The best constant for the Sobolev trace embedding from \( W^{1,1}(\Omega) \) into \( L^1(\partial\Omega) \)

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Abstract

In this paper we study the best constant, \( \lambda_1(\Omega) \) for the trace map from \( W^{1,1}(\Omega) \) into \( L^1(\partial\Omega) \). We show that this constant is attained in \( BV(\Omega) \) when \( \lambda_1(\Omega) < 1 \). Moreover, we prove that this constant can be obtained as limit when \( p \downarrow 1 \) of the best constant of \( W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega) \). To perform the proofs we will look at Neumann problems involving the 1-Laplacian, \( A_1(u) = \text{div}(Du/|Du|) \).

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1. Introduction

Let \( \Omega \) be a bounded set in \( \mathbb{R}^N \) with Lipschitz continuous boundary \( \partial\Omega \). Of importance in the study of boundary value problems for differential operators in \( \Omega \) are the Sobolev trace inequalities. In particular, \( W^{1,1}(\Omega) \hookrightarrow L^1(\partial\Omega) \) and hence the following inequality holds:

\[ \lambda_1 \|u\|_{L^1(\partial\Omega)} \leq \|u\|_{1,1}, \]

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for all $u \in W^{1,1}(\Omega)$. The best constant for this embedding is the largest $\lambda$ such that the above inequality holds, that is,

$$\lambda_1(\Omega) = \inf \left\{ \int_{\Omega} |u| + \int_{\Omega} |\nabla u| : u \in W^{1,1}(\Omega), \int_{\partial \Omega} |u| = 1 \right\}. \quad (1)$$

Our main interest in this paper is to study the dependence of the best constant $\lambda_1(\Omega)$ and extremals (functions where the constant is attained) on the domain. A related problem was studied by Demengel in [8] (see Remark 2). We remark that the existence of extremals is not trivial, due to the lack of compactness of the embedding.

For $1 < p \leq N$, let us consider the variational problem

$$\inf \left\{ \int_{\Omega} |u|^p + \int_{\Omega} |\nabla u|^p : u \in W^{1,p}(\Omega), \int_{\partial \Omega} |u|^p = 1 \right\}. \quad (2)$$

If we denote by $\lambda_p(\Omega)$ the above infimum, we have that

$$\lambda_p(\Omega) = \inf \left\{ \frac{\int_{\Omega} |u|^p + \int_{\partial \Omega} |\nabla u|^p}{\int_{\partial \Omega} |u|^p} : u \equiv 0 \text{ on } \partial \Omega, \ u \in W^{1,p}(\Omega) \right\} \quad (3)$$

is the best constant for the trace map from $W^{1,p}(\Omega)$ into $L^p(\partial \Omega)$. Due to the compactness of the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial \Omega)$, it is well known (see for instance [10]) that problem (3) has a minimizer in $W^{1,p}(\Omega)$. These extremals are weak solutions of the following problem:

$$\begin{cases}
A_p u = |u|^{p-2}u & \text{in } \Omega, \\
|\nabla u|^{p-2} \frac{\partial u}{\partial v} = \lambda |u|^{p-2}u & \text{on } \partial \Omega,
\end{cases} \quad (4)$$

where $A_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian, $\partial / \partial v$ is the outer unit normal derivative and if we use the normalization $\|u\|_{L^p(\partial \Omega)} = 1$, one can check that $\lambda = \lambda_p(\Omega)$, see [11].

Our first result says that $\lambda_1(\Omega)$ is the limit as $p \searrow 1$ of $\lambda_p(\Omega)$ and provides a bound for $\lambda_1(\Omega)$.

**Theorem 1.** We have that

$$\lim_{p \searrow 1} \lambda_p(\Omega) = \lambda_1(\Omega)$$

and

$$\lambda_1(\Omega) \leq \min \left\{ \frac{|\Omega|}{P(\Omega)}, 1 \right\},$$

where $P(\Omega)$ stands for the perimeter of $\Omega$. 
Therefore, it seems natural to search for an extremal for \( \lambda_1(\Omega) \) as the limit of extremals for \( \lambda_p(\Omega) \) when \( p \downarrow 1 \). Formally, if we take limit as \( p \downarrow 1 \) in Eq. (4), we get

\[
\begin{cases}
A_1 u := \text{div} \left( \frac{Du}{|Du|} \right) = \frac{u}{|u|} & \text{in } \Omega, \\
\frac{Du}{|Du|} \cdot v = \lambda_1(\Omega) \frac{u}{|u|} & \text{on } \partial \Omega.
\end{cases}
\]

Hence we will look at Neumann problems involving the 1-Laplacian, \( A_1(u) = \text{div}(Du/|Du|) \), in the context of bounded variation functions (the natural context for this type of problems). To our knowledge the results obtained here have independent interest.

We shall say that \( \Omega \) has the trace-property if there exists a vector field \( z_\Omega \in L^\infty(\Omega, \mathbb{R}^N) \), with \( \|z_\Omega\|_\infty \leq 1 \) such that \( \text{div}(z_\Omega) \in L^\infty(\Omega) \) and

\[
[z_\Omega, v] = \lambda_1(\Omega) \mathcal{H}^{N-1}\text{-a.e.} \quad \text{on } \partial \Omega.
\]

Our main result states that for any domain having the trace property, the best Sobolev trace constant, \( \lambda_1(\Omega) \), is attained by a function in \( L^1(\Omega) \) whose derivatives in the sense of distributions are bounded measures on \( \Omega \), that is a function with bounded variation.

**Theorem 2.** Let \( \Omega \) be a bounded open set in \( \mathbb{R}^N \) with the trace-property. Then, there exists a nonnegative function of bounded variation which is a minimizer of the variational problem (1) and a solution of problem (5).

We will see that every bounded domain \( \Omega \) with \( \lambda_1(\Omega) < 1 \), has the trace-property. Hence we have proved that, if \( \lambda_1(\Omega) < 1 \) then there exists an extremal. Moreover, using results from [12], we can find examples of domains (a ball or an annulus) such that \( \lambda_1(\Omega) = 1 \) and verify the trace property (and therefore they have extremals). We also prove that every planar domain \( \Omega \) with a point of curvature greater than 2 verifies \( \lambda_1(\Omega) < 1 \).

**Organization of the paper.** In Section 2 we collect some preliminary results and prove Theorem 1. In Section 3 we deal with the Neumann problem for the equation \( \text{div}(z) = 1 \). Finally, in Section 4 we use these results to prove the main theorem, Theorem 2. Throughout this paper \( C \) and \( c \) denote constants that may change from one line to another.

