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The total variation flow with nonlinear boundary conditions

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Abstract. We study existence, uniqueness and the asymptotic behaviour of the entropy solutions for the Total Variation Flow with nonlinear boundary conditions. To prove the existence we use the nonlinear semigroup theory and for the uniqueness we apply Kruzhkov's method of doubling variables both in space and in time. We show that when the initial data are in L^2, the entropy solutions are strong solutions. Respect to the asymptotic behaviour, we show that entropy solutions stabilize as $t \to \infty$ by converging to a constant function.

1. Introduction

We are interested in the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}\left(\dfrac{Du}{|Du|}\right) & \text{in } Q = (0,\infty) \times \Omega, \\ -\dfrac{\partial u}{\partial \eta} \in \beta(u) & \text{on } S = (0,\infty) \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{1}$$

where Ω is an open bounded domain in $\mathbb{R}^N$ with a C^1 boundary, $\partial/\partial\eta$ is the Neumann boundary operator associated to $Du/|Du|$, i.e.,

$$\frac{\partial u}{\partial \eta} := \left\langle \frac{Du}{|Du|}, \eta \right\rangle$$

with η the unit outward normal on $\partial\Omega$ and β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$. These nonlinear flows on the boundary occur in some problems in Mechanics and Physics [21] (see also [25,14,15] or [26]). Observe also that the classical Neumann and Dirichlet boundary conditions correspond to $\beta = \mathbb{R} \times \{0\}$ and $\beta = \{0\} \times \mathbb{R}$, respectively. Let us recall that this partial differential equation appears when one uses the steepest descent method to minimize the Total Variation, a method introduced by Rudin and Osher ([28,29]) in the context of image denoising and reconstruction. For the classical Neumann and Dirichlet boundary conditions, existence and uniqueness of solutions of the Total Variation Flow have been studied in [2] and [3]; the asymptotic behaviour of such solutions is analyzed in [5]. A similar problem, the case in which the associated variational energy has a growth at infinity of order p with $p > 1$, has been studied in [6,7] and [8].

Due to the linear growth of the energy functional associated with the problem (1), the natural energy space to study this problem is the space of functions of bounded variation. Recall that a function $u \in L^1(\Omega)$ whose partial derivatives in the sense of distributions are measures with finite total variation in Ω is called a function of bounded variation. The class of such functions will be denoted by $BV(\Omega)$. Thus $u \in BV(\Omega)$ if and only if there are Radon measures $\mu_1, \ldots, \mu_N$ defined in Ω with finite total mass in Ω and

$$\int_\Omega u D_i \varphi \, dx = -\int_\Omega \varphi \, d\mu_i \tag{2}$$

for all $\varphi \in C_0^\infty(\Omega)$, $i = 1, \ldots, N$. Thus the gradient of u is a vector valued measure with finite total variation

$$|Du|(\Omega) = \sup\left\{ \int_\Omega u \operatorname{div}(\varphi) \, dx \colon \varphi \in C_0^\infty(\Omega, \mathbb{R}^N), \ |\varphi(x)| \leqslant 1 \text{ for } x \in \Omega \right\}.$$

The space $BV(\Omega)$ is a Banach space endowed with the norm

$$\|u\|_{BV} = \|u\|_{L^1(\Omega)} + |Du|(\Omega). \tag{3}$$

For further information concerning functions of bounded variation we refer to [1,22] and [31].

We shall need several results from [9]. Following [9], let

$$X(\Omega) = \left\{ z \in L^\infty(\Omega, \mathbb{R}^N) \colon \operatorname{div}(z) \in L^1(\Omega) \right\}. \tag{4}$$

If $z \in X(\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$ we define the functional $(z, Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$\langle (z, Dw), \varphi \rangle = -\int_\Omega w\varphi \operatorname{div}(z) \, dx - \int_\Omega w z \cdot \nabla\varphi \, dx. \tag{5}$$

Then (z, Dw) is a Radon measure in Ω,

$$\int_\Omega (z, Dw) = \int_\Omega z \cdot \nabla w \, dx \tag{6}$$

for all $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and

$$\left| \int_B (z, Dw) \right| \leqslant \int_B |(z, Dw)| \leqslant \|z\|_\infty \int_B |Dw| \tag{7}$$

for any Borel set $B \subseteq \Omega$. Moreover, (z, Dw) is absolutely continuous with respect to $|Dw|$ with Radon–Nikodym derivative $\theta(z, Dw, x)$ which is a $|Dw|$ measurable function from Ω to $\mathbb{R}$ such that

$$\int_B (z, Dw) = \int_B \theta(z, Dw, x) |Dw| \tag{8}$$

for any Borel set $B \subseteq \Omega$. We also have that

$$\|\theta(z, Dw, \cdot)\|_{L^\infty(\Omega, \|Dw\|)} \leqslant \|z\|_{L^\infty(\Omega, \mathbb{R}^N)}. \tag{9}$$

In [9], a weak trace on $\partial\Omega$ of the normal component of $z \in X(\Omega)$ is defined. Concretely, it is proved that there exists a linear operator $\gamma : X(\Omega) \to L^\infty(\partial\Omega)$ such that

$$\|\gamma(z)\|_\infty \leqslant \|z\|_\infty,$$
$$\gamma(z)(x) = z(x) \cdot \nu(x) \quad \text{for all } x \in \partial\Omega \text{ if } z \in C^1(\overline{\Omega}, \mathbb{R}^N).$$

We shall denote $\gamma(z)(x)$ by $[z, \nu](x)$. Moreover, the following *Green's formula*, relating the function $[z, \nu]$ and the measure (z, Dw), for $z \in X(\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$, is established:

$$\int_\Omega w \operatorname{div}(z)\,\mathrm{d}x + \int_\Omega (z, Dw) = \int_{\partial\Omega} [z, \nu] w \,\mathrm{d}\mathcal{H}^{N-1}. \tag{10}$$

This paper is organized as follows. In Section 2 we study the problem from the point of view of nonlinear semigroup theory, showing that for initial data in $L^2(\Omega)$ the semigroup solution is a strong solution. In the next section we compute explicit solutions for a particular β, showing the different behaviour of these explicit solutions respect to the ones corresponding to the Neumann or Dirichlet boundary conditions. Section 4 is devoted to the existence and uniqueness of entropy solutions for initial data in $L^1(\Omega)$. Finally, in the last section we study the asymptotic behaviour of the entropy solutions. Using the Lyapunov method for semigroups of nonlinear contractions introduced by Pazy [27], we show that the entropy solutions stabilize as $t \to \infty$ by converging to a constant function.

2. Strong solutions for data in $L^2(\Omega)$

Let $\beta : D(\beta) \subseteq \mathbb{R} \to [-\infty, +\infty]$ a given maximal monotone graph with $0 \in \beta(0)$ and $D(\beta)$ an interval of extreme points $a \leqslant b$. We can consider that $\beta(r) = -\infty$ if $r \leqslant a$, $r \notin D(\beta)$ and $\beta(r) = +\infty$ if $r \geqslant b$, $r \notin D(\beta)$. For any $k > 0$, we consider the truncature operator $T_k(r) := [k - (k - |r|)^+]\operatorname{sign}_0(r)$, $r \in \mathbb{R}$. Since $|\partial u/\partial\eta| \leqslant 1$, we need to truncate β in the following way: $\hat\beta(x) = \{T_1(y){:}\ y \in \beta(x)\}$ for $x \in \mathbb{R}$. Obviously, $\hat\beta$ is also a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$. Then, as $|\hat\beta(x)| \leqslant 1$, we can find a convex function $j : \mathbb{R} \to \mathbb{R}$ such that $j(0) = 0$, $j \geqslant 0$, $|j(x) - j(y)| \leqslant |x - y|$ for all $x, y \in \mathbb{R}$, with $\hat\beta = \partial j$.

Our concept of solution of problem (1) when the initial datum is in $L^2(\Omega)$ is the following.

Definition 1. Let $u_0 \in L^2(\Omega)$. A function $u : (0, T) \times \Omega \to \mathbb{R}$ is a *strong solution* of (1) in $Q_T := (0, T) \times \Omega$ if $u \in C(0, T; L^2(\Omega))$, $u(0) = u_0$, $u'(t) \in L^2(\Omega)$, $u(t) \in L^2(\Omega) \cap BV(\Omega)$ a.e. $t \in [0, T]$ and there exists $z(t) \in X(\Omega)$, with $\|z(t)\|_\infty \leqslant 1$, satisfying for almost all $t \in [0, T]$:

$$\operatorname{div}(z(t)) = u'(t) \quad \text{in } \mathcal{D}'(\Omega), \tag{11}$$

$$\int_\Omega (z(t), Du(t)) = \int_\Omega |Du(t)| \tag{12}$$

and

$$-\left[z(t),\nu\right] \in \hat{\beta}\big(u(t)\big) \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega. \tag{13}$$

Let us remark that the above definition is consistent with the concept of solution for the Neumann and Dirichlet problem given in [2] and [3].

The main result of this section is the following existence and uniqueness theorem.

Theorem 1. *For any $u_0 \in L^2(\Omega)$, there exists a unique strong solution $u(t)$ of problem* (1) *in Q_T for all $T > 0$. Moreover, if $u(t)$, $\hat{u}(t)$ are the strong solutions corresponding to initial data u_0, $\hat{u}_0$, respectively, then*

$$\left\|\big(u(t)-\hat{u}(t)\big)^+\right\|_2 \leqslant \left\|\big(u_0-\hat{u}_0\big)^+\right\|_2 \quad \textit{and} \quad \left\|u(t)-\hat{u}(t)\right\|_2 \leqslant \left\|u_0-\hat{u}_0\right\|_2 \tag{14}$$

for all $t \geqslant 0$.

To prove Theorem 1 we shall use the nonlinear semigroup theory associated with subdifferentials (see [16]).

Consider the following functional defined in $L^2(\Omega)$:

$$\Phi_\beta(u) := \begin{cases} \displaystyle\int_\Omega |Du| + \int_{\partial\Omega} j(u) & \text{if } u \in BV(\Omega), \\ +\infty & \text{elsewhere.} \end{cases} \tag{15}$$

By a result of Modica (Proposition 1.2 in [26]) we know that the functional Φ_β is lower semicontinuous in $L^2(\Omega)$. We also have that Φ_β is convex. Therefore, the subdifferential $\partial\Phi_\beta$ of Φ_β, i.e., the operator in $L^2(\Omega)$ defined by

$$v \in \partial\Phi_\beta(u) \Leftrightarrow \Phi_\beta(w) - \Phi_\beta(u) \geqslant \int_\Omega v(w-u)\,\mathrm{d}x \quad \forall w \in L^2(\Omega),$$

is a maximal monotone operator in $L^2(\Omega)$ (see [16]). Consequently, the existence and uniqueness of solution of the abstract Cauchy problem

$$\begin{cases} u'(t) + \partial\Phi_\beta\big(u(t)\big) \ni 0, & t \in\,]0,\infty[, \\ u(0) = u_0, \quad u_0 \in L^2(\Omega) \end{cases} \tag{16}$$

follows immediately from the nonlinear semigroup theory (see [16]). Now, to get the full strength of the abstract result derived from semigroup theory we need to characterize $\partial\Phi_\beta$. To get this characterization, we introduce the following operator $\mathcal{A}_\beta$ in $L^2(\Omega)$.

$(u,v) \in \mathcal{A}_\beta$ if and only if $u,v \in L^2(\Omega)$, $u \in BV(\Omega)$ and there exists $z \in X(\Omega)$ with $\|z\|_\infty \leqslant 1$, $v = -\mathrm{div}(z)$ in $\mathcal{D}'(\Omega)$ such that

$$\int_\Omega (w-u)v \leqslant \int_\Omega z\cdot\nabla w - \int_\Omega |Du| + \int_{\partial\Omega} j(w)\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\partial\Omega} j(u)\,\mathrm{d}\mathcal{H}^{N-1} \tag{17}$$

for all $w \in W^{1,1}(\Omega) \cap L^2(\Omega)$.

We have the following characterization of the operator $\mathcal{A}_\beta$.

Lemma 1. *The following assertions are equivalent:*

(a) $(u, v) \in \mathcal{A}_\beta$,

(b) $u, v \in L^2(\Omega)$, $u \in BV(\Omega)$ *and there exists* $z \in X(\Omega)$ *with* $\|z\|_\infty \leqslant 1$, $v = -\mathrm{div}(z)$ *in* $\mathcal{D}'(\Omega)$ *such that*

$$\int_\Omega (w-u)v\,\mathrm{d}x \leqslant \int_\Omega (z, Dw) - \int_\Omega |Du| + \int_{\partial\Omega} j(w)\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\partial\Omega} j(u)\,\mathrm{d}\mathcal{H}^{N-1} \tag{18}$$

for all $w \in BV(\Omega) \cap L^2(\Omega)$,

(c) $u, v \in L^2(\Omega)$, $u \in BV(\Omega)$ *and there exists* $z \in X(\Omega)$, *with* $\|z\|_\infty \leqslant 1$, $v = -\mathrm{div}(z)$ *in* $\mathcal{D}'(\Omega)$ *such that*

$$\int_\Omega |Du| = \int_\Omega (z, Du), \tag{19}$$

$$-[z,\nu] \in \hat{\beta}(u) \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega. \tag{20}$$

Proof. Let $(u, v) \in \mathcal{A}_\beta$. Then, there exists $z \in X(\Omega)$ with $\|z\|_\infty \leqslant 1$, $v = -\mathrm{div}(z)$ in $\mathcal{D}'(\Omega)$, such that

$$\int_\Omega (w-u)v\,\mathrm{d}x \leqslant \int_\Omega z\cdot\nabla w - \int_\Omega |Du| + \int_{\partial\Omega} j(w)\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\partial\Omega} j(u)\,\mathrm{d}\mathcal{H}^{N-1} \tag{21}$$

for every $w \in W^{1,1} \cap L^2(\Omega)$. Let $w \in BV(\Omega) \cap L^2(\Omega)$, applying results from [9] and [10], we know that there exists a sequence $\{w_n\}_{n\in\mathbb{N}} \subset C^\infty(\Omega)$ such that

$$w_n \to w \quad \text{in } L^2(\Omega),$$
$$\int_\Omega |\nabla w_n|\,\mathrm{d}x \to \int_\Omega |Dw|,$$
$$\int_\Omega z\cdot\nabla w_n\,\mathrm{d}x = \int_\Omega (z, Dw_n) \to \int_\Omega (z, Dw).$$

In particular we have that w_n strictly converges to w in $BV(\Omega)$. Then, we have $w_n \to w$ in $L^{N-1}(\partial\Omega)$ (see [1]) and therefore, from the continuity of j we also obtain

$$\lim_{n\to\infty} \int_{\partial\Omega} j(w_n)\,\mathrm{d}\mathcal{H}^{N-1} = \int_{\partial\Omega} j(w)\,\mathrm{d}\mathcal{H}^{N-1}.$$

Then, taking w_n as test functions in (21) and letting $n \to \infty$ we get that (18) holds for all $w \in BV(\Omega) \cap L^2(\Omega)$. Thus (a) and (b) are equivalent.

Let us show that (b) implies (c): Taking $w = u$ as a test function in (18) we obtain

$$\int_\Omega |Du| \leqslant \int_\Omega (z, Du) \leqslant \int_\Omega |Du|$$

and then (19) holds. To prove (20) we multiply the equality $v = -\mathrm{div}(z)$ by $w - u$ and apply Green's formula to obtain

$$\int_\Omega (w-u)v\,\mathrm{d}x = -\int_\Omega (w-u)\,\mathrm{div}(z)\,\mathrm{d}x = \int_\Omega \big(z, D(w-u)\big) - \int_{\partial\Omega} [z,\nu](w-u)\,\mathrm{d}\mathcal{H}^{N-1}$$

$$= \int_{\Omega} (z, Dw) - \int_{\Omega} (z, Du) - \int_{\partial\Omega} [z, \nu](w - u) \, \mathrm{d}\mathcal{H}^{N-1}$$
$$\leqslant \int_{\Omega} (z, Dw) - \int_{\Omega} |Du| + \int_{\partial\Omega} j(w) \, \mathrm{d}\mathcal{H}^{N-1} - \int_{\partial\Omega} j(u) \, \mathrm{d}\mathcal{H}^{N-1}.$$

Taking into account (19) we get

$$-\int_{\partial\Omega} [z, \nu](w - u) \, \mathrm{d}\mathcal{H}^{N-1} \leqslant \int_{\partial\Omega} j(w) \, \mathrm{d}\mathcal{H}^{N-1} - \int_{\partial\Omega} j(u) \, \mathrm{d}\mathcal{H}^{N-1}. \tag{22}$$

Given $w \in BV(\Omega) \cap L^2(\Omega)$ and $0 \leqslant \varphi \in L^\infty(\partial\Omega)$, let $w_\varphi := u + \frac{\varphi}{\|\varphi\|_\infty + 1}(w - u)$. Taking w_φ as test function in (22) we get

$$-\int_{\partial\Omega} [z, \nu] \frac{\varphi}{\|\varphi\|_\infty + 1}(w - u) \, \mathrm{d}\mathcal{H}^{N-1}$$
$$\leqslant \int_{\partial\Omega} j\left(u + \frac{\varphi}{\|\varphi\|_\infty + 1}(w - u)\right) \mathrm{d}\mathcal{H}^{N-1} - \int_{\partial\Omega} j(u) \, \mathrm{d}\mathcal{H}^{N-1}$$

and, by the convexity of j,

$$-\int_{\partial\Omega} [z, \nu] \frac{\varphi}{\|\varphi\|_\infty + 1}(w - u) \, \mathrm{d}\mathcal{H}^{N-1} \leqslant \int_{\partial\Omega} \frac{\varphi}{\|\varphi\|_\infty + 1} \big(j(w) - j(u)\big) \, \mathrm{d}\mathcal{H}^{N-1},$$

which implies

$$\int_{\partial\Omega} \varphi\big(j(w) - j(u) + [z, \nu](w - u)\big) \, \mathrm{d}\mathcal{H}^{N-1} \geqslant 0 \quad \forall \varphi \in L^\infty(\partial\Omega) \; \varphi \geqslant 0,$$

from where we finally obtain $-[z, \nu] \in \partial j(u) = \hat{\beta}(u)$, $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$.

