Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Nonlinear Analysis 59 (2004) 1125–1145
Nonlinear Analysis
www.elsevier.com/locate/na

The best constant for the Sobolev trace embedding from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$

F. Andreu[a], J.M. Mazón[a], J.D. Rossi[b,*]

[a] *Departamento de Análisis Matemático, Universitat de Valencia, Spain*
[b] *Dept. de Matemáticas, Universidad Católica de Chile, Spain*

Received 3 April 2004; accepted 29 July 2004

Abstract

In this paper we study the best constant, $\lambda_1(\Omega)$ for the trace map from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$. We show that this constant is attained in $BV(\Omega)$ when $\lambda_1(\Omega) < 1$. Moreover, we prove that this constant can be obtained as limit when $p \searrow 1$ of the best constant of $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$. To perform the proofs we will look at Neumann problems involving the 1-Laplacian, $\Delta_1(u) = \mathrm{div}(Du/|Du|)$.

MSC: 35J65; 35B33

Keywords: Sobolev trace embedding; Functions of bounded variation

1. Introduction

Let Ω be a bounded set in $\mathbb{R}^N$ with Lipschitz continuous boundary $\partial\Omega$. Of importance in the study of boundary value problems for differential operators in Ω are the Sobolev trace inequalities. In particular, $W^{1,1}(\Omega) \hookrightarrow L^1(\partial\Omega)$ and hence the following inequality holds:

$$\lambda \|u\|_{L^1(\partial\Omega)} \leqslant \|u\|_{1,1},$$

* Corresponding author.
E-mail address: jrossi@riemann.mat.puc.cl (J.D. Rossi).

0362-546X/$ - see front matter © 2004 Published by Elsevier Ltd.
doi:10.1016/j.na.2004.07.051

for all $u \in W^{1,1}(\Omega)$. The best constant for this embedding is the largest λ such that the above inequality holds, that is,

$$\lambda_1(\Omega) = \inf \left\{ \int_\Omega |u| + \int_\Omega |\nabla u| : \; u \in W^{1,1}(\Omega), \int_{\partial\Omega} |u| = 1 \right\}. \tag{1}$$

Our main interest in this paper is to study the dependence of the best constant $\lambda_1(\Omega)$ and extremals (functions where the constant is attained) on the domain. A related problem was studied by Demengel in [8] (see Remark 2). We remark that the existence of extremals is not trivial, due to the lack of compactness of the embedding.

For $1 < p \leqslant N$, let us consider the variational problem

$$\inf \left\{ \int_\Omega |u|^p + \int_\Omega |\nabla u|^p : \; u \in W^{1,p}(\Omega), \int_{\partial\Omega} |u|^p = 1 \right\}. \tag{2}$$

If we denote by $\lambda_p(\Omega)$ the above infimum, we have that

$$\lambda_p(\Omega) = \inf \left\{ \frac{\int_\Omega |u|^p + \int_\Omega |\nabla u|^p}{\int_{\partial\Omega} |u|^p} : u \not\equiv 0 \text{ on } \partial\Omega, \; u \in W^{1,p}(\Omega) \right\} \tag{3}$$

is the best constant for the trace map from $W^{1,p}(\Omega)$ into $L^p(\partial\Omega)$. Due to the compactness of the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$, it is well known (see for instance [10]) that problem (3) has a minimizer in $W^{1,p}(\Omega)$. These extremals are weak solutions of the following problem:

$$\begin{cases} \Delta_p u = |u|^{p-2} u & \text{in } \Omega, \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} = \lambda |u|^{p-2} u & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian, $\partial/\partial\nu$ is the outer unit normal derivative and if we use the normalization $\|u\|_{L^p(\partial\Omega)} = 1$, one can check that $\lambda = \lambda_p(\Omega)$, see [11].

Our first result says that $\lambda_1(\Omega)$ is the limit as $p \searrow 1$ of $\lambda_p(\Omega)$ and provides a bound for $\lambda_1(\Omega)$.

Theorem 1. *We have that*

$$\lim_{p \searrow 1} \lambda_p(\Omega) = \lambda_1(\Omega)$$

and

$$\lambda_1(\Omega) \leqslant \min \left\{ \frac{|\Omega|}{P(\Omega)}, 1 \right\},$$

where $P(\Omega)$ stands for the perimeter of Ω.