## 2. Preliminary results. Proof of Theorem 1

Let us begin with some notation and definitions. Recall that a function \( u \in L^1(\Omega) \) whose partial derivatives in the sense of distributions are measures with finite total variation in \( \Omega \) is called a function of bounded variation. The class of such functions will be denoted by \( BV(\Omega) \). Thus \( u \in BV(\Omega) \) if and only if there are Radon measures \( \mu_1, \ldots, \mu_N \) defined in \( \Omega \) with finite total mass in \( \Omega \) and

\[
\int_\Omega u \, D_i \varphi = - \int_\Omega \varphi \, d\mu_i
\]
for all \( \varphi \in C_0^\infty(\Omega), i = 1, \ldots, N \). Thus the gradient of \( u \) is a vector valued measure with finite total variation

\[
|Du|(\Omega) = \sup \left\{ \int_\Omega u \, \text{div} \varphi : \varphi \in C_0^\infty(\Omega, \mathbb{R}^N), \ |\varphi(x)| \leq 1 \text{ for } x \in \Omega \right\}.
\]

The set of bounded variation functions, \( BV(\Omega) \), is a Banach space endowed with the norm

\[
\|u\|_{BV(\Omega)} := \|u\|_1 + |Du|(\Omega) = \int_\Omega |u| + \int_\Omega |Du|.
\]

A measurable set \( E \subset \mathbb{R}^N \) is said to be of finite perimeter in \( \Omega \) if \( \mathcal{L}_E \in BV(\Omega) \), in this case the perimeter of \( E \) in \( \Omega \) is defined as \( P(E, \Omega) := |D\mathcal{L}_E| \). We shall use the notation \( P(E) := P(E, \mathbb{R}^N) \), and \( \mathcal{P}_f(\Omega) \) shall denote the set of all subset of \( \Omega \) of finite perimeter. For a set of finite perimeter \( E \) one can define the essential boundary \( \partial^* E \), which is countably \((N-1)\) rectifiable with finite \( H^{N-1} \) measure, and compute the unit normal \( \nu^E(x) \) at \( H^{N-1} \) almost all points \( x \) of \( \partial^* E \), where \( H^{N-1} \) is the \((N-1)\) dimensional Hausdorff measure, and \( |D\mathcal{L}_E| \) coincides with the restriction of \( H^{N-1} \) to \( \partial^* E \).

It is well known (see, [1,9] or [13]) that for a given function \( u \in BV(\Omega) \) there exists a sequence \( u_n \in W^{1,1}(\Omega) \) such that \( u_n \) strict converges to \( u \), i.e.,

\[
u_n \to u \text{ in } L^1(\Omega) \text{ and } \int_\Omega |\nabla u_n| \to \int_\Omega |Du|.
\]

Moreover, there is a trace operator \( \tau: BV(\Omega) \to L^1(\partial\Omega) \) such that

\[
\|\tau(u)\|_{L^1(\partial\Omega)} \leq C\|u\|_{BV(\Omega)} \quad \forall u \in BV(\Omega) \tag{6}
\]

for some constant \( C \) depending only on \( \Omega \). The trace operator \( \tau \) is continuous between \( BV(\Omega) \), endowed with the topology induced by the strict convergence, and \( L^1(\partial\Omega) \). In the sequel we write \( \tau(u) = u \).

From the above results, we get

\[
\lambda_1(\Omega) = \inf \left\{ \frac{\int_\Omega |u| + \int_\Omega |Du|}{\int_{\partial\Omega} |u|} : u \neq 0 \text{ on } \partial\Omega, \ u \in BV(\Omega) \right\}.
\]

Let us recall a generalized Green’s formula given in [5] (see also [4]). For \( 1 \leq p < \infty \), let

\[
X_p(\Omega) = \{ z \in L^\infty(\Omega, \mathbb{R}^N) : \text{div}(z) \in L^p(\Omega) \}.
\]

If \( z \in X_p(\Omega) \) and \( w \in BV(\Omega) \cap L^{p'}(\Omega) \) the functional \( (z, Dw): C_0^\infty(\Omega) \to \mathbb{R} \) is defined by the formula

\[
\langle (z, Dw), \varphi \rangle = -\int_\Omega w \varphi \text{div}(z) - \int_\Omega wz \cdot \nabla \varphi.
\]

Then \( (z, Dw) \) is a Radon measure in \( \Omega \),

\[
\int_\Omega (z, Dw) = \int_\Omega z \cdot \nabla w
\]
for all \( w \in W^{1,1}(\Omega) \cap L^\infty(\Omega) \) and
\[
\left| \int_B (z, Dw) \right| \leq \int_B |(z, Dw)| \leq \|z\|_\infty \int_B |Dw| \tag{7}
\]
for any Borel set \( B \subseteq \Omega \).

In [5], a weak trace on \( \partial\Omega \) of the normal component of \( z \in X_p(\Omega) \) is defined. Concretely, it is proved that there exists a linear operator \( \gamma : X_p(\Omega) \rightarrow L^\infty(\partial\Omega) \) such that
\[
\|\gamma(z)\|_\infty \leq \|z\|_\infty,
\]
\( \gamma(z)(x) = z(x) \cdot v(x) \) for all \( x \in \partial\Omega \) if \( z \in C^1(\overline{\Omega}, \mathbb{R}^N) \).

We shall denote \( \gamma(z)(x) \) by \([z, v](x)\). Moreover, the following Green's formula, relating the function \([z, v]\) and the measure \((z, Dw)\), for \( z \in X_p(\Omega) \) and \( w \in BV(\Omega) \cap L^p(\Omega) \), is established:
\[
\int_\Omega w \text{div}(z) \, dx + \int_\Omega (z, Dw) = \int_{\partial\Omega} [z, v]w \, d\mathcal{H}^{N-1}. \tag{8}
\]

Now we can prove Theorem 1.

**Proof of Theorem 1.** We have to prove that
\[
\lim_{p \searrow 1} \lambda_p(\Omega) = \lambda_1(\Omega) \tag{9}
\]
and
\[
\lambda_1(\Omega) \leq \min \left\{ \frac{|\Omega|}{P(\Omega)}, 1 \right\}. \tag{10}
\]

Since
\[
\lambda_p(\Omega) \|u\|_{L^p(\partial\Omega)}^p \leq \|u\|_{1,p}^p \quad \forall u \in W^{1,p}(\Omega)
\]
if we set
\[
\lambda^*: = \limsup_{p \searrow 1} \lambda_p(\Omega)
\]
we have
\[
\lambda^* \|u\|_{L^1(\partial\Omega)} \leq \|u\|_{1,1} \quad \forall u \in W^{1,1}(\Omega)
\]
from where it follows that \( \lambda^* \leq \lambda_1(\Omega) \), that is
\[
\limsup_{p \searrow 1} \lambda_p(\Omega) \leq \lambda_1(\Omega). \tag{11}
\]

Let \( v_p \) be a minimizer of problem (3). Then, if \( u_p := a_p v_p \), with \( a_p \) satisfying
\[
a_p = \left( \int_{\partial\Omega} |v_p|^p \right)^{1/(p-1)} \left( \int_{\partial\Omega} |v_p|^p \right)^{1/(1-p)},
\]
we get
\[ \int_{\partial \Omega} |u_p| = \int_{\partial \Omega} |u_p|^p \]
and consequently
\[ \lambda_p(\Omega) = \frac{\int_{\Omega} |u_p|^p + \int_{\Omega} |\nabla u_p|^p}{\int_{\partial \Omega} |u_p|}. \]  
(12)

Applying Hölder’s inequality and (12), we have
\[ \lambda_1(\Omega) \leq \frac{\int_{\Omega} |u_p| + \int_{\Omega} |\nabla u_p|}{\int_{\partial \Omega} |u_p|} \leq |\Omega|^{1/p'} \left( \frac{\int_{\Omega} |u_p| + \int_{\Omega} |\nabla u_p|^p}{\int_{\partial \Omega} |u_p|} \right)^{1/p} \]
\[ \leq |\Omega|^{1/p'} 2^{(p-1)/p} \left( \frac{\int_{\Omega} |u_p|^p + \int_{\Omega} |\nabla u_p|^p}{\int_{\partial \Omega} |u_p|} \right)^{1/p} \]
\[ = |\Omega|^{1/p'} 2^{(p-1)/p} \frac{\lambda_p(\Omega)^{1/p}}{\left( \int_{\partial \Omega} |u_p| \right)^{1/p'}}. \]

Hence
\[ \lambda_p(\Omega) \geq \lambda_1(\Omega)^p \frac{2^{1-p}}{|\Omega|^{p/p'}} \left( \int_{\partial \Omega} |u_p| \right)^{p/p'} \]
from where it follows that
\[ \liminf_{p \to 1} \lambda_p(\Omega) \geq \lambda_1(\Omega). \]  
(13)

Now, (9) follows from (11) and (13).