To prove (c) implies (b) we only need to apply Green's formula. □

We have the following result.

Theorem 2. *The operator $\partial\Phi_\beta$ has dense domain in $L^2(\Omega)$ and $\partial\Phi_\beta = \mathcal{A}_\beta$. Moreover, $\mathcal{A}_\beta$ is m-completely accretive in $L^2(\Omega)$.*

To prove this theorem we need to introduce the following operator which is related to the p-Laplacian operator with nonlinear boundary conditions (see [6]). For $p > 1$ we define the operator $\mathcal{A}_{\beta,p}$ in $L^p(\Omega)$ as

$(u, v) \in \mathcal{A}_{\beta,p}$ if and only if $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $v \in L^1(\Omega)$ and

$$\int_{\Omega} v(w - u) \, \mathrm{d}x \leqslant \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla(w - u) \, \mathrm{d}x + \int_{\partial\Omega} j(w) \, \mathrm{d}\mathcal{H}^{N-1} - \int_{\partial\Omega} j(u) \, \mathrm{d}\mathcal{H}^{N-1}$$

for all $w \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.

We have the following result (see Theorem 2.1 in [6]).

Theorem 3. *The operator $\mathcal{A}_{\beta,p}$ satisfies the following statements:*

(i) *$\mathcal{A}_{\beta,p}$ is univalued; i.e., if $(u,v) \in \mathcal{A}_{\beta,p}$, then*

$$v = -\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big), \quad \textit{in the sense of distributions}.$$

(ii) *$\mathcal{A}_{\beta,p}$ is completely accretive.*

(iii) *$L^\infty(\Omega) \subseteq R(I + \mathcal{A}_{\beta,p})$.*

Proof of Theorem 2. Firstly, let us see that $\partial\Phi_\beta$ is completely accretive. By Lemma 7.1 in [12], it is enough to prove that for all $u, \hat{u} \in L^2(\Omega)$

$$\Phi_\beta\big(u + p(\hat{u} - u)\big) + \Phi_\beta\big(\hat{u} - p(\hat{u} - u)\big) \leqslant \Phi_\beta(u) + \Phi_\beta(\hat{u}), \tag{23}$$

holds for all $p \in P_0$, where

$$P_0 := \big\{p \in C^\infty(\mathbb{R}):\ 0 \leqslant p' \leqslant 1,\ \mathrm{supp}(p') \text{ compact and } 0 \notin \mathrm{supp}(p)\big\}.$$

In fact, we may assume that $u, \hat{u} \in BV(\Omega) \cap L^2(\Omega)$. If $v = u + p(\hat{u} - u)$ and $\hat{v} = \hat{u} - p(\hat{u} - u)$, then by the chain rule for BV-functions (see [1]), it is easy to see that

$$\int_\Omega |Dv| + \int_\Omega |D\hat{v}| \leqslant \int_\Omega |Du| + \int_\Omega |D\hat{u}|.$$

On the other hand, if

$$\alpha = \chi_{\{u \neq \hat{u}\}} \frac{p(\hat{u} - u)}{\hat{u} - u},$$

having in mind the convexity of j, we have

$$\begin{aligned}
&\int_{\partial\Omega} j(v)\,\mathrm{d}\mathcal{H}^{N-1} + \int_{\partial\Omega} j(\hat{v})\,\mathrm{d}\mathcal{H}^{N-1} \\
&\quad = \int_{\partial\Omega} j\big(\alpha\hat{u} + (1-\alpha)u\big)\,\mathrm{d}\mathcal{H}^{N-1} + \int_{\partial\Omega} j\big(\alpha u + (1-\alpha)\hat{u}\big)\,\mathrm{d}\mathcal{H}^{N-1} \\
&\quad \leqslant \int_{\partial\Omega} j(u)\,\mathrm{d}\mathcal{H}^{N-1} + \int_{\partial\Omega} j(\hat{u})\,\mathrm{d}\mathcal{H}^{N-1}.
\end{aligned}$$

Thus

$$\Phi_\beta(v) + \Phi_\beta(\hat{v}) \leqslant \Phi_\beta(u) + \Phi_\beta(\hat{u}),$$

and (23) holds.

From Lemma 1, we have

$$\mathcal{A}_\beta \subset \partial\Phi_\beta. \tag{24}$$

Let us see that

$$L^\infty(\Omega) \subset R(I + \mathcal{A}_\beta). \tag{25}$$

Let $v \in L^\infty(\Omega)$. We need to find $u \in BV(\Omega)$ such that $(u, v-u) \in \mathcal{A}_\beta$; i.e., there is $z \in X(\Omega)$ with $\|z\|_\infty \leqslant 1$ such that $v - u = -\mathrm{div}(z)$ and

$$\int_\Omega (w-u)(v-u)\,\mathrm{d}x \leqslant \int_\Omega z \cdot \nabla w\,\mathrm{d}x - \int_\Omega |Du| + \int_{\partial\Omega} j(w)\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\partial\Omega} j(u)\,\mathrm{d}\mathcal{H}^{N-1}$$

for every $w \in W^{1,1}(\Omega) \cap L^2(\Omega)$.

For every $1 < p \leqslant 2$, applying Theorem 3, there is $u_p \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that

$$\begin{aligned} &\int_\Omega (w-u_p)(v-u_p)\,\mathrm{d}x \\ &\quad \leqslant \int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla(w-u_p)\,\mathrm{d}x + \int_{\partial\Omega} j(w)\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\partial\Omega} j(u_p)\,\mathrm{d}\mathcal{H}^{N-1} \end{aligned} \tag{26}$$

for all $w \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Moreover, since $\mathcal{A}_{\beta,p}$ is completely accretive, we also get $\|u_p\|_\infty \leqslant \|v\|_\infty$.

Taking $w = 0$ in (26) we get

$$-\int_\Omega u_p v\,\mathrm{d}x + \int_\Omega (u_p)^2\,\mathrm{d}x \leqslant -\int_\Omega |\nabla u_p|^p\,\mathrm{d}x - \int_{\partial\Omega} j(u_p)\,\mathrm{d}\mathcal{H}^{N-1},$$

and consequently,

$$\int_\Omega |\nabla u_p|^p\,\mathrm{d}x + \int_\Omega |u_p|^2\,\mathrm{d}x \leqslant \int_\Omega u_p v\,\mathrm{d}x \leqslant C\big(\Omega, \|v\|_\infty\big) \quad \text{for every } 1 < p \leqslant 2.$$

Thus

$$\int_\Omega |\nabla u_p|^p\,\mathrm{d}x \leqslant M_1 \quad \text{for every } 1 < p \leqslant 2, \tag{27}$$

where M_1 does not depend on p. Hence, applying Young's inequality we also have the boundness of $|\nabla u_p|$ in $L^1(\Omega)$ and so $\{u_p\}_{p>1}$ is bounded in $W^{1,1}(\Omega)$ and then we may extract a subsequence such that u_p converges in $L^1(\Omega)$ and almost everywhere to some $u \in L^1(\Omega)$ as $p \to 1^+$. From the estimates we also get $u_n \to u$ in $L^2(\Omega)$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$.

Let us prove that $\{|\nabla u_p|^{p-2}\nabla u_p\}_{p>1}$ is weakly relatively compact in $L^1(\Omega; \mathbb{R}^N)$. To do that, using (27), we have that

$$\int_\Omega |\nabla u_p|^{p-1}\,\mathrm{d}x \leqslant \left(\int_\Omega |\nabla u_p|^p\,\mathrm{d}x\right)^{(p-1)/p} \mathcal{L}^N(\Omega)^{1/p} \leqslant M_2,$$

where M_2 does not depend on p. On the other hand, for any measurable subset $E \subset \Omega$ such that $\mathcal{L}^N(E) < 1$, we have

$$\left| \int_E |\nabla u_p|^{p-2} \nabla u_p \, \mathrm{d}x \right| \leqslant \int_E |\nabla u_p|^{p-1} \leqslant M_1^{(p-1)/p} \mathcal{L}^N(E)^{1/p} \leqslant M_3 \mathcal{L}^N(E)^{1/p}.$$

Thus, $\{|\nabla u_p|^{p-2}\nabla u_p\}_{p>1}$ being bounded and equiintegrable in $L^1(\Omega, \mathbb{R}^N)$, is weakly relatively compact in $L^1(\Omega, \mathbb{R}^N)$. Hence, we may assume that

$$|\nabla u_p|^{p-2}\nabla u_p \rightharpoonup z \quad \text{as } p \to 1^+, \text{ weakly in } L^1(\Omega; \mathbb{R}^N).$$

Given $\psi \in C_0^\infty(\Omega)$, taking $w = u_p \pm \psi$ as test functions in (26) and letting $p \to 1^+$, we obtain

$$\int_\Omega (v - u)\psi \, \mathrm{d}x = \int_\Omega z \cdot \nabla \psi \, \mathrm{d}x,$$

that is, $v - u = -\mathrm{div}(z)$ in $\mathcal{D}'(\Omega)$. Moreover, we also get $\|z\|_\infty \leqslant 1$ (see the proof of Lemma 1 in [2]).

For every $w \in W^{1,2}(\Omega)$, applying Young's inequality we get

$$\begin{aligned} p \int_\Omega |\nabla u_p| \, \mathrm{d}x &+ \int_{\partial\Omega} j(u_p) \, \mathrm{d}\mathcal{H}^{N-1} \\ &\leqslant (p-1)\mathcal{L}^N(\Omega) - \int_\Omega (w - u_p)(v - u_p) \, \mathrm{d}x \\ &\quad + \int_\Omega |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla w \, \mathrm{d}x + \int_{\partial\Omega} j(w) \, \mathrm{d}\mathcal{H}^{N-1}. \end{aligned}$$

Then, using the semicontinuity of the functional Φ_β and letting $p \to 1^+$ we obtain

$$\int_\Omega |Du| + \int_{\partial\Omega} j(u) \, \mathrm{d}\mathcal{H}^{N-1} \leqslant -\int_\Omega (w - u)(v - u) \, \mathrm{d}x + \int_\Omega z \cdot \nabla w \, \mathrm{d}x + \int_{\partial\Omega} j(w) \, \mathrm{d}\mathcal{H}^{N-1}$$

for every $w \in W^{1,2}(\Omega)$; from where, by approximation we can conclude that $(u, v - u) \in \mathcal{A}_\beta$ and, therefore (25) holds.

We claim now that

$$\mathcal{A}_\beta \text{ is closed in } L^2(\Omega). \tag{28}$$

In fact: let $(u_n, v_n) \in \mathcal{A}_\beta$ such that $(u_n, v_n) \to (u, v)$ in $L^2(\Omega) \times L^2(\Omega)$. Then, there exists $z_n \in X(\Omega)$ such that $\|z_n\|_\infty \leqslant 1$, $v_n = -\mathrm{div}(z_n)$ in $D'(\Omega)$ and

$$\int_\Omega (w - u_n) v_n \, \mathrm{d}x \leqslant \int_\Omega (z_n, Dw) - \int_\Omega |Du_n| + \int_{\partial\Omega} j(w) \, \mathrm{d}\mathcal{H}^{N-1} - \int_{\partial\Omega} j(u_n) \, \mathrm{d}\mathcal{H}^{N-1}$$

for every $w \in BV(\Omega) \cap L^2(\Omega)$. Now, since $\|z_n\|_\infty \leqslant 1$ we may assume that $z_n \rightharpoonup z$ weakly* in $L^\infty(\Omega; \mathbb{R}^N)$ with $\|z\|_\infty \leqslant 1$. Moreover, since $v_n \to v$ in $L^2(\Omega)$ and $v_n = -\mathrm{div}(z_n)$, we get $v = -\mathrm{div}(z)$ in $\mathcal{D}'(\Omega)$ and

$$\lim_{n\to\infty} \int_\Omega (z_n, Dw) = \int_\Omega (z, Dw).$$

Now, letting $n \to \infty$ and having in mind the lower semicontinuity of Φ_β we finally get that $(u, v) \in \mathcal{A}_\beta$ since

$$\int_\Omega (w-u)v \,\mathrm{d}x \leqslant \int_\Omega (z, Dw) - \int_{\partial\Omega} |Du| + \int_\Omega j(w)\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\partial\Omega} j(u)\,\mathrm{d}\mathcal{H}^{N-1}.$$

From (24), (25) and (28), it follows that $\partial\Phi_\beta = \mathcal{A}_\beta$. Finally, since $BV(\Omega) \cap L^2(\Omega) \subset D(\Phi_\beta)$, we get that $D(\partial\Phi_\beta)$ is dense in $L^2(\Omega)$, and the proof concludes. □

Proof of Theorem 1. Let $\{T(t)\}_{t\geqslant 0}$ be the semigroup in $L^2(\Omega)$ generated by the operator $\partial\Phi_\beta$. Then, by the nonlinear semigroup theory, given $u_0 \in L^2(\Omega)$, $u(t) = T(t)u_0$ is the only strong solution of the problem (16). Thus, by Theorem 2, we have that for almost all $t \in [0, \infty[$, $u(t) \in D(\mathcal{A}_\beta)$ and $-u'(t) \in \mathcal{A}_\beta(u(t))$. □

3. Explicit solutions

In [2] and [5] explicit solutions have been obtained for the Neumann and Dirichlet problem. Let us compute now explicit solutions for a particular β. More precisely, in the two following examples we consider

$$\beta(r) := \begin{cases} -1, & r \leqslant -1, \\ r, & |r| \leqslant 1, \\ 1, & r > 1. \end{cases} \tag{29}$$

Example 1. Let $\Omega = B_R(0)$ and let β given by (29). We are going to calculate the solution of problem (1) for the initial datum $u_0 := a\chi_{B_R(0)}$, with $a > 0$. We seek a solution u of the form $u(t) = a(t)\chi_{B_R(0)}$. Since $u'(t) = a'(t)\chi_{B_R(0)} = \mathrm{div}(z(t))$ in $\mathcal{D}'(B_R(0))$, integrating we get

$$a'(t)\left|B_R(0)\right| = \int_{B_R(0)} \mathrm{div}\big(z(t)\big)\,\mathrm{d}x = \int_{\partial B_R(0)} [z(t), \nu]\,\mathrm{d}\mathcal{H}^{N-1} = -\beta\big(a(t)\big)\mathcal{H}^{N-1}\big(\partial B_R(0)\big).$$

Hence, we have

$$a'(t) = \begin{cases} -\dfrac{N}{R} & \text{if } a(t) \geqslant 1, \\ -\dfrac{N}{R}a(t) & \text{if } \left|a(t)\right| \leqslant 1. \end{cases}$$

Thus, $a(t) = a\,\mathrm{e}^{-\frac{N}{R}t}$, if $|a(t)| \leqslant 1$, and $a(t) = (a - \frac{N}{R}t)$, if $a(t) \geqslant 1$. Therefore, the solution $u(t)$ is given by

$$u(t) = a\,\mathrm{e}^{-\frac{N}{R}t}\chi_{B_R(0)}, \quad t \geqslant 0, \text{ if } 0 < a \leqslant 1 \tag{30}$$

and

$$u(t) = \begin{cases} \left(a - \frac{N}{R}t\right)\chi_{B_R(0)}, & 0 \leqslant t \leqslant T := \frac{(a-1)R}{N}, \\ \mathrm{e}^{-\frac{N}{R}(t-T)}\chi_{B_R(0)}, & t \geqslant T \end{cases} \quad \text{if } a \geqslant 1. \tag{31}$$

To see that the function $u(t)$ given by (30) or (31) is the solution, we only need to consider the vector fields: $z(t,x) := -\frac{x}{R}a\,\mathrm{e}^{-\frac{N}{R}t}$, in the case $0 < a \leqslant 1$, and

$$z(t,x) := \begin{cases} -\frac{x}{R}, & 0 \leqslant t \leqslant T := \frac{(a-1)R}{N}, \\ -\frac{x}{R}\,\mathrm{e}^{-\frac{N}{R}(t-T)}, & t \geqslant T, \end{cases}$$

in the case $a > 1$.