Therefore, it seems natural to search for an extremal for $\lambda_1(\Omega)$ as the limit of extremals for $\lambda_p(\Omega)$ when $p \searrow 1$. Formally, if we take limit as $p \searrow 1$ in Eq. (4), we get

$$\begin{cases} \Delta_1 u := \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = \dfrac{u}{|u|} & \text{in } \Omega, \\ \dfrac{Du}{|Du|} \cdot \nu = \lambda_1(\Omega)\dfrac{u}{|u|} & \text{on } \partial\Omega. \end{cases} \tag{5}$$

Hence we will look at Neumann problems involving the 1-Laplacian, $\Delta_1(u)=\operatorname{div}(Du/|Du|)$ in the context of bounded variation functions (the natural context for this type of problems). To our knowledge the results obtained here have independent interest.

We shall say that Ω has the *trace-property* if there exists a vector field $z_\Omega \in L^\infty(\Omega, \mathbb{R}^N)$, with $\|z_\Omega\|_\infty \leqslant 1$ such that $\operatorname{div}(z_\Omega) \in L^\infty(\Omega)$ and

$$[z_\Omega, \nu] = \lambda_1(\Omega)\mathscr{H}^{N-1}\text{-a.e.} \quad \text{on } \partial\Omega.$$

Our main result states that for any domain having the trace property, the best Sobolev trace constant, $\lambda_1(\Omega)$, is attained by a function in $L^1(\Omega)$ whose derivatives in the sense of distributions are bounded measures on Ω, that is a function with bounded variation.

Theorem 2. *Let Ω be a bounded open set in $\mathbb{R}^N$ with the trace-property. Then, there exists a nonnegative function of bounded variation which is a minimizer of the variational problem* (1) *and a solution of problem* (5).

We will see that every bounded domain Ω with $\lambda_1(\Omega) < 1$, has the trace-property. Hence we have proved that, if $\lambda_1(\Omega) < 1$ then there exists an extremal. Moreover, using results from [12], we can find examples of domains (a ball or an annulus) such that $\lambda_1(\Omega) = 1$ and verify the trace property (and therefore they have extremals). We also prove that every planar domain Ω with a point of curvature greater than 2 verifies $\lambda_1(\Omega) < 1$.

Organization of the paper. In Section 2 we collect some preliminary results and prove Theorem 1. In Section 3 we deal with the Neumann problem for the equation $\operatorname{div}(z) = 1$. Finally, in Section 4 we use these results to prove the main theorem, Theorem 2. Throughout this paper C and c denote constants that may change from one line to another.

2. Preliminary results. Proof of Theorem 1

Let us begin with some notation and definitions. Recall that a function $u \in L^1(\Omega)$ whose partial derivatives in the sense of distributions are measures with finite total variation in Ω is called a *function of bounded variation*. The class of such functions will be denoted by $BV(\Omega)$. Thus $u \in BV(\Omega)$ if and only if there are Radon measures $\mu_1, \dots, \mu_N$ defined in Ω with finite total mass in Ω and

$$\int_\Omega u D_i \varphi = -\int_\Omega \varphi \,\mathrm{d}\mu_i$$

for all $\varphi \in C_0^\infty(\Omega)$, $i = 1, \dots, N$. Thus the gradient of u is a vector valued measure with finite total variation

$$|Du|(\Omega) = \sup\left\{\int_\Omega u \,\mathrm{div}\,\varphi \colon\ \varphi \in C_0^\infty(\Omega, \mathbb{R}^N),\ |\varphi(x)| \leqslant 1 \text{ for } x \in \Omega\right\}.$$

The set of bounded variation functions, $BV(\Omega)$, is a Banach space endowed with the norm

$$\|u\|_{BV(\Omega)} := \|u\|_1 + |Du|(\Omega) = \int_\Omega |u| + \int_\Omega |Du|.$$