Taking \( u = \chi_{\Omega} \), we obtain that
\[ \lambda_1(\Omega) \leq \frac{\int_{\Omega} |\chi_{\Omega}| + \int_{\Omega} |\nabla \chi_{\Omega}|}{\int_{\partial \Omega} |\chi_{\Omega}|} = \frac{|\Omega|}{P(\Omega)}. \]

On the other hand, if \( \Omega_\varepsilon := \{ x \in \Omega: d(x, \partial \Omega) < \varepsilon \} \), we have
\[ \lambda_1(\Omega) \leq \frac{\int_{\Omega} |\chi_{\Omega}| + \int_{\Omega} |\nabla \chi_{\Omega_\varepsilon}|}{\int_{\partial \Omega} |\chi_{\Omega_\varepsilon}|} = \frac{|\Omega_\varepsilon| + P(\Omega_\varepsilon, \Omega)}{P(\Omega)}. \]

Hence, taking \( \varepsilon \to 0^+ \), it follows that \( \lambda_1(\Omega) \leq 1 \). Therefore, (10) holds.  \( \square \)

It is well known (see for instance [11]) that, for every \( p > 1 \), there exist an extremal for the embedding \( W^{1,p}(\Omega) \hookrightarrow L^p(\partial \Omega) \) (this embedding is compact). This is a solution \( 0 \leq u_p \in W^{1,p}(\Omega) \) of the equation
\[ \lambda_p(\Omega) = \int_{\Omega} |u_p|^p + \int_{\Omega} |\nabla u_p|^p, \quad \int_{\partial \Omega} |u_p| = 1, \]  
(14)
such that \( u_p > 0 \) and satisfies in the weak sense
\[
\begin{cases}
\mathcal{A}_p u_p := \text{div}(|\nabla u_p|^{p-2} \nabla u_p) = |u_p|^{p-2} u_p & \text{in } \Omega, \\
|\nabla u_p|^{p-2} \nabla u_p \cdot v = \nu_p (\Omega) |u_p|^{p-2} u_p & \text{on } \partial \Omega.
\end{cases}
\] (15)

Now, by Young’s inequality we have
\[
1 \leq \frac{1}{p} \left( \int_{\Omega} |u_p| + \int_{\Omega} |\nabla u_p| \right) \leq \frac{1}{p} \left( \int_{\Omega} |u_p|^p + \int_{\Omega} |\nabla u_p|^p \right) + \frac{2}{p'} |\Omega|.
\]

Then, by (9) it follows that
\[
\lambda_1(\Omega) = \lim_{p \downarrow 1} \frac{1}{p} \int_{\Omega} |u_p| + \int_{\Omega} |\nabla u_p|.
\] (16)

Moreover, by the compact embedding of \( BV(\Omega) \) into \( L^1(\Omega) \), we can suppose that
\[ u_p \to u \in BV(\Omega) \quad \text{in the } L^1 \text{-norm and a.e. in } \Omega \] (17)
and
\[ \nabla u_p \to Du \quad \text{weakly}^* \text{ as measures.} \] (18)

Then, we have
\[
\exists A := \lim_{p \downarrow 1} \int_{\Omega} |\nabla u_p| = \lambda_1(\Omega) - \int_{\Omega} |u|.
\] (19)

On the other hand, by the lower-semi-continuity of the total variation respect the \( L^1 \)-norm, we get
\[
\int_{\Omega} |Du| \leq \liminf_{p \downarrow 1} \int_{\Omega} |\nabla u_p| = \lambda_1(\Omega) - \int_{\Omega} |u|.
\]

Hence,
\[
\int_{\Omega} |Du| + \int_{\Omega} |u| \leq \lambda_1(\Omega).
\] (20)

We are interested in the problem: When is \( u \) a minimizer of the variational problem (1)? In these cases we would find an extremal for our minimization problem (1).

Formally, if we take limit as \( p \searrow 1 \) in Eq. (15), we get
\[
\begin{cases}
\mathcal{A}_1 u := \text{div} \left( \frac{Du}{|Du|} \right) = \frac{u}{|u|} & \text{in } \Omega, \\
\frac{Du}{|Du|} \cdot v = \nu_1(\Omega) \frac{u}{|u|} & \text{on } \partial \Omega.
\end{cases}
\] (21)

Following [2,6] (see also [4]), we give the following definition of solution of problem (21).
**Definition 1.** A function \( u \in BV(\Omega) \) is said to be a solution of problem (21) if there exists \( z \in X_1(\Omega) \) with \( \|z\|_{\infty} \leq 1 \), \( \tau \in L^\infty(\Omega) \) with \( \|\tau\|_{\infty} \leq 1 \) and \( \theta \in L^\infty(\partial\Omega) \) with \( \|\theta\|_{\infty} \leq 1 \) such that

\[
\text{div}(z) = \tau \quad \text{in} \quad \mathcal{D}'(\Omega),
\]

\[
\tau u = |u| \text{ a.e. in } \Omega \quad \text{and} \quad (z, Du) = |Du| \text{ as measures},
\]

\[
[z, v] = \lambda_1(\Omega) \theta \quad \text{and} \quad \theta u = |u| \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega.
\]

Another problem we are interested in is: In what cases is \( u \) a solution of problem (21)? Note that if \( v \) is a solution of problem (21) and \( \int_{\partial\Omega} |v| \neq 0 \), then \( w := v/\int_{\partial\Omega} |v| \) is a minimizer of the variational problem (1). Indeed, multiplying (22) by \( v \) and integrating by parts, we have

\[
\int_{\Omega} |v| = \int_{\Omega} \tau v = \int_{\Omega} \text{div}(z)v = -\int_{\Omega} (z, Dv) + \int_{\partial\Omega} [z, v]v
\]

\[
= -\int_{\Omega} |Dv| + \lambda_1(\Omega) \int_{\partial\Omega} |v|.
\]

From where it follows that

\[
\lambda_1(\Omega) = \int_{\Omega} |w| + \int_{\Omega} |Dw|.
\]

Before we solve the above problems, let us study first the equation \( \text{div}(z) = 1 \) with Neumann boundary conditions.