Example 2. Let $\Omega = B_R(0)$ and let β given by (29). Let $u_0 := a\chi_{B_r(0)}$ with $0 < r < R$. We seek a solution u of the form $u(t) = a(t)\chi_{B_R(0)} + b(t)\chi_{\Omega_{R,r}}$, with $a(t) \geqslant b(t)$, where $\Omega_{R,r} := B_R(0) \setminus \overline{B}_r(0)$. Then we look for a $z(t) \in X(\Omega)$ with $\|z(t)\|_\infty \leqslant 1$, $u_t(t,x) = \operatorname{div}(z(t))$ and such that (12) and (13) hold. From $u_t(t,x) = \operatorname{div}(z(t,x))$, multiplying by $u(t,x)$ and integrating in Ω we get

$$\begin{aligned} \int_\Omega u_t(t,x)u(t,x)\,\mathrm{d}x &= -\int_\Omega \big(z(t), Du(t)\big) + \int_{\partial\Omega} \big[z(t),\nu\big]u(t)\,\mathrm{d}\mathcal{H}^{N-1} \\ &= -\int_\Omega |Du(t)| - \int_{\partial\Omega} \beta\big(b(t)\big)b(t)\,\mathrm{d}\mathcal{H}^{N-1}. \end{aligned}$$

Now, from the coarea formula, if $E_s^t := \{x \in B_R(0)\colon\ u(t,x) > s\}$, we have

$$\int_\Omega |Du(t)| = \int_{-\infty}^{+\infty} |D\chi_{E_s^t}|\big(B_R(0)\big)\,\mathrm{d}s = \int_{b(t)}^{a(t)} |D\chi_{B_r(0)}|\big(B_R(0)\big)\,\mathrm{d}s = \big(a(t)-b(t)\big)\operatorname{Per}\big(B_r(0)\big).$$

Consequently, we get

$$a(t)a'(t)\big|B_r(0)\big| + b'(t)b(t)|\Omega_{R,r}| = \big(b(t)-a(t)\big)\operatorname{Per}\big(B_r(0)\big) - b(t)\beta\big(b(t)\big)\operatorname{Per}\big(B_R(0)\big),$$

from where it follows that

$$\begin{aligned} a'(t) &= -\frac{N}{r}, \\ b'(t) &= \frac{\operatorname{Per}(B_r(0)) - \beta(b(t))\operatorname{Per}(B_R(0))}{|\Omega_{R,r}|}. \end{aligned}$$

Then, since $a(0) = a$, we have

$$a(t) = a - \frac{N}{r}t.$$

On the other hand, assuming that $|b(t)| \leqslant 1$, we have

$$b'(t) = \frac{\mathrm{Per}(B_r(0)) - b(t)\mathrm{Per}(B_R(0))}{|\Omega_{R,r}|}.$$

Then, since $b(0) = 0$, we get

$$b(t) = \frac{r^{N-1}}{R^{N-1}}\left(1 - \mathrm{e}^{-\frac{\mathrm{Per}(B_R(0))}{|\Omega_{R,r}|}t}\right) = \frac{r^{N-1}}{R^{N-1}}\left(1 - \mathrm{e}^{-\frac{NR^{N-1}}{R^N - r^N}t}\right).$$

This is true until time T for which $a(T) = b(T)$, that is, until time $T > 0$, solution of the equation

$$a - \frac{N}{r}T = \frac{r^{N-1}}{R^{N-1}}\left(1 - \mathrm{e}^{-\frac{NR^{N-1}}{R^N - r^N}T}\right). \tag{32}$$

We need to find now the vector field $z(t) \in X(\Omega)$ verifying $\|z(t)\|_\infty \leqslant 1$, $u_t = \mathrm{div}(z(t))$ and (12) and (13). As $\mathrm{div}(z(t,x)) = -N/r$ in $B_r(0)$ we may have $z(t,x) = -x/r$ for $x \in B_r(0)$ and $0 \leqslant t \leqslant T$. To construct $z(t)$ in $\Omega_{R,r}$ first we suppose $z(t,x) = \rho(t, \|x\|)\frac{x}{\|x\|}$. Then, $\rho(t, \|x\|)$ will be the solution of the following EDP:

$$\frac{\partial \rho(t,s)}{\partial s} + \frac{(N-1)\rho(t,s)}{s} = \frac{\mathrm{Per}(B_r(0))}{|\Omega_{R,r}|}\,\mathrm{e}^{-\frac{NR^{N-1}}{R^N - r^N}t} \tag{33}$$

coupled with the initial condition

$$\rho(t,R) = -\frac{r^{N-1}}{R^{N-1}}\left(1 - \mathrm{e}^{-\frac{NR^{N-1}}{R^N - r^N}t}\right).$$

The solution of the EDP with this initial boundary condition is:

$$\rho(t,s) = \frac{r^{N-1}}{s^{N-1}}\left(\frac{\mathrm{e}^{-\frac{NR^{N-1}}{R^N - r^N}t}}{R^N - r^N}\left(s^N - r^N\right) - 1\right).$$

Then we finally have:

$$z(t,x) = \begin{cases} -\dfrac{x}{r} & \text{if } x \in B_r(0), \\[2ex] \dfrac{r^{N-1}}{\|x\|^N}x\left(\dfrac{\mathrm{e}^{-\frac{NR^{N-1}}{R^N - r^N}t}}{R^N - r^N}\left(\|x\|^N - r^N\right) - 1\right) & \text{if } x \in \Omega_{R,r}. \end{cases} \tag{34}$$

We note that $z(t,x) \cdot \frac{x}{r} = -1$, $z(t,x) \cdot \frac{x}{R} = -b(t)$ and $\|z(t)\|_\infty \leqslant 1$. Let us see that $u_t = \mathrm{div}(z)$ in $\mathcal{D}'(\Omega \times [0,T])$. Let $\phi \in \mathcal{D}(\Omega \times [0,T])$,

$$\langle \mathrm{div}(z), \phi \rangle = -\int_0^T \int_\Omega z(t,x) \cdot \nabla \phi(x,t)$$

$$= \int_0^T \int_{B_r(0)} \frac{x}{r} \cdot \nabla\phi(x,t) - \int_0^T \int_{\Omega_{R,r}} \frac{r^{N-1}}{\|x\|^N} x \left(\frac{\mathrm{e}^{-\frac{NR^{N-1}}{R^N - r^N}t}}{R^N - r^N} \left(\|x\|^N - r^N\right) - 1 \right) \cdot \nabla\phi(x,t)$$

$$= -\int_0^T \int_{B_r(0)} \frac{N}{r}\phi(x,t) + \int_0^T \int_{\partial B_r(0)} \phi(x,t) + \int_0^T \int_{\Omega_{R,r}} \frac{Nr^{N-1}}{(R^N - r^N)} \mathrm{e}^{-\frac{NR^{N-1}}{R^N - r^N}t} \phi(x,t)$$

$$+ \int_0^T \int_{\partial\Omega_{R,r}} z(t,x) \cdot \frac{x}{r}\phi(x,t)$$

$$= -\int_0^T \int_{B_r(0)} \frac{N}{r}\phi(x,t) + \int_0^T \int_{\Omega_{R,r}} \frac{Nr^{N-1}}{(R^N - r^N)} \mathrm{e}^{-\frac{NR^{N-1}}{R^N - r^N}t} \phi(x,t) = \langle u_t, \varphi \rangle.$$

Finally, let us see that (12) holds:

$$\int_\Omega \big(z(t), Du(t)\big) = -\int_\Omega u(t)\,\mathrm{div}\big(z(t)\big)\,\mathrm{d}x + \int_{\partial\Omega} [z(t),\nu]\,u(t)\,\mathrm{d}\mathcal{H}^{N-1}$$

$$= -\int_{B_r(0)} a(t)a'(t)\,\mathrm{d}x - \int_{\Omega_{R,r}} b(t)b'(t)\,\mathrm{d}x - \int_{\partial\Omega} b(t)^2\,\mathrm{d}\mathcal{H}^{N-1}$$

$$= -a(t)a'(t)\big|B_r(0)\big| - b'(t)b(t)|\Omega_{R,r}| - b(t)^2 \mathrm{Per}\big(B_R(0)\big)$$

$$= a(t)\mathrm{Per}\big(B_r(0)\big) - \frac{\mathrm{Per}(B_r(0)) - b(t)\mathrm{Per}(B_R(0))}{|\Omega_{R,r}|} b(t)|\Omega_{R,r}| - b(t)^2 \mathrm{Per}\big(B_R(0)\big)$$

$$= \big(a(t) - b(t)\big)\mathrm{Per}\big(B_r(0)\big) = \int_\Omega \big|Du(t)\big|.$$

At time $t = T$ we have that $a(T) = b(T) = C \leqslant 1$ and $u(T) = C\chi_\Omega$. Then, for $t \geqslant T$, $u(t)$ evolves as in the above example, that is, $u(t) = C\mathrm{e}^{-\frac{N}{R}(t-T)}\chi_{B_R(0)}$, for $t \geqslant T$. Consequently, the solution of problem (1) for the initial datum $u_0 := a\chi_{B_r(0)}$ with $0 < r < R$ is given by

$$u(t) = \begin{cases} \left(a - \frac{N}{r}t\right)\chi_{B_r(0)} + \frac{r^{N-1}}{R^{N-1}}\left(1 - \mathrm{e}^{-\frac{NR^{N-1}}{R^N - r^N}t}\right)\chi_{\Omega_{R,r}} & \text{if } 0 \leqslant t \leqslant T, \\ C\mathrm{e}^{-\frac{N}{R}(t-T)}\chi_{B_R(0)} & \text{if } t > T, \end{cases} \tag{35}$$

where T is the solution of (32) and $C = (a - \frac{N}{r}T)$.

4. Solutions for data in $L^1(\Omega)$

Similarly to the case of the Dirichlet problem ([3]), to get existence and uniqueness of solutions for initial data in $L^1(\Omega)$ we need to work with the concept of entropy solution. To make precise this notion of solution let us recall some notations and definitions given in [3]. First we need to introduce a weak trace on $\partial\Omega$ of the normal component of certain vector fields in Ω. We define the space

$$Z(\Omega) := \big\{(z,\xi) \in L^\infty\big(\Omega,\mathbb{R}^N\big) \times BV(\Omega)^*\colon\ \mathrm{div}(z) = \xi \text{ in } \mathcal{D}'(\Omega)\big\}.$$

We denote $R(\Omega) := W^{1,1}(\Omega) \cap L^\infty(\Omega) \cap C(\Omega)$. For $(z,\xi) \in Z(\Omega)$ and $w \in R(\Omega)$ we define

$$\langle (z,\xi), w\rangle_{\partial\Omega} := \langle \xi, w\rangle_{BV(\Omega)^*,BV(\Omega)} + \int_\Omega z \cdot \nabla w \,\mathrm{d}x.$$

Then, working as in the proof of Theorem 1.1 of [9], we obtain that if $w, v \in R(\Omega)$ and $w = v$ on $\partial\Omega$ one has

$$\langle (z,\xi), w\rangle_{\partial\Omega} = \langle (z,\xi), v\rangle_{\partial\Omega} \quad \forall (z,\xi) \in Z(\Omega). \tag{36}$$

As a consequence of (36), we can give the following definition: Given $u \in BV(\Omega) \cap L^\infty(\Omega)$ and $(z,\xi) \in Z(\Omega)$, we define $\langle (z,\xi), u\rangle_{\partial\Omega}$ by setting

$$\langle (z,\xi), u\rangle_{\partial\Omega} := \langle (z,\xi), w\rangle_{\partial\Omega},$$

where w is any function in $R(\Omega)$ such that $w = u$ on $\partial\Omega$. Again, working as in the proof of Theorem 1.1 of [9], we can prove that for every $(z,\xi) \in Z(\Omega)$ there exists $M_{z,\xi} > 0$ such that

$$\left|\langle (z,\xi), u\rangle_{\partial\Omega}\right| \leqslant M_{z,\xi} \|u\|_{L^1(\partial\Omega)} \quad \forall u \in BV(\Omega) \cap L^\infty(\Omega). \tag{37}$$

Now, taking a fixed $(z,\xi) \in Z(\Omega)$, we consider the linear functional $F : L^\infty(\partial\Omega) \to \mathbb{R}$ defined by

$$F(v) := \langle (z,\xi), w\rangle_{\partial\Omega},$$

where $v \in L^\infty(\partial\Omega)$ and $w \in BV(\Omega) \cap L^\infty(\Omega)$ is such that $w|_{\partial\Omega} = v$. By estimate (37), there exists $\gamma_{z,\xi} \in L^\infty(\partial\Omega)$ such that

$$F(v) = \int_{\partial\Omega} \gamma_{z,\xi}(x) v(x) \,\mathrm{d}\mathcal{H}^{N-1}.$$

Consequently there exists a linear operator $\gamma : Z(\Omega) \to L^\infty(\partial\Omega)$, with $\gamma(z,\xi) := \gamma_{z,\xi}$, satisfying

$$\langle (z,\xi), w\rangle_{\partial\Omega} = \int_{\partial\Omega} \gamma_{z,\xi}(x) w(x) \,\mathrm{d}\mathcal{H}^{N-1} \quad \forall w \in BV(\Omega) \cap L^\infty(\Omega).$$

In case $z \in C^1(\overline{\Omega}, \mathbb{R}^N)$, we have $\gamma_z(x) = z(x) \cdot \nu(x)$ for all $x \in \partial\Omega$. Hence, the function $\gamma_{z,\xi}(x)$ is the weak trace of the normal component of (z,ξ). For simplicity of the notation, we shall denote $\gamma_{z,\xi}(x)$ by $[z,\nu](x)$.

We need to consider the space $BV(\Omega)_2$, defined as $BV(\Omega) \cap L^2(\Omega)$ endowed with the norm

$$\|w\|_{BV(\Omega)_2} := \|w\|_{L^2(\Omega)} + |Du|(\Omega).$$

It easy to see that $L^2(\Omega) \subset BV(\Omega)_2^*$ and

$$\|w\|_{BV(\Omega)_2^*} \leqslant \|w\|_{L^2(\Omega)} \quad \forall w \in L^2(\Omega). \tag{38}$$

Now, it is well known (see for instance [30]) that the dual space $(L^1(0,T;BV(\Omega)_2))^*$ is isometric to the space $L^\infty(0,T;BV(\Omega)_2^*, BV(\Omega)_2)$ of all weakly* measurable functions $f:[0,T]\to BV(\Omega)_2^*$, such that $v(f)\in L^\infty([0,T])$, where $v(f)$ denotes the supremum of the set $\{|\langle w,f\rangle|: \|w\|_{BV(\Omega)_2}\leqslant 1\}$ in the vector lattice of measurable real functions. Moreover, the dual pairing of the isometry is defined by

$$\langle w,f\rangle=\int_0^T\langle w(t),f(t)\rangle\,\mathrm{d}t,$$

for $w\in L^1(0,T;BV(\Omega)_2)$ and $f\in L^\infty(0,T;BV(\Omega)_2^*, BV(\Omega)_2)$.

By $L^1_w(0,T,BV(\Omega))$ we denote the space of weakly measurable functions $w:[0,T]\to BV(\Omega)$ (i.e., $t\in[0,T]\to\langle w(t),\phi\rangle$ is measurable for every $\phi\in BV(\Omega)^*$) such that $\int_0^T\|w(t)\|<\infty$. Observe that, since $BV(\Omega)$ has a separable predual (see [1]), it follows easily that the map $t\in[0,T]\to\|w(t)\|$ is measurable.

To make precise our notion of solution we need the following definitions:

Definition 2. Let $\Psi\in L^1(0,T,BV(\Omega))$. We say Ψ admits a *weak derivative* in the space $L^1_w(0,T,BV(\Omega))\cap L^\infty(Q_T)$ if there is a function $\Theta\in L^1_w(0,T,BV(\Omega))\cap L^\infty(Q_T)$ such that $\Psi(t)=\int_0^t\Theta(s)\,\mathrm{d}s$, the integral being taken as a Pettis integral.

Definition 3. Let $\xi\in(L^1(0,T,BV(\Omega)_2))^*$. We say that ξ is *the time derivative* in the space $(L^1(0,T,BV(\Omega)_2))^*$ of a function $u\in L^1((0,T)\times\Omega)$ if

$$\int_0^T\langle\xi(t),\Psi(t)\rangle\,\mathrm{d}t=-\int_0^T\int_\Omega u(t,x)\Theta(t,x)\,\mathrm{d}x\,\mathrm{d}t$$

for all test functions $\Psi\in L^1(0,T,BV(\Omega))$ which admit a weak derivative $\Theta\in L^1_w(0,T,BV(\Omega))\cap L^\infty(Q_T)$ and have compact support in time.

Observe that if $w\in L^1(0,T,BV(\Omega))\cap L^\infty(Q_T)$ and $z\in L^\infty(Q_T,\mathbb{R}^N)$ is such that there exists $\xi\in(L^1(0,T,BV(\Omega)))^*$ with $\mathrm{div}(z)=\xi$ in $\mathcal{D}'(Q_T)$, we can define, associated to the pair (z,ξ), the distribution (z,Dw) in Q_T by

$$\langle(z,Dw),\phi\rangle:=-\int_0^T\langle\xi(t),w(t)\phi(t)\rangle\,\mathrm{d}t-\int_0^T\int_\Omega z(t,x)w(t,x)\nabla_x\phi(t,x)\,\mathrm{d}x\,\mathrm{d}t \tag{39}$$

for all $\phi\in\mathcal{D}(Q_T)$.