A measurable set $E \subset \mathbb{R}^N$ is said to be of *finite perimeter* in Ω if $\chi_E \in BV(\Omega)$, in this case the perimeter of E in Ω is defined as $P(E, \Omega) := |D\chi_E|$. We shall use the notation $P(E) := P(E, \mathbb{R}^N)$, and $\mathscr{P}_f(\Omega)$ shall denote the set of all subset of Ω of finite perimeter. For a set of finite perimeter E one can define the essential boundary $\partial^* E$, which is countably $(N-1)$ rectifiable with finite $\mathscr{H}^{N-1}$ measure, and compute the unit normal $\nu^E(x)$ at $\mathscr{H}^{N-1}$ almost all points x of $\partial^* E$, where $\mathscr{H}^{N-1}$ is the $(N-1)$ dimensional Hausdorff measure, and $|D\chi_E|$ coincides with the restriction of $\mathscr{H}^{N-1}$ to $\partial^* E$.

It is well known (see, [1,9] or [13]) that for a given function $u \in BV(\Omega)$ there exists a sequence $u_n \in W^{1,1}(\Omega)$ such that u_n strict converges to u, i.e.,

$$u_n \to u \text{ in } L^1(\Omega) \text{ and } \int_\Omega |\nabla u_n| \to \int_\Omega |Du|.$$

Moreover, there is a trace operator $\tau\colon\ BV(\Omega) \to L^1(\partial\Omega)$ such that

$$\|\tau(u)\|_{L^1(\partial\Omega)} \leqslant C\|u\|_{BV(\Omega)} \quad \forall u \in BV(\Omega) \tag{6}$$

for some constant C depending only on Ω. The trace operator τ is continuous between $BV(\Omega)$, endowed with the topology induced by the strict convergence, and $L^1(\partial\Omega)$. In the sequel we write $\tau(u) = u$.

From the above results, we get

$$\lambda_1(\Omega) = \inf\left\{\frac{\int_\Omega |u| + \int_\Omega |Du|}{\int_{\partial\Omega} |u|}\colon\ u \not\equiv 0 \text{ on } \partial\Omega,\ u \in BV(\Omega)\right\}.$$

Let us recall a generalized Green's formula given in [5] (see also [4]). For $1 \leqslant p < \infty$, let

$$X_p(\Omega) = \{z \in L^\infty(\Omega, \mathbb{R}^N)\colon\ \mathrm{div}(z) \in L^p(\Omega)\}.$$

If $z \in X_p(\Omega)$ and $w \in BV(\Omega) \cap L^{p'}(\Omega)$ the functional $(z, Dw)\colon\ C_0^\infty(\Omega) \to \mathbb{R}$ is defined by the formula

$$\langle (z, Dw), \varphi\rangle = -\int_\Omega w\varphi\,\mathrm{div}(z) - \int_\Omega wz \cdot \nabla\varphi.$$

Then (z, Dw) is a Radon measure in Ω,

$$\int_\Omega (z, Dw) = \int_\Omega z \cdot \nabla w$$

for all $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and

$$\left| \int_B (z, Dw) \right| \leqslant \int_B |(z, Dw)| \leqslant \|z\|_\infty \int_B |Dw| \tag{7}$$

for any Borel set $B \subseteq \Omega$.

In [5], a weak trace on $\partial\Omega$ of the normal component of $z \in X_p(\Omega)$ is defined. Concretely, it is proved that there exists a linear operator $\gamma\colon X_p(\Omega) \to L^\infty(\partial\Omega)$ such that

$$\|\gamma(z)\|_\infty \leqslant \|z\|_\infty,$$

$$\gamma(z)(x) = z(x) \cdot v(x) \quad \text{for all } x \in \partial\Omega \text{ if } z \in C^1(\overline{\Omega}, \mathbb{R}^N).$$

We shall denote $\gamma(z)(x)$ by $[z, v](x)$. Moreover, the following *Green's formula*, relating the function $[z, v]$ and the measure (z, Dw), for $z \in X_p(\Omega)$ and $w \in BV(\Omega) \cap L^{p'}(\Omega)$, is established:

$$\int_\Omega w \operatorname{div}(z)\, dx + \int_\Omega (z, Dw) = \int_{\partial\Omega} [z, v] w \, d\mathscr{H}^{N-1}. \tag{8}$$

Now we can prove Theorem 1.

