm.$$

(ii) Given $\phi_1, \phi_2 \in L^1(\Omega)$ and $\psi_1, \psi_2 \in L^1(\partial\Omega)$,

$$\int_\Omega (z_{\phi_1,\psi_1} - z_{\phi_2,\psi_2})^+ + \int_{\partial\Omega} (w_{\phi_1,\psi_2} - w_{\phi_2,\psi_2})^+ \leq \int_{\partial\Omega} (\psi_1 - \psi_2)^+ + \int_\Omega (\phi_1 - \phi_2)^+.$$

(iii) For any $[u_1, z_1, w_1]$ entropy solution relative to $D(\beta)$ of problem $(S^{\gamma,\beta}_{\phi_1,\psi_1})$, $\phi_1 \in L^1(\Omega)$, $\psi_1 \in L^1(\partial\Omega)$, and any $[u_2, z_2, w_2]$ entropy solution relative to $D(\beta)$ of problem $(S^{\gamma,\beta}_{\phi_2,\psi_2})$, $\phi_2 \in L^1(\Omega)$, $\psi_2 \in L^1(\partial\Omega)$, we have that

$$\int_\Omega (z_1 - z_2)^+ \leq \int_{\partial\Omega} (\psi_1 - \psi_2)^+ + \int_\Omega (\phi_1 - \phi_2)^+.$$

Remark 5.22 In general, for this concept of solution we do not have uniqueness of w, as the following example shows.

Let γ and β be such that $\gamma(0) = [0, 1]$ and $\beta(0) =]-\infty, 0]$ and let $0 < \phi < 1$ and $\psi \leq 0$. Then, for any w such that $\psi \leq w \leq 0$, $[0, \phi, w]$ is an entropy solution relative to $D(\beta)$.

Acknowledgements. This work has been performed during the visits of the first, third and fourth authors to the Université de Picardie Jules Verne and the visit of the second author to the Universitat de València. They thank these institutions for their support and hospitality. The first, third and fourth authors have been partially supported by PNPGC project, reference BFM2002-01145, also, the first and third authors by the EC through the RTN Programme *Nonlinear Partial Differential Equations Describing Front propagation and other Singular Phenomena*, HPRN-CT-2002-00274.

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