Definition 4. Let $\xi\in(L^1(0,T,BV(\Omega)_2))^*$, $z\in L^\infty(Q_T,\mathbb{R}^N)$. We say that $\xi=\mathrm{div}(z)$ in $(L^1(0,T,BV(\Omega)_2))^*$ if (z,Dw) is a Radon measure in Q_T with normal boundary values $[z,\nu]\in L^\infty((0,T)\times\partial\Omega)$, such that

$$\int_{Q_T}(z,Dw)+\int_0^T\langle\xi(t),w(t)\rangle\,\mathrm{d}t=\int_0^T\int_{\partial\Omega}[z(t,x),\nu]w(t,x)\,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t$$

for all $w\in L^1(0,T,BV(\Omega))\cap L^\infty(Q_T)$.

We need to consider the set of truncature functions:

$$\mathcal{T} := \{T_k, T_k^+, T_k^- :\ k > 0\}$$

and

$$\mathcal{P} := \{p \in W^{1,\infty}(\mathbb{R}):\ 0 \leqslant p' \leqslant 1,\ \operatorname{supp}(p') \text{ compact}\}.$$

Our concept of solution is the following.

Definition 5. A measurable function $u : (0,T)\times\Omega \to \mathbb{R}$ is an *entropy solution* of (1) in $Q_T = (0,T)\times\Omega$ if $u \in C([0,T]; L^1(\Omega))$, $p(u(\cdot)) \in L^1_w(0,T; BV(\Omega))$ $\forall p \in \mathcal{P}$ and there exist $(z(t), \xi(t)) \in Z(\Omega)$ with $\|z(t)\|_\infty \leqslant 1$, and $\xi \in (L^1(0,T; BV(\Omega)_2))^*$ such that

(i) ξ is the time derivative of u in $(L^1(0,T; BV(\Omega)_2))^*$ in the sense of Definition 3,
(ii) $\xi = \operatorname{div}(z)$ in $(L^1(0,T; BV(\Omega)))^*$ in the sense of Definition 4,
(iii) for almost all $t \in [0,T]$, all $w \in L^1(0,T; BV(\Omega))$ and $p \in \mathcal{P}$,

$$\begin{aligned} &-\int_0^T\int_{\partial\Omega} [z(t),\nu]\big(p(w(t)) - p(u(t))\big)\,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t \\ &\quad\leqslant \int_0^T \left(\int_{\partial\Omega} j\big(w(t)\big)\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\partial\Omega} j\big(p(u(t)) - p(0)\big)\,\mathrm{d}\mathcal{H}^{N-1}\right)\mathrm{d}t, \end{aligned}$$

(iv) the following inequality is satisfied

$$\begin{aligned} &-\int_0^T\int_\Omega J_p\big(u(t)\big)\eta_t + \int_0^T\int_\Omega \eta(t)\big|Dp\big(u(t)\big)\big| + \int_0^T\int_{\partial\Omega} j\big(p\big(u(t)\big) - p(0)\big)\eta(t) \\ &\quad+ \int_0^T\int_\Omega z(t)\cdot D\eta(t)p\big(u(t)\big) \leqslant \int_0^T\int_{\partial\Omega} \big[z(t),\nu\big]p(0)\eta(t) \end{aligned} \tag{40}$$

for all $\eta \in C^\infty(\overline{Q_T})$, with $\eta \geqslant 0$, $\eta(t,x) = \phi(t)\psi(x)$, being $\phi \in \mathcal{D}(]0,T[)$, $\psi \in C^\infty(\overline{\Omega})$ and $p \in \mathcal{P}$, where $J_p(r) = \int_0^r p(s)\,\mathrm{d}s$.

The main result of this section is the following one:

Theorem 4. *Given $u_0 \in L^1(\Omega)$, there exists a unique entropy solution of (1) in $(0,T)\times\Omega$ for every $T > 0$ such that $u(0) = u_0$. Moreover, if $u(t), \hat{u}(t)$ are the entropy solutions corresponding to initial data $u_0, \hat{u}_0$, respectively, then*

$$\big\|\big(u(t) - \hat{u}(t)\big)^+\big\|_1 \leqslant \big\|\big(u_0 - \hat{u}_0\big)^+\big\|_1 \quad \textit{and} \quad \big\|u(t) - \hat{u}(t)\big\|_1 \leqslant \big\|u_0 - \hat{u}_0\big\|_1 \tag{41}$$

for all $t \geqslant 0$.

To prove the existence part of the above theorem we shall use Crandall–Liggett's semigroup generation theorem. So we shall associate an m-completely accretive operator $\mathcal{B}_\beta$ to the problem (1). In the line of [11], to do that we need to consider the function space

$$TBV(\Omega) := \{u \in L^1(\Omega):\ T_k(u) \in BV(\Omega)\ \forall k > 0\}.$$

Notice that the function space $TBV(\Omega)$ is closely related to the space $GBV(\Omega)$ of generalized functions of bounded variation introduced by Di Giorgi and Ambrosio ([19], see also [1]), indeed $TBV(\Omega) \subset GBV(\Omega)$. Given $p \in \mathcal{P}$, since $p(u) = p(T_k(u))$ for k large enough, we have that $p(u) \in BV(\Omega)$ for all $u \in TBV(\Omega)$ and $p \in \mathcal{P}$.

Remark 1. Let us remark that we can define the trace on the boundary for functions in $TBV(\Omega)$. In fact: since there is a trace on interior rectifiable sets for functions in $TBV(\Omega)$ (see the remark after Theorem 4.34 in [1]), as for BV-functions, it is sufficient to construct an extension operator from $TBV(\Omega)$ to $TBV(\mathbb{R}^N)$. Let $u \in TBV(\Omega)$ and consider the extension operator T defined in the proof of Proposition 3.21 in [1]. By construction, it is clear that the property $T_k(T(T_r(u))) = T(T_k(u))$ holds for $r > k$. As a consequence, the $\lim_{r\to\infty} T(T_r(u))$ exists (pointwise) and defines a function $E(u)$ in $TBV(\mathbb{R}^N)$ which extends u. Therefore we can define, for $\mathcal{H}^{N-1}$-almost all $x \in \partial\Omega$, the trace of a function $u \in TBV(\Omega)$ as $E(u)^+_{\mathcal{F}\Omega}(x)$. Now, by Theorem 4.34 in [1], we have

$$E(u)^+_{\mathcal{F}\Omega}(x) = \lim_{k\to\infty} \big(T_k\big(E(u)\big)\big)^+_{\mathcal{F}\Omega}(x) = \lim_{k\to\infty} \big(T\big(T_k(u)\big)\big)^+_{\mathcal{F}\Omega}(x) = \lim_{k\to\infty} \big(T_k(u)\big)^{\Omega}(x).$$

We define the following operator in $L^1(\Omega)$:
$(u,v) \in \mathcal{B}_\beta$ if and only if $u, v \in L^1(\Omega)$, $u \in TBV(\Omega)$ and there exists $z \in X(\Omega)$ such that $\|z\|_\infty \leqslant 1$, $v = -\mathrm{div}(z)$ in $\mathcal{D}'(\Omega)$ and

$$\begin{aligned}&\int_\Omega \big(p(w) - p(u)\big) v \,\mathrm{d}x \\ &\quad \leqslant \int_\Omega \big(z, Dp(w)\big) - \int_\Omega |Dp(u)| + \int_{\partial\Omega} j(w)\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\partial\Omega} j\big(p(u) - p(0)\big)\,\mathrm{d}\mathcal{H}^{N-1}\end{aligned} \tag{42}$$

for all $w \in BV(\Omega) \cap L^\infty(\Omega)$ and for all $p \in \mathcal{P}$. We have the following characterization of $\mathcal{B}_\beta$:

Lemma 2. *The following are equivalent:*

(i) $(u,v) \in \mathcal{B}_\beta$,
(ii) $u, v \in L^1(\Omega)$, $u \in TBV(\Omega)$ *and there exists* $z \in X(\Omega)$ *such that* $\|z\|_\infty \leqslant 1$, $v = -\mathrm{div}(z)$ *in* $\mathcal{D}'(\Omega)$ *and*

$$\int_\Omega \big(z, Dp(u)\big) = \int_\Omega |Dp(u)| \quad \forall p \in \mathcal{P}; \tag{43}$$

$$-\int_{\partial\Omega} [z,\nu]\big(p(w) - p(u)\big)\,\mathrm{d}\mathcal{H}^{N-1} \leqslant \int_{\partial\Omega} j(w)\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\partial\Omega} j\big(p(u) - p(0)\big)\,\mathrm{d}\mathcal{H}^{N-1} \tag{44}$$

for all $w \in BV(\Omega) \cap L^\infty(\Omega)$ *and for all* $p \in \mathcal{P}$.

Proof. (i) $\Rightarrow$ (ii) If we take $p = T_k$ and $w = T_k(u)$ in (42), we get

$$0 \leqslant \int_\Omega \big(z, DT_k(u)\big) - \int_\Omega |DT_k(u)|,$$

from where (43) holds for $p = T_k$. Then, $\theta(z, DT_k(u), x) = 1$ $|DT_k(u)|$-a.e., hence, using Corollary 1.6 in [9], we obtain that (43) holds for all $p \in \mathcal{P}$.

On the other hand, multiplying the equality $v = -\mathrm{div}(z)$ by $p(w)-p(u)$, integrating in Ω and applying Green's formula, we get

$$\int_\Omega \big(p(w) - p(u)\big)v\,\mathrm{d}x = \int_\Omega \big(z, Dp(w)\big) - \int_\Omega |Dp(u)| - \int_{\partial\Omega} [z,\nu]\big(p(w) - p(u)\big)\,\mathrm{d}\mathcal{H}^{N-1}.$$

Then, since $(u, v) \in \mathcal{B}_\beta$ we obtain that

$$\begin{aligned} &-\int_{\partial\Omega} [z,\nu]\big(p(w) - p(u)\big)\,\mathrm{d}x \\ &\quad\leqslant \int_{\partial\Omega} j(w)\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\partial\Omega} j\big(p(u) - p(0)\big)\,\mathrm{d}\mathcal{H}^{N-1} \quad \forall w \in BV(\Omega) \cap L^\infty(\Omega), \end{aligned}$$

and (44) holds.

(ii) $\Rightarrow$ (i) Let $u, v \in L^1(\Omega)$ verifying assumptions (ii). Since $v = -\mathrm{div}(z)$, working as before, for every $w \in BV(\Omega) \cap L^\infty(\Omega)$ and $p \in \mathcal{P}$, we get

$$\int_\Omega \big(p(w) - p(u)\big)v\,\mathrm{d}x = \int_\Omega \big(z, Dp(w)\big) - \int_\Omega |Dp(u)| - \int_{\partial\Omega} [z,\nu]\big(p(w) - p(u)\big)\,\mathrm{d}\mathcal{H}^{N-1}.$$

Then, using (44), we obtain that $(u, v) \in \mathcal{B}_\beta$. $\square$

Remark 2. Suppose that $(u, v) \in \mathcal{B}_\beta$. Taking $p \in \mathcal{T}$ and

$$w_\varphi := p(u) + \frac{\varphi}{\|\varphi\|_\infty + 1}\big(p(w) - p(u)\big)$$

as test function in (44), where $0 \leqslant \varphi \in L^\infty(\partial\Omega)$, $w \in BV(\Omega) \cap L^\infty(\Omega)$, and working as in the proof of Lemma 1 we get

$$-[z,\nu]\big(p(w) - p(u)\big) \leqslant j\big(p(w)\big) - j\big(p(u)\big) \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega.$$

In particular,

$$-[z,\nu]\big(T_k(w) - T_k(u)\big) \leqslant j\big(T_k(w)\big) - j\big(T_k(u)\big) \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega.$$

Then having in mind Remark 1, letting $k \to \infty$ in the above inequality, we obtain that

$$-[z,\nu](w - u) \leqslant j(w) - j(u) \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega, \tag{45}$$

where the case $u(x) = \pm\infty$ is not excluded in the above inequality.

Observe that if the trace of u is finite $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$, then (45) is equivalent to

$$-[z,\nu] \in \hat{\beta}(u) \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega. \tag{46}$$

Similarly, if u is an entropy solution of (1), for every $w \in L^1(0,T;BV(\Omega))$ and $k > 0$,

$$-[z(t),\nu]\big(T_k(w(t)) - T_k(u(t))\big) \leqslant j\big(T_k(w(t))\big) - j\big(T_k(u(t))\big) \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega, \text{ a.e. } t > 0.$$

Lemma 3. *We have the following inequality,*

$$y\big(p(r) - p(x)\big) \leqslant j(r) - j\big(p(x) - p(0)\big) \quad \forall y \in \hat{\beta}(x),\ r \in \mathbb{R} \text{ and } p \in \mathcal{P}. \tag{47}$$

In particular, $\mathcal{A}_\beta \subset \mathcal{B}_\beta$.

Proof. We first note that, as $p \in \mathcal{P}$ and j is increasing in $]0,+\infty[$ and decreasing in $]-\infty,0[$, we have

$$j(x) \geqslant j\big(p(x) - p(0)\big) \quad \forall x \in \mathbb{R}. \tag{48}$$

Let $y \in \hat{\beta}(x)$ and $r \in \mathbb{R}$. Suppose $y \geqslant 0$. Then $x \geqslant 0$. If $r > x$, we have $p(r) - p(x) \leqslant r - x$. Hence, having in mind (48), we have

$$y\big(p(r) - p(x)\big) \leqslant y(r - x) \leqslant j(r) - j(x) \leqslant j(r) - j\big(p(x) - p(0)\big).$$

Suppose now that $r < x$. Then, $r - x \leqslant p(r) - p(x) \leqslant 0$. Moreover, since $p(x) - p(0) \leqslant x$, if $z \in \hat{\beta}(p(x) - p(0))$, we have $y \geqslant z \geqslant 0$, and consequently, having in mind (48), we have

$$\begin{aligned} y\big(p(r) - p(x)\big) &\leqslant z\big(p(r) - p(x)\big) = z\big(\big(p(r) - p(0)\big) - \big(p(x) - p(0)\big)\big) \\ &\leqslant j\big(p(r) - p(0)\big) - j\big(p(x) - p(0)\big) \leqslant j(r) - j\big(p(x) - p(0)\big). \end{aligned}$$

This concludes the proof of (47) in the case $y \geqslant 0$; the case $y < 0$ is similar. □

Theorem 5. *The operator $\mathcal{B}_\beta$ is m-completely accretive in $L^1(\Omega)$ with dense domain.*

Proof. First we are going to prove that the operator $\mathcal{B}_\beta$ is accretive in $L^1(\Omega)$. To do that we have to show that

$$\int_\Omega |u - \hat{u}|\,\mathrm{d}x \leqslant \int_\Omega |f - \hat{f}|\,\mathrm{d}x \tag{49}$$

whenever $f \in u + \mathcal{B}_\beta(u)$, $\hat{f} \in \hat{u} + \mathcal{B}_\beta(\hat{u})$. In fact: there exist $z, \hat{z} \in X(\Omega)$ with $\|z\|_\infty \leqslant 1$, $\|\hat{z}\|_\infty \leqslant 1$, $f - u = -\mathrm{div}(z)$, $\hat{f} - \hat{u} = -\mathrm{div}(\hat{z})$, $\int_\Omega |Dp(u)| = \int_\Omega (z, Dp(u))$ and $\int_\Omega |Dp(\hat{u})| = \int_\Omega (\hat{z}, Dp(\hat{u}))$. Multiplying $f - u = -\mathrm{div}(z)$ by $T_r(T_k(\hat{u}) - T_k(u))$ and $\hat{f} - \hat{u} = -\mathrm{div}(\hat{z})$ by $T_r(T_k(u) - T_k(\hat{u}))$ for $r, k > 0$ and integrating in Ω, we get

$$\begin{aligned} &\int_\Omega T_r\big(T_k(\hat{u}) - T_k(u)\big)(f - u)\,\mathrm{d}x \\ &\quad = \int_\Omega \big(z, DT_r\big(T_k(\hat{u}) - T_k(u)\big)\big) - \int_{\partial\Omega} [z,\nu] T_r\big(T_k(\hat{u}) - T_k(u)\big)\,\mathrm{d}\mathcal{H}^{N-1} \end{aligned}$$

and

$$\int_\Omega T_r\big(T_k(u)-T_k(\hat{u})\big)\big(\hat{f}-\hat{u}\big)\,\mathrm{d}x$$
$$=\int_\Omega \big(\hat{z},DT_r\big(T_k(u)-T_k(\hat{u})\big)\big)-\int_{\partial\Omega}[\hat{z},\nu]T_r\big(T_k(u)-T_k(\hat{u})\big)\,\mathrm{d}\mathcal{H}^{N-1}.$$

Adding both equalities and having in mind that $T_k(-r)=-T_k(r)$ we have

$$\int_\Omega T_r\big(T_k(u)-T_k(\hat{u})\big)\big(\hat{f}-\hat{u}-f+u\big)\,\mathrm{d}x$$
$$=\int_\Omega\big(\hat{z}-z,DT_r\big(T_k(u)-T_k(\hat{u})\big)\big)-\int_{\partial\Omega}\big([\hat{z},\nu]-[z,\nu]\big)T_r\big(T_k(u)-T_k(\hat{u})\big)\,\mathrm{d}\mathcal{H}^{N-1}. \qquad (50)$$