### 3. The equation \( \text{div}(z) = 1 \) with Neumann boundary conditions

Throughout this section we shall denote by \( \Omega \) a bounded connected open set in \( \mathbb{R}^N, N \geq 2 \), with Lipschitz continuous boundary \( \partial\Omega \). Given \( g \in L^\infty(\partial\Omega) \) with \( \|g\|_{\infty} < 1 \), consider the functional \( \mathcal{E}_g : L^2(\Omega) \to ]-\infty, +\infty] \) defined by

\[
\mathcal{E}_g(u) := \begin{cases} 
\int_{\Omega} |Du| - \int_{\partial\Omega} gu & \text{if } u \in BV(\Omega) \cap L^2(\Omega), \\
+\infty & \text{if } u \notin BV(\Omega).
\end{cases}
\]

In [6] it is proved that

\[
\partial \mathcal{E}_g = \mathcal{A}_g,
\]

where \( \mathcal{A}_g \) is the operator in \( L^2(\Omega) \) defined by

\[
(u, v) \in \mathcal{A}_g \iff u \in BV(\Omega) \cap L^2(\Omega),
\]

\[
v \in L^2(\Omega) \text{ and } \exists z \in X_2(\Omega), \|z\|_{\infty} \leq 1.
\]
such that
\[-\text{div}(z) = v \quad \text{in } \mathcal{D}'(\Omega),\]
\[(z, Du) = |Du| \quad \text{as measures}\]
and
\[[z, v] = g \mathcal{H}^{N-1}\text{-a.e. on } \partial \Omega.\]

Let \(\psi : L^2(\Omega) \to \mathbb{R}\) the operator defined by
\[\psi(u) := \frac{1}{2} \int_\Omega (u(x) + 1)^2 \, dx.\]

We have
\[0 = \arg\min (\mathcal{E}_g + \psi) \iff 0 \in \partial(\mathcal{E}_g + \psi)(0),\]
\[\iff -1 \in \partial \mathcal{E}_g(0) \iff (0, -1) \in \mathcal{A}_g.\]

Then, by (25), it follows that
\[0 = \arg\min (\mathcal{E}_g + \psi) \iff \exists z \in X_2(\Omega), \|z\|_\infty \leq 1, \text{ such that}\]
\[\text{div}(z) = 1 \quad \text{in } \mathcal{D}'(\Omega),\]
\[[z, v] = g \mathcal{H}^{N-1}\text{-a.e. on } \partial \Omega.\]

On the other hand, \(0 = \arg\min (\mathcal{E}_g + \psi)\) if and only if
\[\int_\Omega |Du| - \int_{\partial \Omega} gu + \frac{1}{2} \int_\Omega (u + 1)^2 \geq \frac{1}{2} |\Omega| \quad \forall u \in BV(\Omega) \cap L^2(\Omega). \tag{26}\]
Replacing \(u\) by \(\varepsilon u\) (where \(\varepsilon > 0\)), expanding the \(L^2\) norm, dividing by \(\varepsilon\), and letting \(\varepsilon \to 0^+\) we get
\[\int_{\partial \Omega} gu \leq \int_\Omega |Du| + \int_\Omega u \quad \forall u \in BV(\Omega) \cap L^2(\Omega). \tag{27}\]

Consequently we have obtained the following result.

**Lemma 1.** Let \(g \in L^\infty(\partial \Omega)\) with \(\|g\|_\infty < 1.\) Then the following are equivalent:

(i) there exists \(z \in X_2(\Omega), \|z\|_\infty \leq 1, \) such that
\[\text{div}(z) = 1 \quad \text{in } \mathcal{D}'(\Omega),\]
\[[z, v] = g \mathcal{H}^{N-1}\text{-a.e. on } \partial \Omega.\]

(ii) Eq. (27) holds.
Working as above but using the functional
\[
\phi(u) := \frac{1}{2} \int_{\Omega} (|u(x)| + 1)^2 \, dx
\]
instead of \( \psi \), we obtain the following result.

**Lemma 2.** Let \( g \in L^\infty(\partial\Omega) \) with \( \|g\|_\infty < 1 \). Then the following are equivalent:

(i) there exist \( z \in X_2(\Omega) \), \( \|z\|_\infty \leq 1 \), and \( \tau \in L^\infty(\Omega) \), \( \|\tau\|_\infty \leq 1 \) such that
\[
\text{div}(z) = \tau \quad \text{in } \mathcal{D}'(\Omega),
\]
\[
[z, v] = g \mathcal{H}^{N-1}-a.e. \text{ on } \partial\Omega.
\]

(ii) the following inequality holds
\[
\int_{\partial\Omega} gu \leq \int_{\Omega} |Du| + \int_{\Omega} |u| \quad \forall u \in BV(\Omega) \cap L^2(\Omega).
\] (28)

We state now the main result of this section.

**Theorem 3.** Let \( \Omega \) be a bounded connected open set in \( \mathbb{R}^N \), \( N \geq 2 \), with Lipschitz continuous boundary \( \partial\Omega \). Assume that \( |\Omega|/P(\Omega) \leq \lambda < 1 \). Then the following are equivalent:

(i) there exist \( z \in X_2(\Omega) \), \( \|z\|_\infty \leq 1 \), such that
\[
\text{div}(z) = 1 \quad \text{in } \mathcal{D}'(\Omega),
\]
\[
[z, v] = \lambda \mathcal{H}^{N-1}-a.e. \text{ on } \partial\Omega,
\]

(ii)
\[
\lambda \int_{\partial\Omega} u \leq \int_{\Omega} |Du| + \int_{\Omega} u \quad \forall u \in BV(\Omega) \cap L^2(\Omega),
\] (29)

(iii)
\[
\lambda \int_{\partial\Omega} u \leq \int_{\Omega} |Du| + \int_{\Omega} |u| \quad \forall u \in BV(\Omega) \cap L^2(\Omega),
\] (30)

(iv)
\[
\lambda \leq \lambda_1(\Omega),
\] (31)

(v)
\[
|E| - \lambda \mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega) \leq P(E, \Omega) \quad \forall E \in \mathcal{P}_f(\Omega).
\] (32)

**Proof.** By Lemma 1 with \( g = \lambda \), (i) and (ii) are equivalent. Obviously, (iii) and (iv) are equivalent, and (ii) implies (iii). Let us see that (iii) implies (v). Taking \( u = \chi_E \) in (30), with \( E \subset \Omega \) a set of finite perimeter, it follows that
\[
-|E| - \lambda \mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega) \leq P(E, \Omega)
\]
and taking \( u = \mathbb{I}_\Omega \setminus \mathbb{I}_E \), we get
\[
\lambda \mathcal{H}^{N-1}(\partial^* (\Omega, E) \cap \partial \Omega) \leq \nu (\Omega \setminus E, \Omega) + |\Omega \setminus E|.
\]

Now, since \( \nu (\Omega \setminus E, \Omega) = \nu (E, \Omega) \) and \( \mathcal{H}^{N-1}(\partial^* (\Omega \setminus E) \cap \partial \Omega) = \nu (\Omega) - \mathcal{H}^{N-1}(\partial^* E \cap \partial \Omega) \), we have
\[
\lambda [\nu (\Omega) - \mathcal{H}^{N-1}(\partial^* E \cap \partial \Omega)] \leq \nu (E, \Omega) + |\Omega| - |E|.
\]