Proof of Theorem 1. We have to prove that

$$\lim_{p \searrow 1} \lambda_p(\Omega) = \lambda_1(\Omega) \tag{9}$$

and

$$\lambda_1(\Omega) \leqslant \min\left\{ \frac{|\Omega|}{P(\Omega)}, 1 \right\}. \tag{10}$$

Since

$$\lambda_p(\Omega) \|u\|^p_{L^p(\partial\Omega)} \leqslant \|u\|^p_{1,p} \quad \forall u \in W^{1,p}(\Omega)$$

if we set

$$\lambda^* := \limsup_{p \searrow 1} \lambda_p(\Omega)$$

we have

$$\lambda^* \|u\|_{L^1(\partial\Omega)} \leqslant \|u\|_{1,1} \quad \forall u \in W^{1,1}(\Omega)$$

from where it follows that $\lambda^* \leqslant \lambda_1(\Omega)$, that is

$$\limsup_{p \searrow 1} \lambda_p(\Omega) \leqslant \lambda_1(\Omega). \tag{11}$$

Let v_p be a minimizer of problem (3). Then, if $u_p := a_p v_p$, with a_p satisfying

$$a_p = \left(\int_{\partial\Omega} |v_p| \right)^{1/(p-1)} \left(\int_{\partial\Omega} |v_p|^p \right)^{1/(1-p)},$$

we get

$$\int_{\partial\Omega} |u_p| = \int_{\partial\Omega} |u_p|^p$$

and consequently

$$\lambda_p(\Omega) = \frac{\int_\Omega |u_p|^p + \int_\Omega |\nabla u_p|^p}{\int_{\partial\Omega} |u_p|}. \tag{12}$$

Applying Hölder's inequality and (12), we have

$$\begin{aligned}
\lambda_1(\Omega) &\leqslant \frac{\int_\Omega |u_p| + \int_\Omega |\nabla u_p|}{\int_{\partial\Omega} |u_p|} \leqslant |\Omega|^{1/p'} \frac{\left(\int_\Omega \left(|u_p| + |\nabla u_p|\right)^p\right)^{1/p}}{\int_{\partial\Omega} |u_p|} \\
&\leqslant |\Omega|^{1/p'} 2^{(p-1)/p} \frac{\left(\int_\Omega |u_p|^p + \int_\Omega |\nabla u_p|^p\right)^{1/p}}{\int_{\partial\Omega} |u_p|} \\
&= |\Omega|^{1/p'} 2^{(p-1)/p} \frac{\lambda_p(\Omega)^{1/p}}{\left(\int_{\partial\Omega} |u_p|\right)^{1/p'}}.
\end{aligned}$$

Hence

$$\lambda_p(\Omega) \geqslant \lambda_1(\Omega)^p \frac{2^{1-p}}{|\Omega|^{p/p'}} \left(\int_{\partial\Omega} |u_p|\right)^{p/p'}$$

from where it follows that

$$\liminf_{p \searrow 1} \lambda_p(\Omega) \geqslant \lambda_1(\Omega). \tag{13}$$

Now, (9) follows from (11) and (13).

Taking $u = \chi_\Omega$, we obtain that

$$\lambda_1(\Omega) \leqslant \frac{\int_\Omega |\chi_\Omega| + \int_\Omega |D\chi_\Omega|}{\int_{\partial\Omega} |\chi_\Omega|} = \frac{|\Omega|}{P(\Omega)}.$$

On the other hand, if $\Omega_\varepsilon := \{x \in \Omega\colon\ d(x, \partial\Omega) < \varepsilon\}$, we have

$$\lambda_1(\Omega) \leqslant \frac{\int_\Omega |\chi_{\Omega_\varepsilon}| + \int_\Omega |D\chi_{\Omega_\varepsilon}|}{\int_{\partial\Omega} |\chi_{\Omega_\varepsilon}|} = \frac{|\Omega_\varepsilon| + P(\Omega_\varepsilon, \Omega)}{P(\Omega)}.$$

Hence, taking $\varepsilon \to 0^+$, it follows that $\lambda_1(\Omega) \leqslant 1$. Therefore, (10) holds. □