As consequence of Lemma 2, $\theta(z,DT_k(u),x)=1$ $|DT_k(u)|$-a.e., hence, using Corollary 1.6 in [9], we obtain that

$$\int_B\big(z,DT_k(u)\big)=\int_B\theta\big(z,DT_k(u),x\big)\big|DT_k(u)\big|=\int_B\big|DT_k(u)\big|,$$
$$\left|\int_B\big(\hat{z},DT_k(u)\big)\right|\leqslant\int_B\big|DT_k(u)\big|$$

for any Borel set $B\subseteq\Omega$. Similarly,

$$\int_B\big(\hat{z},DT_k(\hat{u})\big)=\int_B\big|DT_k(\hat{u})\big|,\qquad\left|\int_B\big(z,DT_k(\hat{u})\big)\right|\leqslant\int_B\big|DT_k(\hat{u})\big|$$

for any Borel set $B\subseteq\Omega$. Thus, it follows that

$$\int_B\big(z-\hat{z},D\big(T_k(u)-T_k(\hat{u})\big)\big)\geqslant 0$$

for any Borel set $B\subseteq\Omega$. This implies that

$$\theta\big(z-\hat{z},D\big(T_k(u)-T_k(\hat{u})\big),x\big)\geqslant 0\quad\big|D\big(T_k(u)-T_k(\hat{u})\big)\big|\text{-a.e.}$$

Since, according to Proposition 2.8 in [9], we have that

$$\theta\big(z-\hat{z},DT_r\big(T_k(u)-T_k(\hat{u})\big),x\big)=\theta\big(z-\hat{z},D\big(T_k(u)-T_k(\hat{u})\big),x\big)$$

a.e. with respect to the measures $|D(T_k(u)-T_k(\hat{u}))|$ and $|DT_r(T_k(u)-T_k(\hat{u}))|$, we conclude that

$$\theta\big(z-\hat{z},DT_r\big(T_k(u)-T_k(\hat{u})\big),x\big)\geqslant 0\quad\big|DT_r\big(T_k(u)-T_k(\hat{u})\big)\big|\text{-a.e.} \qquad (51)$$

From (50) and (51), it follows that

$$\begin{aligned}
&\int_\Omega T_r\big(T_k(u)-T_k(\hat{u})\big)\big(\hat{f}-\hat{u}-f+u\big)\,\mathrm{d}x \\
&\quad = -\int_\Omega \theta\big(\hat{z}-z, DT_r\big(T_k(u)-T_k(\hat{u})\big),x\big)\big|DT_r\big(T_k(u)-T_k(\hat{u})\big)\big| \\
&\qquad -\int_{\partial\Omega}\big([\hat{z},\nu]-[z,\nu]\big)T_r\big(T_k(u)-T_k(\hat{u})\big)\,\mathrm{d}\mathcal{H}^{N-1} \\
&\quad \leqslant -\int_{\partial\Omega}\big([\hat{z},\nu]-[z,\nu]\big)T_r\big(T_k(u)-T_k(\hat{u})\big)\,\mathrm{d}\mathcal{H}^{N-1}.
\end{aligned}$$

Now, by Remark 2, we have

$$-[z,\nu]\big(T_k(\hat{u})-T_k(u)\big) \leqslant j\big(T_k(\hat{u})\big)-j\big(T_k(u)\big) \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega \tag{52}$$

and

$$-[\hat{z},\nu]\big(T_k(u)-T_k(\hat{u})\big) \leqslant j\big(T_k(u)\big)-j\big(T_k(\hat{u})\big) \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega. \tag{53}$$

From (52) and (53) we get

$$\big([\hat{z},\nu]-[z,\nu]\big)\big(T_k(u)-T_k(\hat{u})\big) \geqslant 0 \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega,\ \forall k>0.$$

Thus, we obtain that

$$\frac{1}{r}\int_\Omega T_r\big(T_k(u)-T_k(\hat{u})\big)(u-\hat{u})\,\mathrm{d}x \leqslant \frac{1}{r}\int_\Omega T_r\big(T_k(u)-T_k(\hat{u})\big)\big(f-\hat{f}\big)\,\mathrm{d}x \leqslant \int_\Omega \big|f-\hat{f}\big|\,\mathrm{d}x.$$

Then, letting $k\to+\infty$, we get

$$\frac{1}{r}\int_\Omega T_r(u-\hat{u})(u-\hat{u})\,\mathrm{d}x \leqslant \int_\Omega \big|f-\hat{f}\big|\,\mathrm{d}x.$$

Finally, letting $r\to 0^+$, we get (49) and the proof of the accretivity of the operator $\mathcal{B}_\beta$ concludes.

In view of Theorem 2, to prove that $\mathcal{B}_\beta$ satisfies the range condition, it is enough to prove that $\overline{\mathcal{A}_\beta}^{L^1(\Omega)} \subset \mathcal{B}_\beta$. Let $(u_n,v_n)\in\mathcal{A}_\beta$, such that $(u_n,v_n)\to(u,v)$ in $L^1(\Omega)\times L^1(\Omega)$. Let us see that $(u,v)\in\mathcal{B}_\beta$. Since $(u_n,v_n)\in\mathcal{A}_\beta$, by Lemma 3, there exists $z_n\in X(\Omega)$, $\|z_n\|_\infty\leqslant 1$ with $v_n=-\operatorname{div}(z_n)$ in $\mathcal{D}'(\Omega)$ such that

$$\begin{aligned}
&\int_\Omega \big(p(w)-p(u_n)\big)v_n\,\mathrm{d}x \\
&\quad\leqslant \int_\Omega \big(z_n, Dp(w)\big) - \int_\Omega \big|Dp(u_n)\big| + \int_{\partial\Omega} j(w)\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\partial\Omega} j\big(p(u_n)-p(0)\big)\,\mathrm{d}\mathcal{H}^{N-1}
\end{aligned} \tag{54}$$

for all $w \in BV(\Omega) \cap L^\infty(\Omega)$ and $p \in \mathcal{P}$. Then, taking $w = 0$ and $p = T_k$ in (54), we obtain that

$$\int_\Omega |DT_k(u_n)| + \int_{\partial\Omega} j\big(T_k(u_n)\big)\,\mathrm{d}\mathcal{H}^{N-1} \leqslant \int_\Omega T_k(u_n)v_n\,\mathrm{d}x \quad \forall n \in \mathbb{N} \text{ and } k > 0. \tag{55}$$

From (55), it follows that $u \in TBV(\Omega)$.

Since $\|z_n\|_\infty \leqslant 1$ we may assume that $z_n \rightharpoonup z$ in the weak* topology of $L^\infty(\Omega, \mathbb{R}^N)$ with $\|z\|_\infty \leqslant 1$. Moreover, since $v_n \to v$ in $L^1(\Omega)$, we have $v = -\mathrm{div}(z)$ in $\mathcal{D}'(\Omega)$, and

$$\lim_{n\to\infty} \int_\Omega (z_n, Dw) = \int_\Omega (z, Dw).$$

Then, letting $n \to +\infty$ in (54), and having in mind the lower semicontinuity of the operator Φ_β, we get

$$\begin{aligned}
&\int_\Omega |Dp(u)| + \int_{\partial\Omega} j\big(p(u) - p(0)\big)\,\mathrm{d}\mathcal{H}^{N-1} \leqslant \liminf_{n\to\infty} \int_\Omega |Dp(u_n)| + \int_{\partial\Omega} j\big(p(u_n) - p(0)\big)\,\mathrm{d}\mathcal{H}^{N-1} \\
&\quad \leqslant \limsup_{n\to\infty} \int_\Omega |Dp(u_n)| + \int_{\partial\Omega} j\big(p(u_n) - p(0)\big)\,\mathrm{d}\mathcal{H}^{N-1} \\
&\quad \leqslant \lim_{n\to\infty} \int_\Omega \big(p(u_n) - p(w)\big)v_n\,\mathrm{d}x + \int_\Omega \big(z_n, Dp(w)\big) + \int_{\partial\Omega} j(w)\,\mathrm{d}\mathcal{H}^{N-1} \\
&\quad = \int_\Omega \big(p(u) - p(w)\big)v\,\mathrm{d}x + \int_\Omega \big(z, Dp(w)\big) + \int_{\partial\Omega} j(w)\,\mathrm{d}\mathcal{H}^{N-1}.
\end{aligned} \tag{56}$$

Observe that from (56), we obtain that $(u, v) \in \mathcal{B}_\beta$. □

4.1. Proof of Theorem 4. Existence

Let $u_0 \in L^1(\Omega)$ and $\{S(t)\}_{t\geqslant 0}$ the contraction semigroup in $L^1(\Omega)$ generated by $\mathcal{B}_\beta$. We shall prove that $u(t) := S(t)u_0$ is an entropy solution of problem (1). We divide the proof in different steps.

Step 1. Since $\mathcal{D}(\mathcal{B}_\beta) \cap L^\infty(\Omega)$ is dense in $L^1(\Omega)$, given $u_0 \in L^1(\Omega)$ there exists a sequence $u_{0,n} \in \mathcal{D}(\mathcal{B}_\beta) \cap L^\infty(\Omega)$ such that $u_{0,n} \to u_0$ in $L^1(\Omega)$. Then, if $u_n(t) := S(t)u_{0,n}$, we have that $u_n \to u$ in $C([0,T]; L^1(\Omega))$ for every $T > 0$. As a consequence of Theorem 1, $u_n(t), u_n'(t) \in L^2(\Omega)$, $p(u_n(t)) \in BV(\Omega)$ for all $p \in \mathcal{P}$ and there exist $z_n(t) \in X(\Omega)$, $\|z_n(t)\|_\infty \leqslant 1$ and $u_n'(t) = \mathrm{div}(z_n(t))$ in $\mathcal{D}'(\Omega)$ a.e. $t \in [0, +\infty[$, satisfying

$$\begin{aligned}
&-\int_\Omega \big(p(w) - p\big(u_n(t)\big)\big)u_n'(t) \\
&\quad \leqslant \int_\Omega \big(z_n(t), Dp(w)\big) - \int_\Omega \big|Dp\big(u_n(t)\big)\big| + \int_{\partial\Omega} j(w) - \int_{\partial\Omega} j\big(p\big(u_n(t)\big) - p(0)\big)
\end{aligned} \tag{57}$$

for every $w \in BV(\Omega) \cap L^\infty(\Omega)$ and $p \in \mathcal{P}$. Moreover

$$\int_\Omega \big(z_n(t), Dp\big(u_n(t)\big)\big) = \int_\Omega \big|Dp\big(u_n(t)\big)\big| \quad \forall p \in \mathcal{P} \tag{58}$$

and

$$-\big[z_n(t), \nu\big]\big(p(w) - p\big(u_n(t)\big)\big) \leqslant j(w) - j\big(p\big(u_n(t)\big) - p(0)\big) \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega, \tag{59}$$

for all $w \in BV(\Omega) \cap L^\infty(\Omega)$ and for all $p \in \mathcal{P}$.

Since $\|[z_n(t),\nu]\|_\infty \leqslant \|z_n(t)\|_\infty \leqslant 1$, we can suppose (up to extraction of a subsequence, if necessary) that

$$[z_n(\cdot),\nu] \to \rho \qquad \sigma\big(L^\infty(S_T), L^1(S_T)\big).$$

Step 2. *Convergence of the derivatives and identification of the limit.* Since the map $t \mapsto u_n'(t)$ is strongly measurable from $[0,T]$ into $L^2(\Omega)$, and by (38),

$$\|u_n'(t)\|_{BV(\Omega)_2^*} \leqslant \|u_n'(t)\|_{L^2(\Omega)},$$

it follows that this map is strongly measurable from $[0,T]$ into $BV(\Omega)_2^*$. Moreover, for every $w \in BV(\Omega)_2$, by Green's formula we have

$$\int_\Omega u_n'(t)w = \int_\Omega \operatorname{div}\big(z_n(t)\big)w = -\int_\Omega \big(z_n(t), Dw\big) + \int_{\partial\Omega} \big[z_n(t),\nu\big]w.$$

Hence

$$\left|\int_\Omega u_n'(t)w\right| \leqslant \int_\Omega |Dw| + \int_{\partial\Omega} |w| \leqslant M\|w\|_{BV(\Omega)_2} \quad \forall n \in \mathbb{N}.$$

Thus,

$$\|u_n'(t)\|_{BV(\Omega)_2^*} \leqslant M \quad \forall n \in \mathbb{N} \text{ and } t \in [0,T].$$

Consequently, $\{u_n'\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^\infty(0,T;BV(\Omega)_2^*)$. Since $L^\infty(0,T;BV(\Omega)_2^*)$ is a vector subspace of the dual space $(L^1(0,T;BV(\Omega)_2))^*$, we can find a net $\{u_\alpha'\}$ such that

$$u_\alpha' \to \xi \in \big(L^1\big(0,T;BV(\Omega)_2\big)\big)^* \quad \text{weakly}^*. \tag{60}$$

Since $\|z_n(t)\|_\infty \leqslant 1$ for all $n \in \mathbb{N}$ and a.e. $t \in [0,T]$, we can suppose that

$$z_n \to z \in L^\infty\big(Q_T,\mathbb{R}^N\big) \quad \text{weakly}^*. \tag{61}$$

Given $\eta \in \mathcal{D}(Q_T)$, since $\eta \in L^1(0,T;BV(\Omega)_2)$, we have

$$\begin{aligned}
\langle \xi,\eta\rangle &= \lim_\alpha \langle u_\alpha',\eta\rangle = \lim_\alpha \int_0^T \big\langle u_\alpha'(t),\eta(t)\big\rangle\,\mathrm{d}t \\
&= \lim_\alpha \int_0^T\!\!\int_\Omega u_\alpha'(t)\eta(t)\,\mathrm{d}x\,\mathrm{d}t = \lim_\alpha \int_0^T\!\!\int_\Omega \operatorname{div}\big(z_\alpha(t)\big)\eta(t)\,\mathrm{d}x\,\mathrm{d}t \\
&= -\lim_\alpha \int_0^T\!\!\int_\Omega z_\alpha(t)\cdot\nabla\eta(t)\,\mathrm{d}x\,\mathrm{d}t = -\int_{Q_T} z\cdot\nabla\eta\,\mathrm{d}x\,\mathrm{d}t = \big\langle \operatorname{div}_x(z),\eta\big\rangle.
\end{aligned}$$

Hence,

$$\xi = \operatorname{div}_x(z) \quad \text{in } \mathcal{D}'(Q_T). \tag{62}$$

On the other hand, if we take $\eta(t,x) = \phi(t)\psi(x)$ with $\phi \in \mathcal{D}(]0,T[)$ and $\psi \in \mathcal{D}(\Omega)$, the same calculation as above shows that

$$\xi(t) = \mathrm{div}_x\bigl(z(t)\bigr) \quad \text{in } \mathcal{D}'(\Omega) \text{ a.e. } t \in [0,T]. \tag{63}$$

Consequently, $(z(t),\xi(t)) \in Z(\Omega)$ for almost all $t \in [0,T]$, therefore we can consider $[z(t),\nu]$.

Lemma 4. *ξ is the time derivative of u in the sense of the Definition* 3.