Then, since \( |\Omega| \geq |\Omega| \), we obtain
\[
|E| - \lambda \mathcal{H}^{N-1}(\partial^* E \cap \partial \Omega) \leq \nu (E, \Omega)
\]
and (v) holds. Finally, let us see that (v) implies (ii). Given \( u \in BV(\Omega) \), since for all \( x \in \Omega \),
\[
uu(\omega) = \int_{0}^{+\infty} \chi_{[u > t]}(x) \, dt - \int_{-\infty}^{0} \chi_{[u \leq t]}(x) \, dt,
\]
using (32) and the coarea formula we get
\[
\int_{\Omega} \nuu(\omega) \, dx = \int_{0}^{+\infty} \int_{\Omega} \chi_{[u > t]}(x) \, dx \, dt - \int_{-\infty}^{0} \int_{\Omega} \chi_{[u \leq t]}(x) \, dx \, dt
\]
\[
\geq \int_{0}^{+\infty} (\lambda \mathcal{H}^{N-1}(\partial^* [u > t] \cap \partial \Omega) - \nu ([u > t], \Omega)) \, dt
\]
\[
- \int_{-\infty}^{0} (\lambda \mathcal{H}^{N-1}(\partial^* [u \leq t] \cap \partial \Omega) + \nu ([u \leq t], \Omega)) \, dt
\]
\[
= \lambda \int_{\partial \Omega} u \, d\mathcal{H}^{N-1} - \int_{-\infty}^{+\infty} \nu ([u > t], \Omega) \, dt
\]
\[
= \lambda \int_{\partial \Omega} u \, d\mathcal{H}^{N-1} - \int_{\Omega} |Du|
\]
and (29) holds. \( \square \)

Taking \( \lambda = |\Omega| / \nu (\Omega) \) in the above theorem we obtain the following result.

**Corollary 1.** Let \( \Omega \) be a bounded connected open set in \( \mathbb{R}^N \), \( N \geq 2 \), with Lipschitz continuous boundary \( \partial \Omega \). If \( |\Omega| / \nu (\Omega) < 1 \), then the following are equivalent:

(i) \( \exists z \in X_2(\Omega), \|z\|_\infty \leq 1 \), such that\( \text{div}(z) = 1 \) in \( \mathcal{D}'(\Omega) \),
\[
[z, v] = \frac{|\Omega|}{\nu (\Omega)} \mathcal{H}^{N-1}\text{-a.e. on } \partial \Omega,
\]
\[ (33) \]
(ii) \[
\lambda_1(\Omega) = \frac{|\Omega|}{P(\Omega)},
\]

(iii) \[
\left|E - \frac{|\Omega|}{P(\Omega)} \mathcal{H}^{N-1}(\partial^* E \cap \partial \Omega)\right| \leq P(E, \Omega) \quad \forall E \in \mathcal{P}_f(\Omega).
\]

We do not know if the assumption \(\Omega\) connected in Corollary 1 is necessary. So, a natural question is the following: Is there a nonconnected open bounded set \(\Omega\) such that \(|\Omega|/P(\Omega) < 1\), verifying (33) and \(\lambda_1(\Omega) < |\Omega|/P(\Omega)\)?

There are open sets \(\Omega\) for which (33) holds and \(|\Omega|/P(\Omega) = 1\), as the following examples show.

**Example 1.** Let \(\Omega = B_R(0) \subset \mathbb{R}^N\) the ball in \(\mathbb{R}^N\) centered in 0 of radius \(R\). Then, if \(z(x) := x/N\), we have
\[
\text{div}(z) = 1 \quad \text{in } \mathcal{D}'(\Omega)
\]
and
\[
[z, v] = \frac{R}{N} = \frac{|\Omega|}{P(\Omega)} \mathcal{H}^{N-1}-\text{a.e. on } \partial \Omega.
\]

Moreover,
\[
\|z\|_\infty = \frac{R}{N} \leq 1 \iff \frac{|\Omega|}{P(\Omega)} \leq 1.
\]

**Example 2.** Let \(\Omega = B_R(0) \setminus B_r(0) \subset \mathbb{R}^N\) the annulus in \(\mathbb{R}^N\) centered in 0 of radius \(R\) and \(r\). Then, it is easy to see that if
\[
z(x) := \left[(R^{N-1} + r^{N-1}) - (R + r) \frac{r^{N-1}R^{N-1}}{\|x\|^N}\right] \frac{x}{N(R^{N-1} + r^{N-1})}
\]
we have
\[
\text{div}(z) = 1 \quad \text{in } \mathcal{D}'(\Omega)
\]
and
\[
[z, v] = \frac{|\Omega|}{P(\Omega)} \mathcal{H}^{N-1}-\text{a.e. on } \partial \Omega.
\]

Moreover,
\[
\|z\|_\infty \leq 1 \iff \frac{|\Omega|}{P(\Omega)} \leq 1.
\]
Remark 1. Motron in [12] proves that if \( \Omega = B_R(0) \) is the ball in \( \mathbb{R}^N \) centered in 0 of radius \( R \) or \( \Omega = B_R(0) \setminus B_r(0) \) the annulus in \( \mathbb{R}^N \) centered in 0 of radius \( R \) and \( r \), then

\[
\int_{\partial \Omega} |u| \leq \frac{P(\Omega)}{|\Omega|} \int_{\Omega} |u| + \int_{\Omega} |\nabla u| \quad \forall u \in W^{1,1}(\Omega)
\]  

(34)

and equality holds in (34) if and only if \( u \) is constant.

From (10) and (34), it follows that if \( \Omega = B_R(0) \subset \mathbb{R}^N \) or \( \Omega = B_R(0) \setminus B_r(0) \subset \mathbb{R}^N \), then

\[
\lambda_1(\Omega) = \begin{cases} \frac{|\Omega|}{P(\Omega)} & \text{if } \frac{|\Omega|}{P(\Omega)} \leq 1, \\ 1 & \text{if } \frac{|\Omega|}{P(\Omega)} > 1. \end{cases}
\]

Moreover, if \( |\Omega|/P(\Omega) \leq 1 \), then \( u = (1/P(\Omega))\chi_{\Omega} \) is a minimizer of the variational problem (1), being the only minimizer in the case \( |\Omega|/P(\Omega) = 1 \), and if \( |\Omega|/P(\Omega) > 1 \), the variational problem (1) does not have minimizer.

In the following example we show that there exists bounded connected open sets \( \Omega \), with \( |\Omega|/P(\Omega) < 1 \), for which \( \lambda_1(\Omega) < |\Omega|/P(\Omega) \).

Example 3. For \( \delta > 0 \) and \( 0 < \alpha \leq \pi/2 \), let

\[
\Omega_{\delta, \alpha} := B_1(0) \cup (B_{2+\delta}(0) \setminus B_2(0)) \\
\cup \left\{ (x, y) \in \mathbb{R}^+ \times \mathbb{R}^+: x^2 + y^2 \leq 2, \arctg \left( \frac{y}{x} \right) < \alpha \right\}.
\]

We have

\[
\frac{|\Omega_{\delta, \alpha}|}{P(\Omega_{\delta, \alpha})} = \frac{\pi + \pi(\delta^2 + 4\delta) + \frac{3}{2}\alpha}{10\pi + 2\pi\delta + 2 - 3\alpha}.
\]

Thus, for \( 0 < \delta \leq 1 \) and \( 0 < \alpha \leq \pi/2 \), we have

\[
\frac{|\Omega_{\delta, \alpha}|}{P(\Omega_{\delta, \alpha})} < 1.
\]

Now, if we take \( u := \chi_{B_{2+\delta}(0) \setminus B_2(0)} \),

\[
\lambda_1(\Omega_{\delta, \alpha}) \leq \frac{\int_{\Omega_{\delta, \alpha}} |Du| + \int_{\Omega_{\delta, \alpha}} |u|}{\int_{\partial \Omega_{\delta, \alpha}} |u|} = \frac{2\alpha + \pi(\delta^2 + 4\delta)}{8\pi + 2\pi\delta - 2\alpha}.
\]