It is well known (see for instance [11]) that, for every $p > 1$, there exist an extremal for the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$ (this embedding is compact). This is a solution $0 \leqslant u_p \in W^{1,p}(\Omega)$ of the equation

$$\lambda_p(\Omega) = \int_\Omega |u_p|^p + \int_\Omega |\nabla u_p|^p, \quad \int_{\partial\Omega} |u_p| = 1, \tag{14}$$

such that $u_p > 0$ and satisfies in the weak sense

$$\begin{cases} \Delta_p u_p := \operatorname{div}(|\nabla u_p|^{p-2}\nabla u_p) = |u_p|^{p-2}u_p & \text{in } \Omega, \\ |\nabla u_p|^{p-2}\nabla u_p \cdot \nu = \lambda_p(\Omega)|u_p|^{p-2}u_p & \text{on } \partial\Omega. \end{cases} \tag{15}$$

Now, by Young's inequality we have

$$\lambda_1(\Omega) \leqslant \int_\Omega |u_p| + \int_\Omega |\nabla u_p| \leqslant \frac{1}{p}\left(\int_\Omega |u_p|^p + \int_\Omega |\nabla u_p|^p\right) + \frac{2}{p'}|\Omega|$$
$$= \frac{1}{p}\lambda_p(\Omega) + \frac{2}{p'}|\Omega|.$$

Then, by (9) it follows that

$$\lambda_1(\Omega) = \lim_{p\downarrow 1} \int_\Omega |u_p| + \int_\Omega |\nabla u_p|. \tag{16}$$

Moreover, by the compact embedding of $BV(\Omega)$ into $L^1(\Omega)$, we can suppose that

$$u_p \to u \in BV(\Omega) \quad \text{in the } L^1\text{-norm and a.e. in } \Omega \tag{17}$$

and

$$\nabla u_p \to Du \quad \text{weakly}^* \text{ as measures.} \tag{18}$$

Then, we have

$$\exists A := \lim_{p\downarrow 1} \int_\Omega |\nabla u_p| = \lambda_1(\Omega) - \int_\Omega |u|. \tag{19}$$

On the other hand, by the lower-semi-continuity of the total variation respect the L^1-norm, we get

$$\int_\Omega |Du| \leqslant \liminf_{p\downarrow 1} \int_\Omega |\nabla u_p| = \lambda_1(\Omega) - \int_\Omega |u|.$$

Hence,

$$\int_\Omega |Du| + \int_\Omega |u| \leqslant \lambda_1(\Omega). \tag{20}$$

We are interested in the problem: When is u a minimizer of the variational problem (1)? In these cases we would find an extremal for our minimization problem (1).

Formally, if we take limit as $p \searrow 1$ in Eq. (15), we get

$$\begin{cases} \Delta_1 u := \operatorname{div}\left(\frac{Du}{|Du|}\right) = \frac{u}{|u|} & \text{in } \Omega, \\ \frac{Du}{|Du|} \cdot \nu = \lambda_1(\Omega)\frac{u}{|u|} & \text{on } \partial\Omega. \end{cases} \tag{21}$$

Following [2,6] (see also [4]), we give the following definition of solution of problem (21).

Definition 1. A function $u \in BV(\Omega)$ is said to be a *solution* of problem (21) if there exists $z \in X_1(\Omega)$ with $\|z\|_\infty \leqslant 1$, $\tau \in L^\infty(\Omega)$ with $\|\tau\|_\infty \leqslant 1$ and $\theta \in L^\infty(\partial\Omega)$ with $\|\theta\|_\infty \leqslant 1$ such that

$$\operatorname{div}(z) = \tau \quad \text{in } \mathscr{D}'(\Omega), \tag{22}$$

$$\tau u = |u| \text{ a.e. in } \Omega \quad \text{and} \quad (z, Du) = |Du| \text{ as measures}, \tag{23}$$

$$[z, v] = \lambda_1(\Omega)\theta \quad \text{and} \quad \theta u = |u|\mathscr{H}^{N-1}\text{-a.e. on } \partial\Omega. \tag{24}$$

Another problem we are interested in is: In what cases is u a solution of problem (21)?