Proof. Let $\Psi \in L^1(0,T,BV(\Omega))$ be the weak derivative of $\Theta \in L^1_w(0,T,BV(\Omega)) \cap L^\infty(Q_T)$, i.e., $\Psi(t) = \int_0^t \Theta(s)\,\mathrm{d}s$, the integral being taken as a Pettis integral. By (60) we have that

$$\int_0^T \bigl\langle \xi(t),\Psi(t)\bigr\rangle\,\mathrm{d}t = \lim_\alpha \int_0^T \bigl\langle u'_\alpha(t),\Psi(t)\bigr\rangle\,\mathrm{d}t.$$

Now,

$$\begin{aligned}\int_0^T \bigl\langle u'_\alpha(t),\Psi(t)\bigr\rangle\,\mathrm{d}t &= \lim_h \int_0^T\!\!\int_\Omega \Psi(t)\frac{u_\alpha(t+h)-u_\alpha(t)}{h}\,\mathrm{d}x\,\mathrm{d}t\\ &= \lim_h \int_0^T\!\!\int_\Omega \frac{\Psi(t-h)-\Psi(t)}{h}u_\alpha(t)\,\mathrm{d}x\,\mathrm{d}t\\ &= -\lim_h \int_0^T\!\!\int_\Omega \frac{1}{h}\int_{t-h}^t \Theta(s)\,\mathrm{d}s u_\alpha(t)\,\mathrm{d}x\,\mathrm{d}t = -\int_0^T\!\!\int_\Omega \Theta(t,x)u_\alpha(t,x)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Passing to the limit in α in the above expression, we obtain

$$\int_0^T \bigl\langle \xi(t),\Psi(t)\bigr\rangle\,\mathrm{d}t = -\int_0^T\!\!\int_\Omega \Theta(t,x)u(t,x)\,\mathrm{d}x\,\mathrm{d}s. \quad \square \tag{64}$$

Let see now that

$$\rho(t) = \bigl[z(t),\nu\bigr] \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega, \text{ a.e. } t \in [0,T]. \tag{65}$$

In fact: If $w \in BV(\Omega) \cap L^\infty(\Omega)$, and $v \in R(\Omega)$ such that $v|_{\partial\Omega} = w|_{\partial\Omega}$, we have that

$$\int_0^t \bigl\langle z_\alpha(s),w\bigr\rangle_{\partial\Omega}\,\mathrm{d}s = \int_0^t \bigl\langle \mathrm{div}\bigl(z_\alpha(s)\bigr),v\bigr\rangle\,\mathrm{d}s + \int_0^t\!\!\int_\Omega z_\alpha(s)\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}s.$$

Hence

$$\begin{aligned}\lim_\alpha \int_0^t \bigl\langle z_\alpha(s),w\bigr\rangle_{\partial\Omega}\,\mathrm{d}s &= \int_0^t \bigl\langle \xi(s),v\bigr\rangle\,\mathrm{d}s + \int_0^t\!\!\int_\Omega z(s)\cdot\nabla v\,\mathrm{d}x\,\mathrm{d}s\\ &= \int_0^t \bigl\langle z(s),w\bigr\rangle_{\partial\Omega} = \int_0^t\!\!\int_{\partial\Omega} \bigl[z(s),\nu\bigr]w\,\mathrm{d}\mathcal{H}^{N-1}.\end{aligned} \tag{66}$$

On the other hand, since $z_\alpha(s) \in X(\Omega)$, if we apply Green's formula we have that

$$\int_0^t \langle \operatorname{div}(z_\alpha(s)), v\rangle \, \mathrm{d}s = -\int_0^t\int_\Omega z_\alpha(s)\cdot \nabla v \, \mathrm{d}x \, \mathrm{d}s + \int_0^t\int_{\partial\Omega} [z_\alpha(s),\nu] w \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}s.$$

Consequently,

$$\int_0^t \langle z_\alpha(s), w\rangle_{\partial\Omega} \, \mathrm{d}s = \int_0^t\int_{\partial\Omega} [z_\alpha(s),\nu] w \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}s.$$

From here, taking limits in α, we get

$$\int_0^t\int_{\partial\Omega} \rho(s) w \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}s = \int_0^t\int_{\partial\Omega} [z(s),\nu] w \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}s \tag{67}$$

for all $w \in BV(\Omega) \cap L^\infty(\Omega)$, and $t \in [0,T]$. Now, if $w \in L^1(\partial\Omega)$, we take $w_k \in BV(\Omega) \cap L^\infty(\Omega)$ such that $w_k|_{\partial\Omega} = T_k(w)$. By (67), we have

$$\int_0^t\int_{\partial\Omega} \rho(s) w_k \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}s = \int_0^t\int_{\partial\Omega} [z(s),\nu] w_k \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}s.$$

Letting $k \to \infty$, it follows that

$$\int_0^t\int_{\partial\Omega} \rho(s) w \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}s = \int_0^t\int_{\partial\Omega} [z(s),\nu] w \, \mathrm{d}\mathcal{H}^{N-1} \, \mathrm{d}s \quad \forall w \in L^1(\partial\Omega) \text{ and } t \in [0,T],$$

and consequently (65) holds.

Step 3. Next, we prove that $\xi = \operatorname{div}(z)$ in $(L^1(0,T,BV(\Omega)_2))^*$ in the sense of the Definition 4. To do that let us first observe that (z, Dw), defined by (39), is a Radon measure in Q_T for all $w \in L^1_w(0,T,BV(\Omega)) \cap L^\infty(Q_T)$. Let $\phi \in \mathcal{D}(Q_T)$, then

$$\begin{aligned}\langle (z,Dw),\phi\rangle = &-\int_0^T \langle \xi(t) - u_\alpha'(t), w(t)\phi(t)\rangle \, \mathrm{d}t - \int_{Q_T} w(z - z_\alpha)\cdot\nabla_x\phi \, \mathrm{d}x \, \mathrm{d}t \\ &+ \int_0^T \langle (z_\alpha(t), Dw(t)), \phi(t)\rangle \, \mathrm{d}t.\end{aligned}$$

Then by (60), taking limits in α, we get

$$\langle (z,Dw),\phi\rangle = \lim_\alpha \int_0^T \langle (z_\alpha(t), Dw(t)), \phi(t)\rangle \, \mathrm{d}t. \tag{68}$$

Therefore

$$\left|\langle (z,Dw),\phi\rangle\right| \leqslant \|\phi\|_\infty \int_0^T\int_\Omega \left|Dw(t)\right| \mathrm{d}t,$$

from where it follows that (z, Dw) is a Radon measure in Q_T. Moreover, from (68), applying Green's formula we obtain that

$$\begin{aligned}\int_{Q_T} (z, Dw) &= \lim_{\alpha} \int_0^T \big(z_\alpha(t), Dw(t)\big)\,\mathrm{d}t \\ &= \lim_{\alpha}\Big(-\int_0^T\!\!\int_\Omega \operatorname{div}\big(z_\alpha(t)\big)w(t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_{\partial\Omega} \big[z_\alpha(t),\nu\big]w(t)\,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t\Big) \\ &= -\int_0^T \big\langle \xi(t), w(t)\big\rangle\,\mathrm{d}t + \int_0^T\!\!\int_{\partial\Omega} \big[z(t),\nu\big]w(t)\,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t.\end{aligned}$$

Consequently

$$\int_{Q_T} (z, Dw) + \int_0^T \big\langle \xi(t), w(t)\big\rangle\,\mathrm{d}t = \int_0^T\!\!\int_{\partial\Omega} \big[z(t),\nu\big]w(t)\,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t. \tag{69}$$

Step 4. *The boundary condition.* Taking $w = 0$ in (57) we get

$$\int_\Omega \big|Dp\big(u_n(t)\big)\big| + \int_{\partial\Omega} j\big(p\big(u_n(t)\big) - p(0)\big)\,\mathrm{d}\mathcal{H}^{N-1} \leqslant -\int_\Omega \big(p\big(u_n(t)\big) - p(0)\big)u_n'(t)\,\mathrm{d}x.$$

Then integrating from 0 to T, it follows that

$$\begin{aligned}&\int_0^T\!\!\int_\Omega \big|Dp\big(u_n(t)\big)\big|\,\mathrm{d}t + \int_0^T\!\!\int_{\partial\Omega} j\big(p\big(u_n(t)\big) - p(0)\big)\,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t \\ &\quad\leqslant -\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega J_p\big(u_n(t)\big)\,\mathrm{d}x + \int_0^T\!\!\int_\Omega p(0)u_n'(t)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad= \int_\Omega \big(J_p(u_{0,n}) - J_p(u_n(T)\big)\,\mathrm{d}x + \int_\Omega p(0)\big(u_n(T) - u_{0,n}\big)\,\mathrm{d}x \leqslant M_p.\end{aligned}$$

Since the functional $\Psi_p : L^1(\Omega) \to]-\infty, +\infty]$, defined by

$$\Psi_p(w) := \begin{cases} \displaystyle\int_\Omega \big|Dp(w)\big| + \int_{\partial\Omega} j\big(p(w) - p(0)\big)\,\mathrm{d}\mathcal{H}^{N-1} & \text{if } w \in BV(\Omega), \\ +\infty & \text{if } w \in L^1(\Omega) \setminus BV(\Omega)\end{cases}$$

is lower semicontinuous in $L^1(\Omega)$, we have

$$\Psi_p\big(u(t)\big) \leqslant \liminf_{n\to\infty} \Psi_p\big(u_n(t)\big) = \liminf_{n\to\infty}\Big(\int_\Omega \big|Dp\big(u_n(t)\big)\big| + \int_{\partial\Omega} j\big(p\big(u_n(t)\big) - p(0)\big)\,\mathrm{d}\mathcal{H}^{N-1}\Big). \tag{70}$$

On the other hand, by Fatou's Lemma, it follows that

$$\begin{aligned}&\int_0^T \liminf_{n\to\infty}\Big(\int_\Omega \big|Dp\big(u_n(t)\big)\big| + \int_{\partial\Omega} j\big(p\big(u_n(t)\big) - p(0)\big)\,\mathrm{d}\mathcal{H}^{N-1}\Big)\,\mathrm{d}t \\ &\quad\leqslant \liminf_{n\to\infty}\int_0^T \Big(\int_\Omega \big|Dp\big(u_n(t)\big)\big| + \int_{\partial\Omega} j\big(p\big(u_n(t)\big) - p(0)\big)\,\mathrm{d}\mathcal{H}^{N-1}\Big)\,\mathrm{d}t \leqslant M_p.\end{aligned} \tag{71}$$

As a consequence of (70) and (71), we obtain that $p(u(t)) \in BV(\Omega)$ for almost all $t \in [0, T]$.

By Lemma 5 in [4], it follows that the map $t \mapsto p(u(t))$ from $[0,T]$ into $BV(\Omega)$ is weakly measurable, and also, if $0 \leqslant \eta \in \mathcal{D}(]0,T[)$, the map $t \mapsto p(u(t))\eta(t)$, from $[0,T]$ into $BV(\Omega)$ is weakly measurable.

Using the same technique that in the proofs of Lemmas 4 and 5 of [3] we obtain the following two results.

Lemma 5. *For any $\tau > 0$, we define the function ψ^τ, as the Dunford integral* (*see* [20])

$$\psi^\tau(t) := \frac{1}{\tau}\int_{t-\tau}^{t} \eta(s)p\big(u(s)\big)\,\mathrm{d}s \in BV(\Omega)^{**},$$

that is,

$$\big\langle \psi^\tau(t), w\big\rangle = \frac{1}{\tau}\int_{t-\tau}^{t} \big\langle \eta(s)p\big(u(s)\big), w\big\rangle\,\mathrm{d}s$$

for any $w \in BV(\Omega)^*$. Then $\psi^\tau \in C([0,T];BV(\Omega))$. Moreover, $\psi^\tau(t) \in L^2(\Omega)$, and, thus, $\psi^\tau(t) \in BV(\Omega)_2$.

Lemma 6. *For $\tau > 0$ small enough, we have*

$$\int_0^T \big\langle \psi^\tau(t), \xi(t)\big\rangle\,\mathrm{d}t \leqslant -\int_0^T\!\!\int_\Omega \frac{\eta(t-\tau)-\eta(t)}{-\tau} J_p\big(u(t)\big)\,\mathrm{d}x\,\mathrm{d}t. \tag{72}$$

Now, we can conclude the proof of Step 4. As a consequence of (72), using Green's formula, we have

$$\begin{aligned}
&\int_0^T\!\!\int_\Omega \frac{\eta(t-\tau)-\eta(t)}{-\tau} J_p\big(u(t)\big)\,\mathrm{d}x\,\mathrm{d}t \leqslant -\int_0^T \big\langle \psi^\tau(t), \xi(t)\big\rangle\,\mathrm{d}t = -\lim_\alpha \int_0^T \big\langle \psi^\tau(t), u'_\alpha(t)\big\rangle\,\mathrm{d}t \\
&\quad = -\lim_\alpha \int_0^T \left(\frac{1}{\tau}\int_{t-\tau}^{t} \eta(s)\big\langle p\big(u(s)\big), u'_\alpha(t)\big\rangle\,\mathrm{d}s\right)\mathrm{d}t \\
&\quad = -\lim_\alpha \int_0^T \left(\frac{1}{\tau}\int_{t-\tau}^{t} \eta(s)\left(\int_\Omega p\big(u(s)\big)\,\mathrm{div}\big(z_\alpha(t)\big)\right)\mathrm{d}s\right)\mathrm{d}t \\
&\quad = \lim_\alpha \left[\int_0^T \left(\frac{1}{\tau}\int_{t-\tau}^{t} \eta(s)\left(\int_\Omega \big(z_\alpha(t), Dp\big(u(s)\big)\big)\,\mathrm{d}s\right)\mathrm{d}t\right.\right. \\
&\qquad \left.- \int_0^T \left(\frac{1}{\tau}\int_{t-\tau}^{t} \eta(s)\left(\int_{\partial\Omega} \big[z_\alpha(t), \nu\big] p\big(u(s)\big)\right)\mathrm{d}s\right)\mathrm{d}t\right] \\
&\quad \leqslant \int_0^T \left(\frac{1}{\tau}\int_{t-\tau}^{t} \eta(s)\int_\Omega \big|Dp\big(u(s)\big)\big|\,\mathrm{d}s\right)\mathrm{d}t - \int_0^T \left(\frac{1}{\tau}\int_{t-\tau}^{t} \eta(s)\left(\int_{\partial\Omega} \rho(t)p\big(u(s)\big)\right)\mathrm{d}s\right)\mathrm{d}t.
\end{aligned}$$

Then, taking limit as $\tau \to 0^+$, we get

$$\int_0^T\!\!\int_\Omega \eta'(t)J_p\big(u(t)\big)\,\mathrm{d}x\,\mathrm{d}t \leqslant \int_0^T \eta(t)\int_\Omega \big|Dp\big(u(t)\big)\big|\,\mathrm{d}t - \int_0^T \eta(t)\int_{\partial\Omega} \rho(t)p\big(u(t)\big)\,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t.$$

Now, since this is true for all $0 \leqslant \eta \in \mathcal{D}(]0,T[)$, it follows that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega J_p\big(u(t)\big)\,\mathrm{d}x \leqslant \int_\Omega \big|Dp\big(u(t)\big)\big|\,\mathrm{d}t - \int_{\partial\Omega} \rho(t)p\big(u(t)\big)\,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t,$$

and consequently

$$\int_\Omega \big(J_p(u_0) - J_p\big(u(T)\big)\,\mathrm{d}x \leqslant \int_0^T\!\!\int_\Omega \big|Dp\big(u(t)\big)\big|\,\mathrm{d}t - \int_0^T\!\!\int_{\partial\Omega} \rho(t)p\big(u(t)\big)\,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t. \tag{73}$$

Finally, using (69), (70) and (73), if $w \in L^1(0,T;BV(\Omega))$, we obtain

$$\begin{aligned}
&\int_0^T\!\!\int_\Omega \big|Dp\big(u(t)\big)\big|\,\mathrm{d}t + \int_0^T\!\!\int_{\partial\Omega} j\big(p\big(u(t)\big) - p(0)\big)\,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t \\
&\quad\leqslant \liminf_{n\to\infty} \int_0^T\!\!\int_\Omega \big|Dp\big(u(t)\big)\big|\,\mathrm{d}t + \int_0^T\!\!\int_{\partial\Omega} j\big(p\big(u_n(t)\big) - p(0)\big)\,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t \\
&\quad\leqslant \liminf_{n\to\infty} \int_0^T \left(\int_\Omega \big(p\big(w(t)\big) - p\big(u_n(t)\big)\big)u_n'(t)\,\mathrm{d}x + \int_\Omega \big(z_n(t), Dp\big(w(t)\big)\big) + \int_{\partial\Omega} j\big(w(t)\big)\right)\mathrm{d}t \\
&\quad= \int_\Omega J_p(u_0) - J_p\big(u(T)\big)\ \mathrm{d}x + \int_0^T \left(\big\langle \xi(t), p\big(w(t)\big)\big\rangle + \int_\Omega \big(z(t), Dp\big(w(t)\big)\big) + \int_{\partial\Omega} j\big(w(t)\big)\right)\mathrm{d}t \\
&\quad\leqslant \int_0^T\!\!\int_\Omega \big|Dp\big(u(t)\big)\big|\,\mathrm{d}t - \int_0^T\!\!\int_{\partial\Omega} \big[z(t),\nu\big]\big(p\big(u(t)\big) - p\big(w(t)\big)\big) + j\big(w(t)\big)\,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t.
\end{aligned}$$

Then, it follows

$$\begin{aligned}
&-\int_0^T\!\!\int_{\partial\Omega} \big[z(t),\nu\big]\big(p\big(w(t)\big) - p\big(u(t)\big)\big)\,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t \\
&\quad\leqslant \int_0^T\!\!\int_{\partial\Omega} \big(j\big(w(t)\big) - j\big(p\big(u(t)\big) - p(0)\big)\big)\,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t.
\end{aligned}$$

Step 5. *Conclusion.* Finally, we are going to prove that u verifies (40).