Then, it is easy to see that for \( \delta \) and \( \alpha \) small enough, we get

\[
\lambda_1(\Omega_{\delta, \alpha}) < \frac{|\Omega_{\delta, \alpha}|}{P(\Omega_{\delta, \alpha})} < 1.
\]

In the next example we will see that even we can take \( \Omega \) convex.
Example 4. Let $\Omega$ be the set in $\mathbb{R}^2$ with the boundary isosceles triangle with height $k$, base of length $2a$ and the two equal sides of length $l$. Let $t$ the angle between the height and one of the equal side (see Fig. 1). Then,

$$\frac{|\Omega|}{P(\Omega)} = \frac{ak}{2(a + l)} = \frac{ak}{2a(a + a/\sin t)} = \frac{k \sin t}{2(1 + \sin t)}.$$ 

Let $E \subset \Omega$ the set with boundary the isosceles triangle with height $k - r$, base of length $2b$ and the two equal sides of length $\tilde{l}$. Then, if $u := \chi_E$, we have

$$\lambda_1(\Omega) \leq \frac{\int_{\partial \Omega} |D u| + \int_{\overline{\Omega}} |u|}{\int_{\overline{\Omega}} |u|} = \frac{2b + b(k - r)}{2\tilde{l}} = \frac{b(k + 2 - r)}{2b/\sin t} = \frac{\sin t}{2} (k + 2 - r).$$

Hence,

$$\lambda_1(\Omega) < \frac{|\Omega|}{P(\Omega)} < 1$$

if

$$k < \min \left\{ (r - 2) \frac{1 + \sin t}{\sin t}, 2 \frac{1 + \sin t}{\sin t} \right\}.$$ 

Now, obviously, we can find $k, r$, and $t$ satisfying the above inequality, and consequently, we can obtain a convex, bounded open set $\Omega$ satisfying

$$\lambda_1(\Omega) < \frac{|\Omega|}{P(\Omega)} < 1.$$ 

The next example show the necessity of the assumption $\Omega$ connected in Lemma 2.

Example 5. For $0 < \rho < r$ and $\delta > 0$, let

$$\Omega_{\rho, r, \delta} := B_{\rho}(0) \cup (B_{r+\delta}(0) \setminus B_r(0)) \subset \mathbb{R}^2.$$
We have
\[
\frac{|\Omega_{\rho,r,\delta}|}{P(\Omega_{\rho,r,\delta})} = \frac{\delta^2 + \rho^2 + 2r\delta}{2(2r + \delta + \rho)}.
\]
If we take \(u := \chi_{B_\rho(0)}\) and \(v := \chi_{B_{r+\delta}(0)\setminus B_r(0)}\), then
\[
\Phi(u) := \frac{\int_{\Omega_{\rho,r,\delta}} |Du| + \int_{\Omega_{\rho,r,\delta}} |u|}{\int_{\partial \Omega_{\rho,r,\delta}} |u|} = \frac{\rho}{2}
\]
and
\[
\Phi(v) := \frac{\int_{\Omega_{\rho,r,\delta}} |Dv| + \int_{\Omega_{\rho,r,\delta}} |v|}{\int_{\partial \Omega_{\rho,r,\delta}} |v|} = \frac{\delta}{2}.
\]
Suppose that \(0 < \rho < \delta \leq 2\). If we consider the vector field \(z\) in \(\Omega_{\rho,r,\delta}\) defined by
\[
z(x,y) := \begin{cases} \frac{\delta(x,y)}{2\rho} & \text{if } (x,y) \in B_\rho(0), \\ \frac{\delta - (\delta + 2r)}{2(2r + \delta)} \frac{r(r + \delta)}{\|x,y\|} & \text{if } (x,y) \in B_{r+\delta}(0) \setminus B_r(0), \end{cases}
\]
we have \(\|z\|_\infty \leq 1\),
\[
\text{div}(z) = \tau \quad \text{in } \mathcal{D}'(\Omega_{\rho,r,\delta})
\]
with
\[
\tau = \frac{\delta}{\rho} \chi_{\partial B_\rho(0)} + \chi_{B_{r+\delta}(0)\setminus B_r(0)}
\]
and
\[
[z,v] = \frac{\delta}{2} H^1\text{-a.e. on } \partial \Omega_{\rho,r,\rho}.
\]
Now,
\[
\lambda_1(\Omega) \leq \Phi(u) = \frac{\rho}{2}.
\]
Hence,
\[
\lambda_1(\Omega) \leq \frac{\delta}{2}.
\]
Consequently, in general, Lemma 2 it is not true if \(\Omega\) is not connected.
If \(\delta = \rho\), we have
\[
\lambda_1(\Omega_{\rho,r,\rho}) \leq \Phi(u) = \Phi(v) = \Phi(\chi_{\Omega_{\rho,r,\rho}}) = \frac{|\Omega_{\rho,r,\rho}|}{P(\Omega_{\rho,r,\rho})} = \frac{\rho}{2}.
\]
Suppose that $\rho \leq 1$. Then, if we consider the vector field $z$ in $\Omega_{\rho,r,\rho}$ defined by

$$z(x, y) := \begin{cases} (x, y) & \text{if } (x, y) \in B_\rho(0), \\ 0 & \text{if } (x, y) \in B_{r+\rho}(0) \setminus B_r(0), \end{cases}$$

we have

$$\text{div}(z) = \mathcal{Z}_{B_\rho(0)} \quad \text{in } \mathcal{D}'(\Omega_{\rho,r,\rho})$$

and

$$[z, v] = \frac{\rho}{2} \mathcal{Z}_{\partial B_\rho(0)} \mathcal{H}^1 \text{-a.e. on } \partial \Omega_{\rho,r,\rho}.$$

Therefore, $u$ is a solution of the problem

$$\begin{cases} \mathcal{A}_1 w := \text{div} \left( \frac{Du}{|Du|} \right) = \frac{w}{|w|} & \text{in } \Omega_{\rho,r,\rho}, \\ \frac{Du}{|Du|} \cdot \nu = \frac{\rho}{2} \frac{w}{|w|} & \text{on } \partial \Omega_{\rho,r,\rho}. \end{cases} \quad (35)$$

Now, if we consider the vector field $z$ in $\Omega_{\rho,r,\rho}$ defined by

$$z(x, y) := \begin{cases} 0 & \text{if } (x, y) \in B_\rho(0), \\ \rho - (\rho + 2r) \frac{r(r + \rho)}{\| (x, y) \|} & \text{if } (x, y) \in B_{r+\rho}(0) \setminus B_r(0), \end{cases}$$

we have

$$\text{div}(z) = \mathcal{Z}_{B_{r+\rho}(0) \setminus B_r(0)} \quad \text{in } \mathcal{D}'(\Omega_{\rho,r,\rho})$$

and

$$[z, v] = \frac{\rho}{2} \mathcal{Z}_{\partial (B_{r+\rho}(0) \setminus B_r(0))} \mathcal{H}^1 \text{-a.e. on } \partial \Omega_{\rho,r,\rho}.$$

Therefore, $v$ is also a solution of problem (35). Moreover, in this case also $\mathcal{Z}_{\Omega_{\rho,r,\rho}}$ is a solution of problem (35).

**Problem.** Is $\lambda_1(\Omega_{\rho,r,\rho}) = \rho/2$?

The next example shows that there are bounded connected open sets, $\Omega$ for which $\lambda_1(\Omega) < 1$ and $|\Omega|/P(\Omega) > 1$.