Let $\eta \in C^\infty(\overline{Q_T})$, with $\eta \geqslant 0$, $\eta(t,x) = \phi(t)\psi(x)$, being $\phi \in \mathcal{D}(]0,T[)$, $\psi \in C^\infty(\overline{\Omega})$, and $p \in \mathcal{P}$. Let $J_p(r) := \int_0^r p(s)\,\mathrm{d}s$. Since $u_n'(t) = \operatorname{div}(z_n(t))$, multiplying by $p(u_n(t))\eta(t)$, integrating and having in mind (59), we obtain that

$$\begin{aligned}
&\int_0^T\!\!\int_\Omega \frac{\mathrm{d}}{\mathrm{d}t} J_p\big(u_n(t)\big)\eta(t) = \int_0^T\!\!\int_\Omega p\big(u_n(t)\big)u_n'(t)\eta(t) = \int_0^T\!\!\int_\Omega \operatorname{div}\big(z_n(t)\big)p\big(u_n(t)\big)\eta(t) \\
&\quad= -\int_0^T\!\!\int_\Omega \big(z_n(t), D\big(p\big(u_n(t)\big)\eta(t)\big)\big) + \int_0^T\!\!\int_{\partial\Omega} \big[z_n(t),\nu\big]p\big(u_n(t)\big)\eta(t) \\
&\quad= -\int_0^T\!\!\int_\Omega \eta(t)\big|Dp\big(u(t)\big)\big| - \int_0^T\!\!\int_\Omega z_n(t)\cdot\nabla\eta(t)p\big(u_n(t)\big) + \int_0^T\!\!\int_{\partial\Omega} \big[z_n(t),\nu\big]p\big(u_n(t)\big)\eta(t) \\
&\quad\leqslant -\int_0^T\!\!\int_\Omega \eta(t)\big|Dp\big(u_n(t)\big)\big| - \int_0^T\!\!\int_\Omega z_n(t)\cdot\nabla\eta(t)p\big(u_n(t)\big)
\end{aligned}$$

$$- \int_0^T \int_{\partial\Omega} j\big(p(u_n(t)) - p(0)\big)\eta(t) + \int_0^T \int_{\partial\Omega} [z_n(t), \nu] p(0)\eta(t).$$

Hence, having in mind that $\eta(0) = \eta(T) = 0$, we get

$$\begin{aligned}
&\int_0^T \int_\Omega \eta(t)\big|Dp(u(t))\big| + \int_0^T \int_{\partial\Omega} j\big(p(u_n(t)) - p(0)\big)\eta(t) \\
&\quad \leqslant - \int_0^T \int_\Omega z_n(t) \cdot \nabla\eta(t) p(u_n(t)) + \int_0^T \int_{\partial\Omega} [z_n(t), \nu] p(0)\eta(t) - \int_0^T \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t} J_p(u_n(t))\eta(t) \\
&\quad = - \int_0^T \int_\Omega z_n(t) \cdot \nabla\eta(t) p(u_n(t)) + \int_0^T \int_{\partial\Omega} [z_n(t), \nu] p(0)\eta(t) \\
&\qquad - \int_0^T \int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}\big(J_p(u_n(t))\eta(t)\big) + \int_0^T \int_\Omega J_p(u_n(t))\eta_t \\
&\quad = - \int_0^T \int_\Omega z_n(t) \cdot \nabla\eta(t) p(u_n(t)) + \int_0^T \int_{\partial\Omega} [z_n(t), \nu] p(0)\eta(t) + \int_0^T \int_\Omega J_p(u_n(t))\eta_t.
\end{aligned}$$

Letting $n \to \infty$, it follows that

$$\begin{aligned}
&\int_0^T \int_\Omega \eta(t)\big|Dp(u(t))\big| + \int_0^T \int_{\partial\Omega} j\big(p(u(t)) - p(0)\big)\eta(t) \\
&\quad \leqslant \liminf_{n\to\infty} \left[\int_0^T \int_\Omega \eta(t)\big|Dp(u(t))\big| + \int_0^T \int_{\partial\Omega} j\big(p(u_n(t)) - p(0)\big)\eta(t) \right] \\
&\quad = \liminf_{n\to\infty} \left[- \int_0^T \int_\Omega z_n(t) \cdot \nabla\eta(t) p(u_n(t)) + \int_0^T \int_{\partial\Omega} [z_n(t), \nu] p(0)\eta(t) + \int_0^T \int_\Omega J_p(u_n(t))\eta_t \right] \\
&\quad = - \int_0^T \int_\Omega z(t) \cdot \nabla\eta(t) p(u(t)) + \int_0^T \int_{\partial\Omega} [z(t), \nu] p(0)\eta(t) + \int_0^T \int_\Omega J_p(u(t))\eta_t.
\end{aligned}$$

We have then

$$\begin{aligned}
&- \int_0^T \int_\Omega J_p(u(t))\eta_t + \int_0^T \int_\Omega \eta(t)\big|Dp(u(t))\big| + \int_0^T \int_{\partial\Omega} j\big(p(u(t)) - p(0)\big)\eta(t) \\
&\quad + \int_0^T \int_\Omega z(t) \cdot \nabla\eta(t) p(u(t)) \leqslant \int_0^T \int_{\partial\Omega} [z(t), \nu] p(0)\eta(t)
\end{aligned} \tag{74}$$

and the proof of the existence is finished.

4.2. Proof of Theorem 4. Uniqueness

To prove uniqueness we shall show that the entropy solutions and semigroup solutions coincide. As a consequence of the semigroup theory, (41) will be then satisfied. We use the same technique that the one introduced in [3] to prove uniqueness for the Dirichlet problem. This technique is inspired by a method introduced by Kruzhkov [24] to prove L^1-contraction for entropy solutions for scalar conservation laws: the doubling of variables (see also [17] or [23]).

Let $u(t)$ be an entropy solution with initial datum $u_0 \in L^1(\Omega)$ and $\bar{u}(t) = S(t)\bar{u}_0$ the semigroup solution with initial datum $\bar{u}_0 \in L^\infty(\Omega)$. Then, there exist $z(t), \overline{z}(t) \in Z(\Omega)$ with $\|z(t)\|_\infty \leqslant 1$, $\|\bar{z}(t)\|_\infty \leqslant 1$ and such that, if $r, \bar{r} \in \mathbb{R}^N$, with $\|r\| \leqslant 1$, $\|\bar{r}\| \leqslant 1$ and $l_1, l_2 \in \mathbb{R}$, then

$$\begin{aligned}
&-\int_0^T\int_\Omega J_k^+\big(u(t)-l_1\big)\eta_t + \int_0^T\int_\Omega \eta(t)\big|DT_k^+\big(u(t)-l_1\big)\big| \\
&\quad + \int_0^T\int_{\partial\Omega} j\big(T_k^+\big(u(t)-l_1\big) - T_k^+(-l_1)\big)\eta(t) \\
&\quad + \int_0^T\int_\Omega \big(z(t)-r\big)\cdot D\eta(t)T_k^+\big(u(t)-l_1\big) + \int_0^T\int_\Omega r\cdot D\eta(t)T_k^+\big(u(t)-l_1\big) \\
&\leqslant \int_0^T\int_{\partial\Omega} \big[z(t),\nu\big]T_k^+(-l_1)\eta(t),
\end{aligned} \tag{75}$$

and

$$\begin{aligned}
&-\int_0^T\int_\Omega J_k^-\big(\bar{u}(t)-l_2\big)\eta_t + \int_0^T\int_\Omega \eta(t)\big|DT_k^-\big(\bar{u}(t)-l_2\big)\big| \\
&\quad + \int_0^T\int_{\partial\Omega} j\big(T_k^-\big(\hat{u}(t)-l_2\big) - T_k^-(-l_2)\big)\eta(t) \\
&\quad + \int_0^T\int_\Omega \big(\bar{z}(t)-\bar{r}\big)\cdot D\eta(t)T_k^-\big(\bar{u}(t)-l_2\big) + \int_0^T\int_\Omega \bar{r}\cdot D\eta(t)T_k^-\big(\bar{u}(t)-l_2\big) \\
&\leqslant \int_0^T\int_{\partial\Omega} \big[\bar{z}(t),\nu\big]T_k^-(-l_2)\eta(t)
\end{aligned} \tag{76}$$

for all $\eta \in C^\infty(\overline{Q_T})$, with $\eta \geqslant 0$, $\eta(t,x) = \phi(t)\psi(x)$, being $\phi \in \mathcal{D}(]0,T[)$, $\psi \in C^\infty(\overline{\Omega})$, $J_k^+(r) = \int_0^r T_k^+(s)\,\mathrm{d}s$ and $J_k^-(r) = \int_0^r T_k^-(s)\,\mathrm{d}s$.

We choose two different pairs of variables (t,x), (s,y) and consider u, z as functions in (t,x), $\bar{u}$, $\bar{z}$ in (s,y). Let $0 \leqslant \phi \in \mathcal{D}(]0,T[)$, $0 \leqslant \psi \in \mathcal{D}(\Omega)$, ρ_n a classical sequence of mollifiers in $\mathbb{R}^N$ and $\tilde{\rho}_n$ a sequence of mollifiers in $\mathbb{R}$. Define

$$\eta_n(t,x,s,y) := \rho_n(x-y)\tilde{\rho}_n(t-s)\phi\left(\frac{t+s}{2}\right)\psi\left(\frac{x+y}{2}\right).$$

Note that for n sufficiently large,

$$\begin{aligned}
&(t,x) \mapsto \eta_n(t,x,s,y) \in \mathcal{D}\big(]0,T[\times\Omega\big) \quad \forall (s,y) \in Q_T, \\
&(s,y) \mapsto \eta_n(t,x,s,y) \in \mathcal{D}\big(]0,T[\times\Omega\big) \quad \forall (t,x) \in Q_T.
\end{aligned}$$

Hence, for (s,y) fixed, if we take in (75) $l_1 = \bar{u}(s,y)$ and $r = \bar{z}(s,y)$, we get

$$-\int_0^T\int_\Omega J_k^+\big(u(t,x)-\bar{u}(s,y)\big)(\eta_n)_t\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega \eta_n\big|D_xT_k^+\big(u(t,x)-\bar{u}(s,y)\big)\big|\,\mathrm{d}t$$

$$+ \int_0^T\int_\Omega \big(z(t,x) - \bar{z}(s,y)\big) \cdot \nabla_x \eta_n T_k^+\big(u(t,x) - \bar{u}(s,y)\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^T\int_\Omega \bar{z}(s,y) \cdot \nabla_x \eta_n T_k^+\big(u(t,x) - \bar{u}(s,y)\big)\,\mathrm{d}x\,\mathrm{d}t \leqslant 0. \tag{77}$$

Similarly, for (t,x) fixed, if we take in (76) $l_2 = u(t,x)$ and $\bar{r} = z(t,x)$, we get

$$- \int_0^T\int_\Omega J_k^-\big(\bar{u}(s,y) - u(t,x)\big)(\eta_n)_s\,\mathrm{d}y\,\mathrm{d}s + \int_0^T\int_\Omega \eta_n\big|D_y T_k^-\big(\bar{u}(s,y) - u(t,x)\big)\big|\,\mathrm{d}s$$
$$+ \int_0^T\int_\Omega \big(\bar{z}(s,y) - z(t,x)\big) \cdot \nabla_y \eta_n T_k^-\big(\bar{u}(s,y) - u(t,x)\big)\,\mathrm{d}y\,\mathrm{d}s$$
$$+ \int_0^T\int_\Omega z(t,x) \cdot \nabla_y \eta_n T_k^-\big(\bar{u}(s,y) - u(t,x)\big)\,\mathrm{d}y\,\mathrm{d}s \leqslant 0. \tag{78}$$

Now, since $T_k^-(r) = -T_k^+(-r)$ and $J_k^-(r) = J_k^+(-r)$, we can rewrite (78) as

$$- \int_0^T\int_\Omega J_k^+\big(u(t,x) - \bar{u}(s,y)\big)(\eta_n)_s\,\mathrm{d}y\,\mathrm{d}s + \int_0^T\int_\Omega \eta_n\big|D_y T_k^+\big(u(t,x) - \bar{u}(s,y)\big)\big|\,\mathrm{d}s$$
$$+ \int_0^T\int_\Omega \big(z(t,x) - \bar{z}(s,y)\big) \cdot \nabla_y \eta_n T_k^+\big(u(t,x) - \bar{u}(s,y)\big)\,\mathrm{d}y\,\mathrm{d}s$$
$$- \int_0^T\int_\Omega z(s,y) \cdot \nabla_y \eta_n T_k^+\big(u(t,x) - \bar{u}(s,y)\big)\,\mathrm{d}y\,\mathrm{d}s \leqslant 0. \tag{79}$$

Integrating (77) in (s,y), (79) in (t,x) and taking their sum yields

$$- \int_{Q_T\times Q_T} J_k^+\big(u(t,x) - \bar{u}(s,y)\big)\big((\eta_n)_t + (\eta_n)_s\big) + \int_{Q_T\times Q_T} \eta_n\big|D_x T_k^+\big(u(t,x) - \bar{u}(s,y)\big)\big|$$
$$+ \int_{Q_T\times Q_T} \eta_n\big|D_y T_k^+\big(u(t,x) - \bar{u}(s,y)\big)\big|$$
$$+ \int_{Q_T\times Q_T} \big(z(t,x) - \bar{z}(s,y)\big) \cdot (\nabla_x \eta_n + \nabla_y \eta_n) T_k^+\big(u(t,x) - \bar{u}(s,y)\big)$$
$$+ \int_{Q_T\times Q_T} \bar{z}(s,y) \cdot \nabla_x \eta_n T_k^+\big(u(t,x) - \bar{u}(s,y)\big)$$
$$- \int_{Q_T\times Q_T} z(t,x) \cdot \nabla_y \eta_n T_k^+\big(u(t,x) - \bar{u}(s,y)\big) \leqslant 0. \tag{80}$$

Now, by Green's formula we have

$$\int_{Q_T\times Q_T} \bar{z}(s,y) \cdot \nabla_x \eta_n T_k^+\big(u(t,x) - \bar{u}(s,y)\big) + \int_{Q_T\times Q_T} \eta_n\big|D_x T_k^+\big(u(t,x) - \bar{u}(s,y)\big)\big|$$
$$= - \int_{Q_T\times Q_T} \eta_n \bar{z}(s,y) \cdot D_x T_k^+\big(u(t,x) - \bar{u}(s,y)\big) + \int_{Q_T\times Q_T} \eta_n\big|D_x T_k^+\big(u(t,x) - \bar{u}(s,y)\big)\big| \geqslant 0,$$

and

$$-\int_{Q_T\times Q_T} z(t,x)\cdot\nabla_y\eta_n T_k^+\big(u(t,x)-\bar u(s,y)\big)+\int_{Q_T\times Q_T}\eta_n\big|D_yT_k^+\big(u(t,x)-\bar u(s,y)\big)\big|$$
$$=\int_{Q_T\times Q_T}\eta_n z(t,x)\cdot D_yT_k^+\big(u(t,x)-\bar u(s,y)\big)+\int_{Q_T\times Q_T}\eta_n\big|D_yT_k^+\big(u(t,x)-\bar u(s,y)\big)\big|\geqslant 0.$$

Hence, from (80), it follows that

$$-\int_{Q_T\times Q_T} J_k^+\big(u(t,x)-\bar u(s,y)\big)\big((\eta_n)_t+(\eta_n)_s\big)$$
$$+\int_{Q_T\times Q_T}\big(z(t,x)-\bar z(s,y)\big)\cdot\big(\nabla_x\eta_n+\nabla_y\eta_n\big)T_k^+\big(u(t,x)-\bar u(s,y)\big)\leqslant 0. \tag{81}$$

Since,

$$(\eta_n)_t+(\eta_n)_s=\rho_n(x-y)\tilde\rho_n(t-s)\phi'\Big(\frac{t+s}{2}\Big)\psi\Big(\frac{x+y}{2}\Big)$$

and

$$\nabla_x\eta_n+\nabla_y\eta_n=\rho_n(x-y)\tilde\rho_n(t-s)\phi\Big(\frac{t+s}{2}\Big)\nabla\psi\Big(\frac{x+y}{2}\Big),$$

passing to the limit in (81), it yields

$$-\int_{Q_T} J_k^+\big(u(t,x)-\bar u(t,x)\big)\phi'(t)\psi(x)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{Q_T}\big(z(t,x)-\bar z(t,x)\big)\cdot\nabla\psi(x)\phi(t)T_k^+\big(u(t,x)-\bar u(t,x)\big)\,\mathrm{d}x\,\mathrm{d}t\leqslant 0. \tag{82}$$

We have to prove that

$$\lim_n\int_{Q_T}\big(z(t,x)-\bar z(t,x)\big)\cdot\nabla\psi_n(x)\phi(t)T_k^+\big(u(t,x)-\bar u(t,x)\big)\,\mathrm{d}x\,\mathrm{d}t\geqslant 0$$

for any sequence $\psi_n\uparrow\chi_\Omega$. Working as in the uniqueness proof of Theorem 1 in [3], we obtain that

$$\lim_n\int_{Q_T}\big(z(t)-\bar z(t)\big)\nabla\psi_n\phi T_k^+\big(u(t)-\bar u(t)\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$\geqslant-\int_0^T\!\!\int_{\partial\Omega}\big[z(t)-\bar z(t),\nu\big]\phi T_k^+\big(u(t)-\bar u(t)\big)\,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t.$$

Thus, having in mind that for m large enough, we have

$$T_k^+\big(u(t)-\bar u(t)\big)=T_k^+\big(T_m\big(u(t)\big)-T_m\big(\bar u(t)\big)\big)$$

and by Remark 2,

$$\big[z(t)-\bar z(t),\nu\big]\big(T_m\big(u(t)\big)-T_m\big(\bar u(t)\big)\big)\leqslant 0 \quad \text{for almost all } t\in[0,T] \text{ and } \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega,$$

we obtain from (82) that

$$\int_{Q_T} J_k^+\big(u(t,x)-\bar u(t,x)\big)\phi'(t)\,\mathrm{d}x\,\mathrm{d}t \geqslant -\int_0^T\!\!\int_{\partial\Omega}[z-\bar z,\nu]\phi T_k^+(u-\bar u)\,\mathrm{d}\mathcal{H}^{N-1}\,\mathrm{d}t\geqslant 0. \tag{83}$$

Since this is true for all $0\leqslant\phi\in\mathcal{D}(]0,T[)$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega J_k^+\big(u(t,x)-\bar u(t,x)\big)\,\mathrm{d}x\,\mathrm{d}t\leqslant 0.$$

Hence

$$\int_\Omega J_k^+\big(u(t,x)-\bar u(t,x)\big)\,\mathrm{d}x\leqslant\int_\Omega J_k^+(u_0-\bar u_0)\,\mathrm{d}x.$$

Then, dividing the last inequality by k and letting $k\to 0$, we obtain

$$\int_\Omega\big(u(t,x)-\bar u(t,x)\big)^+\leqslant\int_\Omega(u_0-\bar u_0)^+.$$

From here we deduce that

$$\big\|u(t)-\bar u(t)\big\|_1\leqslant\big\|u_0-\bar u_0\big\|_1\quad\forall t\geqslant 0.$$

Hence, taking $u_n(t)=S(t)u_{0,n}$, $u_{0,n}\in L^\infty(\Omega)$ and $u_{0,n}\to u_0$ in $L^1(\Omega)$, we have

$$\big\|u(t)-u_n(t)\big\|_1\leqslant\big\|u_0-u_{0,n}\big\|_1\quad\forall t\geqslant 0.$$

Consequently, letting $n\to\infty$, $u(t)=S(t)u_0$, and the proof of the uniqueness concludes.

5. Asymptotic behaviour

In this section we establish that the entropy solutions of problem (1) stabilize as $t\to\infty$ by converging to a constant function. We use the Lyapunov method for semigroups of nonlinear contractions introduced by Pazy [27].

We use some terminology and notations from classical topological dynamics. For a continuous semigroup $(T(t))_{t\geqslant 0}$ on a metric space X, the *orbit or trajectory* of $u\in X$ is the set $\gamma(u)=\{T(t)u\colon\ t\geqslant 0\}$, and the *ω-limit set* of u is

$$\omega(u)=\Big\{v\in X\colon\ v=\lim_{n\to\infty}T(t_n)u \text{ for some sequence } t_n\to\infty\Big\}.$$

This set is possibly empty. Now, it is well-known that if $\gamma(u)$ is relatively compact, then $\omega(u)$ is a nonempty, compact and connected subset of X. Furthermore, $\omega(u)$ is positive invariant under $T(t)$, i.e., $T(t)\omega(u) = \omega(u)$ for any $t \geqslant 0$. An *equilibrium or stationary point* is a point $u \in X$ such that $\gamma(u) = \omega(u) = \{u\}$, or equivalently, $T(t)u = u$ for all $t \geqslant 0$.

In order to prove the stabilization theorem we need the orbits to be relatively compact.

Theorem 6. *Let $(S(t))_{t\geqslant 0}$ be the semigroup generated by $\mathcal{B}_\beta$ and J_λ its resolvent. Then,*

(i) *$J_\lambda(B)$ is a relatively compact subset of $L^1(\Omega)$ if B is a bounded subset of $L^\infty(\Omega)$.*
(ii) *For every $u_0 \in L^1(\Omega)$ the orbit $\gamma(u_0) = \{S(t)u_0\colon t \geqslant 0\}$ is a relatively compact subset of $L^1(\Omega)$.*

Proof. (i) Let B a bounded subset of $L^\infty(\Omega)$. Take $(f_n) \subset B$ and let $u_n := J_\lambda f_n$. Set $M := \sup_{n\in\mathbb{N}} \|f_n\|_\infty < \infty$. Since $\mathcal{B}_\beta$ is completely accretive, we have

$$\|u_n\|_\infty \leqslant M \quad \forall n \in \mathbb{N}. \tag{84}$$

On the other hand, since $(u_n, \frac{1}{\lambda}(f_n - u_n)) \in \mathcal{A}_\beta$, it follows that

$$\int_\Omega |Du_n| \leqslant \frac{1}{\lambda} M_1 \mathcal{L}^N(\Omega) \quad \forall n \in \mathbb{N}. \tag{85}$$

Thus, $\{u_n\colon n \in \mathbb{N}\}$ is a bounded sequence in $BV(\Omega)$, and consequently, we have that $\{u_n\colon n \in \mathbb{N}\}$ is a relatively compact subset of $L^1(\Omega)$.

(ii) Consider first $u_0 \in \mathcal{D}(\mathcal{B}_\beta) \cap L^\infty(\Omega)$. Then, since

$$\|S(t)u_0\|_\infty \leqslant \|u_0\|_\infty \quad \text{for all } t \geqslant 0,$$

as a consequence of (i), we have that $J_\lambda(\gamma(u_0))$ is a relatively compact subset of $L^1(\Omega)$ for all $\lambda > 0$. Moreover,

$$\|S(t)u_0 - J_\lambda S(t)u_0\|_1 \leqslant \lambda \inf\{\|v\|_1\colon v \in \mathcal{A}_\beta(u_0)\}.$$

Hence, $\gamma(u_0)$ is relatively compact in $L^1(\Omega)$.

Finally, since $\mathcal{D}(\mathcal{B}_\beta) \cap L^\infty(\Omega)$ is dense in $L^1(\Omega)$, given $u_0 \in L^1(\Omega)$ and $\varepsilon > 0$, there exists $v_0 \in \mathcal{D}(\mathcal{B}_\beta) \cap L^\infty(\Omega)$ such that $\|u_0 - v_0\|_1 < \varepsilon$. So we have,

$$\sup_{t\geqslant 0} \inf_{s\geqslant 0} \|S(t)u_0 - S(s)v_0\|_1 \leqslant \sup_{t\geqslant 0} \|S(t)u_0 - S(t)v_0\|_1 \leqslant \|u_0 - v_0\|_1 < \varepsilon.$$

From where it follows that $\gamma(u_0)$ is relatively compact in $L^1(\Omega)$. □

In [2] it is proved that in the particular case of Neumann boundary condition (i.e., for $\beta = \mathbb{R} \times \{0\}$) the solutions stabilize at $t \to +\infty$ converging to the average of the initial datum. For general β, we have the following result.

Theorem 7. *Let $u_0 \in L^1(\Omega)$ and $u(x,t)$ be the entropy solution of problem* (1) *with initial datum u_0. Then, there exists a constant K_{u_0}, $K_{u_0} \in \hat{\beta}^{-1}\{0\}$ such that*

$$\|u(\cdot,t) - K_{u_0}\|_1 \to 0 \quad \textit{as } t \to \infty.$$

Moreover, if $u_0 \geqslant 0$ *and* $d_\beta := \sup\{r \geqslant 0\colon\ 0 \in \hat{\beta}(r)\}$*, then*

$$\overline{\inf\{d_\beta, u_0\}} \leqslant K_{u_0} \leqslant \inf\{d_\beta, \overline{u_0}\},$$

where

$$\overline{w} = \frac{1}{\mathcal{L}^N(\Omega)} \int_\Omega w \, \mathrm{d}x.$$

Proof. Suppose first that $u_0 \in L^\infty(\Omega)$. Let $(S(t))_{t \geqslant 0}$ be the semigroup generated by $\mathcal{B}_\beta$ and J_λ its resolvent. Let $\mathcal{V}\colon L^1(\Omega) \to [0, +\infty]$ be defined by

$$\mathcal{V}(w) = \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega w^2 \, \mathrm{d}x & \text{if } w \in L^2(\Omega), \\ +\infty & \text{if } w \notin L^2(\Omega). \end{cases}$$

It is well-known that $\mathcal{V}$ is lower semicontinuous (see [15], p. 160). On the other hand, since $\mathcal{B}_\beta$ is completely accretive, we have

$$\frac{1}{2} \int_\Omega \bigl(J^n_{t/n} f\bigr)^2 \, \mathrm{d}x \leqslant \frac{1}{2} \int_\Omega f^2 \, \mathrm{d}x \quad \text{for } f \in L^2(\Omega),\ t > 0 \text{ and } n \in \mathbb{N}.$$

Now, by the Crandall–Liggett Theorem, since $\mathcal{V}$ is lower semicontinuous, we have

$$\mathcal{V}\bigl(S(t)f\bigr) \leqslant \liminf_{n \to \infty} \mathcal{V}\bigl(J^n_{t/n} f\bigr) \leqslant \mathcal{V}(f) \quad \text{for } t \geqslant 0.$$

Therefore, $\mathcal{V}$ is a Lyapunov functional for the semigroup $(S(t))_{t \geqslant 0}$.

Let $\mathcal{W}\colon L^1(\Omega) \to]-\infty, +\infty]$ defined by

$$\mathcal{W}(u) = \begin{cases} \displaystyle\int_\Omega |Du|, & \text{if } u \in BV(\Omega), \\ +\infty, & \text{if } u \notin BV(\Omega). \end{cases}$$

Since $u_0 \in L^\infty(\Omega)$ and $\mathcal{B}_\beta$ is completely accretive, $J_\lambda u_0 = (I + \lambda \mathcal{B}_\beta)^{-1} u_0 \in \mathcal{D}(\mathcal{B}_\beta) \subset BV(\Omega) \cap L^\infty(\Omega)$. Then, $(J_\lambda u_0, \frac{1}{\lambda}(u_0 - J_\lambda u_0)) \in \mathcal{A}_\beta$. Thus, taking $w = 0$ as a test function in the definition of the operator $\mathcal{A}_\beta$, we have

$$\int_\Omega |D J_\lambda u_0| \leqslant \frac{1}{\lambda} \int_\Omega (u_0 - J_\lambda u_0) J_\lambda u_0 \, \mathrm{d}x - \int_{\partial\Omega} j(J_\lambda u_0) \, \mathrm{d}\mathcal{H}^{N-1}.$$

Hence, since $j(J_\lambda u_0) \geqslant 0$ $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$, we obtain

$$\mathcal{W}(J_\lambda u_0) \leqslant \frac{1}{\lambda} \int_\Omega (u_0 - J_\lambda u_0) J_\lambda u_0 \, \mathrm{d}x.$$

Now, since

$$\mathcal{V}(J_\lambda u_0) - \mathcal{V}(u_0) = \frac{1}{2} \int_\Omega (J_\lambda u_0)^2 \, \mathrm{d}x - \frac{1}{2} \int_\Omega u_0^2 \, \mathrm{d}x \leqslant - \int_\Omega (u_0 - J_\lambda u_0) J_\lambda u_0 \, \mathrm{d}x,$$

we get

$$\mathcal{V}(J_\lambda u_0) + \lambda \mathcal{W}(J_\lambda u_0) - \mathcal{V}(u_0) \leqslant 0. \tag{86}$$

Replacing u_0 by $J_\lambda^{k-1} u_0$ in (86) we find

$$\mathcal{V}\big(J_\lambda^k u_0\big) + \lambda \mathcal{W}\big(J_\lambda^k u_0\big) - \mathcal{V}\big(J_\lambda^{k-1} u_0\big) \leqslant 0.$$

Adding these inequalities from $k = 1$ to $k = n$ and choosing $\lambda = t/n$, it yields

$$\mathcal{V}\big(J_{t/n}^n u_0\big) + \sum_{k=1}^n \frac{t}{n} \mathcal{W}\big(J_{t/n}^k u_0\big) - \mathcal{V}(u_0) \leqslant 0. \tag{87}$$

Next we define a piecewise constant function

$$F_n(\tau) = \mathcal{W}\big(J_{t/n}^k u_0\big) \quad \text{for } (k-1)t/n < \tau \leqslant kt/n.$$

Then

$$\sum_{k=1}^n \frac{t}{n} \mathcal{W}\big(J_{t/n}^k u_0\big) = \int_0^t F_n(\tau)\,\mathrm{d}\tau.$$

On the other hand, by the Crandall–Liggett Theorem,

$$\lim_{n\to\infty} J_{t/n}^k u_0 = S(\tau) u_0 \quad \text{in } L^1(\Omega),$$

where $k = k_n(\tau) = [n\tau/t] + 1$. Since $\mathcal{W}$ is lower semicontinuous in $L^1(\Omega)$, we have

$$\mathcal{W}\big(S(t)u_0\big) \leqslant \liminf_{n\to\infty} \mathcal{W}\big(J_{t/n}^k u_0\big) = \liminf_{n\to\infty} F_n(\tau).$$

Thus, by Fatou's Lemma, we obtain

$$\int_0^t \mathcal{W}\big(S(\tau)u_0\big)\,\mathrm{d}\tau \leqslant \liminf_{n\to\infty} \int_0^t F_n(\tau)\,\mathrm{d}\tau = \liminf_{n\to\infty} \sum_{k=1}^n \frac{t}{n} \mathcal{W}\big(J_{t/n}^k u_0\big). \tag{88}$$

Passing to the limit as $n \to \infty$ in (87) and taking into account (88) and the lower semicontinuity of $\mathcal{V}$, we get

$$\mathcal{V}\big(S(t)u_0\big) + \int_0^t \mathcal{W}\big(S(\tau)u_0\big)\,\mathrm{d}\tau - \mathcal{V}(u_0) \leqslant 0.$$

Consequently

$$\int_0^\infty \mathcal{W}\big(S(\tau)u_0\big)\,\mathrm{d}\tau \leqslant \mathcal{V}(u_0). \tag{89}$$

Thus, there exists a sequence $t_n \to \infty$, such that $\mathcal{W}(S(t_n)u_0) \to 0$ as $n \to \infty$. Now by Theorem 6, there exists a subsequence (t_{n_k}) such that

$$\lim_{k\to\infty} S(t_{n_k})u_0 = v \in \omega(u_0).$$

Hence, by the lower semicontinuity of $\mathcal{W}$, it follows that

$$\mathcal{W}(v) \leqslant \liminf_{k\to\infty} \mathcal{W}\big(S(t_{n_k})u_0\big) = 0.$$

Therefore, v is a constant K_{u_0}. If $K_{u_0} = 0$, since 0 is an equilibrium, $\omega(u_0) = \{0\}$. Suppose $K_{u_0} > 0$. Then, since $\|S(t)K_{u_0}\|_\infty \leqslant \|K_{u_0}\|_\infty = K_{u_0}$ and $S(t)$ is order preserving,

$$0 \leqslant S(t)K_{u_0} \leqslant K_{u_0}. \tag{90}$$

Since, $S(t)K_{u_0}, K_{u_0} \in \omega(u_0)$ and $\mathcal{V}$ is a Lyapunov functional, it follows from the Invariance Principle of Dafermos ([18]) that $\mathcal{V}(S(t)K_{u_0}) = \mathcal{V}(K_{u_0})$ for all $t \geqslant 0$. Consequently, by (90) and the definition of $\mathcal{V}$, $S(t)K_{u_0} = K_{u_0}$ for all $t \geqslant 0$, so as $S(t)$ are contractions, we get $\omega(u_0) = [K_{u_0}]$ and the proof for the case $u_0 \in L^\infty(\Omega)$ concludes. Now, since $L^\infty(\Omega)$ is dense in $\overline{\mathcal{D}(\mathcal{B}_\beta)} = L^1(\Omega)$ and $S(t)$ is an order preserving contraction, from the above we obtain easily the conclusion in the general case $u_0 \in L^1(\Omega)$. Finally, as K_{u_0} is an equilibrium, it follows that $K_{u_0} \in \hat{\beta}^{-1}\{0\}$.

Suppose now that $u_0 \geqslant 0$. Since the operator $\mathcal{B}_\beta$ is completely accretive, we have $0 \leqslant S(t)u_0 \leqslant u_0$ and $\|S(t)u_0\|_1 \leqslant \|u_0\|_1$. Hence, $K_{u_0} \leqslant \overline{u_0}$. Moreover, by the definition of d_β, we have $K_{u_0} \leqslant d_\beta$, so $K_{u_0} \leqslant \inf\{d_\beta, \overline{u_0}\}$. Consider $v_0 = \inf\{d_\beta, u_0\}$. Then, since $v_0 \leqslant u_0$, we have $\omega(v_0) \leqslant \omega(u_0)$. Now, if $\eta = \mathbb{R} \times \{0\}$, it is easy to see that

$$\mathrm{e}^{-t\mathcal{B}_\beta} v_0 = \mathrm{e}^{-t\mathcal{B}_\eta} v_0 \quad \forall t \geqslant 0.$$

Thus, $\omega(v_0) = \overline{v_0}$, and consequently, $K_{u_0} \geqslant \overline{v_0} = \overline{\inf\{d_\beta, u_0\}}$. □

Remark 3. Note that in the above theorem if $u_0 \geqslant 0$, then $K_{u_0} = \overline{u_0}$ if $\beta(r) = 0$ for all $r > 0$ and $K_{u_0} = 0$ if $\beta(r) > 0$ for all $r > 0$. Therefore, for the Dirichlet problem (i.e., for $\beta = \{0\} \times \mathbb{R}$), the solutions stabilize as $t \to \infty$ by converging to zero in the L^1-norm. Now, in [5] it is proved the existence of a finite extinction time for the solutions of the Dirichlet problem and also that the solutions of the Neumann problem reach the average of the initial data in finite time in the two dimensional case. This property of reaching the asymptotic state in finite time is not true for general nonlinear boundary conditions. For example, when β is given by (29) explicit solutions are obtained in Examples 1 and 2 which converge to zero at $t \to +\infty$, but are strictly positives for all time.

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