**Example 6.** Let $\Omega := [-k, 0] \times [0, k] \cup [0, k] \times [0, \delta] \cup [\delta, 0] \times [0, 1] \cup [0, \delta] \subset \mathbb{R}^2$ be. Then

$$\frac{|\Omega|}{P(\Omega)} = \frac{k^2 + \delta}{4k + 2} > 1 \quad \text{if } k > 2 + \sqrt{6 - \delta}.$$  

Now, if we take $u := \chi_{[0,1] \times [0, \delta]}$, we have

$$\lambda_1(\Omega) \leq \frac{\int_{\Omega} |Du| + \int_{\partial \Omega} |u|}{\int_{\partial \Omega} |u|} = \frac{2\delta}{2 + \delta} < 1 \iff 0 < \delta < 2.$$

Therefore, for instance, if $\delta = 1$ and $k = 5$, we have $|\Omega|/P(\Omega) > 1$ and $\lambda_1(\Omega) < 1$.  

4. Proof of Theorem 2

Let $\Omega$ be a bounded connected open set in $\mathbb{R}^N$, $N \geq 2$, with Lipschitz continuous boundary $\partial \Omega$. By Lemma 2, if we assume that $\lambda_1(\Omega) < 1$, there exist $\bar{z} \in X_2(\Omega)$, $\|\bar{z}\|_\infty \leq 1$, and $\|\text{div}(\bar{z})\|_\infty \leq 1$ such that

$$[\bar{z}, v] = \lambda_1(\Omega) \mathcal{H}^{N-1} \text{-a.e. on } \partial \Omega.$$  

We recall the following definition.

**Definition 2.** Let $\Omega$ be a bounded open set in $\mathbb{R}^N$ with Lipschitz continuous boundary $\partial \Omega$. We shall say that $\Omega$ has the trace-property if there exists a vector field $z_\Omega \in L^\infty(\Omega, \mathbb{R}^N)$, with $\|z_\Omega\|_\infty \leq 1$ such that $\text{div}(z_\Omega) \in L^\infty(\Omega)$ and

$$[z_\Omega, v] = \lambda_1(\Omega) \mathcal{H}^{N-1} \text{-a.e. on } \partial \Omega.$$  

By the above, we have that every bounded connected open set $\Omega$ in $\mathbb{R}^N$, $N \geq 2$, with Lipschitz continuous boundary and $\lambda_1(\Omega) < 1$, has the trace-property. Also, as a consequence of Examples 1, 2, and Remark 1, we have that if $\Omega = B_R(0) \subset \mathbb{R}^N$ or $\Omega = B_R(0) \setminus B_r(0) \subset \mathbb{R}^N$, and $|\Omega|/P(\Omega) \leq 1$, then $\Omega$ has the trace-property. Therefore there exists $\Omega$ with $\lambda_1(\Omega) = 1$ satisfying the trace-property.

Let us present some examples of planar domains that verify $\lambda_1(\Omega) < 1$.

**Example 7.** Assume that $\Omega \subset \mathbb{R}^2$ is a bounded open set such that there exists some point, $x_0 \in \partial \Omega$ (we may assume $x_0 = 0$), with curvature of the boundary at that point greater than 2, we will show that in this case $\lambda_1(\Omega) < 1$. So, let us assume that locally near the origin $\Omega$ can be described as $\Omega \cap B_r(0) = \{(x, y): y > ax^2\}$. As we are assuming that the curvature at the origin is greater than 2 we have $a > 1$. Let us consider the function $u_\varepsilon = \mathcal{I}_{\Omega \cap \{y < \varepsilon\}}$ as a test function to estimate $\lambda_1$. We have

$$\lambda_1(\Omega) \leq \frac{\int_{\Omega} |Du_\varepsilon| + \int_{\Omega} |u_\varepsilon|}{\int_{\partial \Omega} |u_\varepsilon|} = \frac{\sqrt{\varepsilon/a} + \int_0^\sqrt{\varepsilon/a} (\varepsilon - as^2) \, ds}{\int_0^\sqrt{\varepsilon/a} \sqrt{1 + (2a)^2s^2} \, ds} = \frac{\sqrt{\varepsilon/a} + \frac{2}{3} \sqrt{\varepsilon/a}}{\int_0^\sqrt{\varepsilon/a} (1 + 2a^2s^2 + O(s^3)) \, ds} < 1$$

if $\varepsilon$ is small enough.

Remark that if $\Omega = B_R(0)$ we have that the curvature is $1/R$ and, by Example 1, we have $\lambda_1(B_R(0)) < 1$ if and only if $R < 2$. Hence, some restriction on the curvature must be imposed.

Next we prove that on every domain that enjoys the trace-property the best Sobolev trace constant is attained, Theorem 2.

**Proof of Theorem 2.** First let us see that the function $u$ obtained in (17) is a minimizer of the variational problem (1). Let $z_\Omega$ be the vector field given in Definition 2. Then, by (18),
we have
\[
\int_\Omega (z_\Omega, Du) = \lim_{p \searrow 1} \int_\Omega (z_\Omega, \nabla u_p) = \lim_{p \searrow 1} \left( - \int_\Omega \text{div}(z_\Omega) u_p + \int_\partial \Omega [z_\Omega, v] u_p \right) \\
= - \int_\Omega \text{div}(z_\Omega) u + \lambda_1(\Omega) = \int_\Omega (z_\Omega, Du) - \lambda_1(\Omega) \int_\partial \Omega u + \lambda_1(\Omega)
\]
from where it follows that
\[
\int_{\partial \Omega} |u| = 1. \tag{36}
\]
Therefore, by (20) we get that \(u\) is a minimizer of (1). Moreover, by (19), it follows that
\[
\lim_{p \searrow 1} \int_{\Omega} |\nabla u_p| = \lambda_1(\Omega) - \int_{\Omega} |u| = \int_{\Omega} |Du|.
\tag{37}
Hence \(u_p \to u\) respect to the strict convergence, and consequently
\[
u_p \to u \quad \text{in } L^1(\partial \Omega) \text{ as } p \searrow 1. \tag{38}
\]
Let us see now that the function \(u\) is a solution of problem (21). By Hölder’s inequality, we have
\[
\int_{\Omega} |u_p|^{p-1} \leq |\Omega|^{1/p} \left( \int_{\Omega} |u_p|^p \right)^{(p-1)/p} \leq |\Omega|^{1/p} \lambda_p(\Omega)^{(p-1)/p} \leq M_1. \tag{39}
\]
On the other hand, if \(E\) is a measurable subset of \(\Omega\) with \(|E| < 1\), we have
\[
\left| \int_{E} |u_p|^{p-2} u_p \right| \leq \int_{E} |u_p|^{p-1} \leq M_2 |E|^{1/p}. \tag{40}
\]
By (39) and (40), it follows that \(\{ |u_p|^{p-2} u_p : 1 < p \leq 2 \}\) is a weakly relatively compact subset of \(L^1(\Omega)\). Hence, we can assume that there exists \(\tau \in L^1(\Omega)\) such that
\[
|u_p|^{p-2} u_p \to \tau \quad \text{weakly in } L^1(\Omega) \text{ as } p \searrow 1. \tag{41}
\]
In a similar way, it is easy to see that there exists \(z \in L^1(\Omega, \mathbb{R}^N)\) such that
\[
|\nabla u_p|^{p-2} \nabla u_p \to z \quad \text{weakly in } L^1(\Omega, \mathbb{R}^N) \text{ as } p \searrow 1. \tag{42}
\]
Now, given \(\varphi \in \mathcal{D}(\Omega)\), from (41) and (42), it follows that
\[
\langle \text{div}(z), \varphi \rangle = - \int \Omega z \cdot \nabla \varphi = - \lim_{p \searrow 1} \int \Omega |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \varphi \\
= \lim_{p \searrow 1} \int \Omega \text{div}(|\nabla u_p|^{p-2} \nabla u_p) \varphi = \lim_{p \searrow 1} \int |u_p|^{p-2} u_p \varphi = \int \Omega \tau \varphi.
\]
Thus,
\[
\text{div}(z) = \tau \quad \text{in } \mathcal{D}'(\Omega). \tag{43}
\]
We claim now that
\[ \|z\|_{\infty} \leq 1. \]  
(44)

In fact, for any \( k > 0 \), let
\[ B_{p,k} := \{ x \in \Omega : |\nabla u_p(x)| > k \}. \]

As above, there exists some \( g_k \in L^1(\Omega, \mathbb{R}^N) \) such that
\[ |\nabla u_p|^{p-2}\nabla u_p \chi_{B_{p,k}} \rightharpoonup g_k \quad \text{weakly in } L^1(\Omega, \mathbb{R}^N) \text{ as } p \searrow 1. \]  
(45)

Now, since \[ |B_{p,k}| = \int_{\Omega} \chi_{B_{p,k}}(x) \, dx \leq \int_{\Omega} \frac{|\nabla u_p(x)|^p}{kp} \, dx \leq \frac{\lambda_p(\Omega)}{kp} \]
for any \( \phi \in L^{\infty}(\Omega) \) with \( \|\phi\|_{\infty} \leq 1 \), we have
\[
\left| \int_{\Omega} |\nabla u_p|^{p-2}\nabla u_p \cdot \phi \chi_{B_{p,k}} \right| \leq \left( \int_{\Omega} |\nabla u_p|^p \right)^{(p-1)/p} |B_{p,k}|^{1/p} \\
\leq \lambda_p(\Omega)^{(p-1)/p} \left( \frac{\lambda_p(\Omega)}{kp} \right)^{1/p} = \frac{\lambda_p(\Omega)}{k}.
\]

Letting \( p \searrow 1 \), we get that
\[ \int_{\Omega} |g_k| \leq \frac{\lambda_1(\Omega)}{k} \quad \text{for every } k > 0. \]  
(46)

On the other hand, since we have
\[ |\nabla u_p|^{p-2}\nabla u_p \chi_{\Omega \setminus B_{p,k}} \rightharpoonup k^{-1} \quad \text{for any } p > 1, \]
letting \( p \searrow 1 \), we obtain that there exists some \( f_k \in L^1(\Omega, \mathbb{R}^N) \) with \( \|f_k\|_{\infty} \leq 1 \) such that
\[ |\nabla u_p|^{p-2}\nabla u_p \chi_{\Omega \setminus B_{p,k}} \rightharpoonup f_k \quad \text{weakly in } L^1(\Omega, \mathbb{R}^N) \text{ as } p \searrow 1. \]  
(47)

Hence, for any \( k > 0 \), we may write \( z = f_k + g_k \), with \( \|f_k\|_{\infty} \leq 1 \) and \( g_k \) satisfying (46).

From where (44) follows.

Since \( u_p \to u \) a.e. in \( \Omega \), by (41) it follows that
\[ \tau u = |u| \quad \text{a.e. in } \Omega \quad \text{and} \quad \|\tau\|_{\infty} \leq 1. \]

On the other hand, given a measurable subset \( E \subset \partial \Omega \), by Hölder’s inequality we have
\[
\int_E u_p^{-1} \, d\mathcal{H}^{N-1} \leq \left( \int_{\partial \Omega} u_p \, d\mathcal{H}^{N-1} \right)^{p-1} \mathcal{H}^{N-1}(E)^2 - p \leq \mathcal{H}^{N-1}(E)^{2-p}
\]
from where it follows that \( \{u_p^{p-1} : 1 < p \leq 2\} \) is a weakly relatively compact subset of \( L^1(\partial \Omega) \). Hence, we can assume that there exists \( \theta \in L^1(\partial \Omega) \), such that
\[ u_p^{p-1} \to \theta \quad \text{weakly in } L^1(\partial \Omega). \]  
(48)
Moreover, by (48), (14) and applying Fatou’s Lemma, it is easy to see that
\[ \|\theta\|_\infty \leq 1. \] (49)
On the other hand, by (38) and (48), we get
\[ \theta u = |u| H^{N-1} \text{-a.e. on } \partial \Omega. \] (50)
Now, since \( u_p \) is a weak solution of (15), having in mind (41), (9) and (48), if \( w \in W^{1,1}(\Omega) \cap C(\Omega) \cap L^\infty(\Omega) \), we have
\[
\langle z, w \rangle := \int_\Omega \text{div}(z)w + \int_\Omega z \cdot \nabla w = \int_\Omega \tau w + \int_\Omega z \cdot \nabla w
\]
\[
= \lim_{p \uparrow 1} \int_\Omega u_p^{p-1}w + \int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \nabla w
\]
\[
= \lim_{p \uparrow 1} \int_\Omega u_p^{p-1}w - \int_\Omega \text{div}(|\nabla u_p|^{p-2}\nabla u_p)w + \int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot vw
\]
\[
= \lim_{p \uparrow 1} \lambda_p(\Omega) \int_{\partial \Omega} u_p^{p-1}w = \lambda_1(\Omega) \int_{\partial \Omega} \theta w.
\]
Thus, having in mind the definition of the weak trace on \( \partial \Omega \) of the normal component of \( z \) given in [5], we get
\[ [z, v] = \lambda_1(\Omega) \theta. \] (51)
Finally, since
\[
\int_\Omega |Du| = \lambda_1(\Omega) - \int_\Omega |u| = \lambda_1(\Omega) - \int_\Omega \tau u
\]
\[
= \lambda_1(\Omega) - \int_\Omega \text{div}(z)u = \lambda_1(\Omega) + \int_\Omega (z, Du) - \int_{\partial \Omega} [z, v]u
\]
\[
= \lambda_1(\Omega) + \int_\Omega (z, Du) - \lambda_1(\Omega) \int_{\partial \Omega} \theta u = \int_\Omega (z, Du),
\]
we have \( (z, Du) = |Du| \) as measures. \( \square \)

**Remark 2.** Let us remark that as a consequence of Theorem 1 in [8] it is obtained the above theorem in the particular case that \( \Omega \) is a bounded open subset of \( \mathbb{R}^N \), whose boundary \( \partial \Omega \) is at least piecewise \( C^2 \) and \( \lambda_1(\Omega) < 1 \).

**Remark 3.** Note that in the above theorem we have proved that if \( u \) is the limit as \( p \searrow 1 \) of the minimizers \( u_p \) of the variational problem
\[
\inf \left\{ \int_\Omega |u|^p + \int_\Omega |\nabla u|^p : u \in W^{1,p}(\Omega), \int_{\partial \Omega} |u| = 1 \right\}
\] (52)
then, if \( \int_{\partial \Omega} |u| = 1 \), we have that \( u \) is a minimizer of the variational problem (1) and a solution of problem (21).
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