Note that if v is a solution of problem (21) and $\int_{\partial\Omega} |v| \neq 0$, then $w := v / \int_{\partial\Omega} |v|$ is a minimizer of the variational problem (1). Indeed, multiplying (22) by v and integrating by parts, we have

$$\begin{aligned}\int_\Omega |v| = \int_\Omega \tau v = \int_\Omega \operatorname{div}(z) v = -\int_\Omega (z, Dv) + \int_{\partial\Omega} [z, v] v \\ = -\int_\Omega |Dv| + \lambda_1(\Omega) \int_{\partial\Omega} |v|.\end{aligned}$$

From where it follows that

$$\lambda_1(\Omega) = \int_\Omega |w| + \int_\Omega |Dw|.$$

Before we solve the above problems, let us study first the equation $\operatorname{div}(z) = 1$ with Neumann boundary conditions.

3. The equation $\operatorname{div}(z) = 1$ with Neumann boundary conditions

Throughout this section we shall denote by Ω a bounded connected open set in $\mathbb{R}^N$, $N \geqslant 2$, with Lipschitz continuous boundary $\partial\Omega$. Given $g \in L^\infty(\partial\Omega)$ with $\|g\|_\infty < 1$, consider the functional $\mathscr{E}_g : L^2(\Omega) \to]-\infty, +\infty]$ defined by

$$\mathscr{E}_g(u) := \begin{cases} \int_\Omega |Du| - \int_{\partial\Omega} gu & \text{if } u \in BV(\Omega) \cap L^2(\Omega), \\ +\infty & \text{if } u \notin BV(\Omega). \end{cases}$$

In [6] it is proved that

$$\partial\mathscr{E}_g = \mathscr{A}_g, \tag{25}$$

where $\mathscr{A}_g$ is the operator in $L^2(\Omega)$ defined by

$$\begin{aligned}&(u, v) \in \mathscr{A}_g \iff u \in BV(\Omega) \cap L^2(\Omega), \\ &v \in L^2(\Omega) \text{ and } \exists z \in X_2(\Omega),\ \|z\|_\infty \leqslant 1\end{aligned}$$

such that

$$-\mathrm{div}(z) = v \quad \text{in } \mathscr{D}'(\Omega),$$

$$(z, Du) = |Du| \quad \text{as measures}$$

and

$$[z, \nu] = g \ \mathscr{H}^{N-1}\text{-a.e. on } \partial\Omega.$$

Let $\psi : L^2(\Omega) \to \mathbb{R}$ the operator defined by

$$\psi(u) := \frac{1}{2}\int_\Omega (u(x) + 1)^2 \, \mathrm{d}x.$$

We have

$$0 = \mathrm{argmin}(\mathscr{E}_g + \psi) \iff 0 \in \partial(\mathscr{E}_g + \psi)(0),$$

$$\iff -1 \in \partial\mathscr{E}_g(0) \iff (0, -1) \in \mathscr{A}_g.$$

Then, by (25), it follows that

$$0 = \mathrm{argmin}(\mathscr{E}_g + \psi) \iff \exists z \in X_2(\Omega),\ \|z\|_\infty \leqslant 1, \text{ such that}$$

$$\mathrm{div}(z) = 1 \quad \text{in } \mathscr{D}'(\Omega),$$

$$[z, \nu] = g \ \mathscr{H}^{N-1}\text{-a.e. on } \partial\Omega.$$

On the other hand, $0 = \mathrm{argmin}(\mathscr{E}_g + \psi)$ if and only if

$$\int_\Omega |Du| - \int_{\partial\Omega} gu + \frac{1}{2}\int_\Omega (u+1)^2 \geqslant \frac{1}{2}|\Omega| \quad \forall u \in BV(\Omega) \cap L^2(\Omega). \tag{26}$$

Replacing u by εu (where $\varepsilon > 0$), expanding the L^2 norm, dividing by ε, and letting $\varepsilon \to 0^+$ we get

$$\int_{\partial\Omega} gu \leqslant \int_\Omega |Du| + \int_\Omega u \quad \forall u \in BV(\Omega) \cap L^2(\Omega). \tag{27}$$

Consequently we have obtained the following result.

Lemma 1. *Let $g \in L^\infty(\partial\Omega)$ with $\|g\|_\infty < 1$. Then the following are equivalent*:

(i) *there exists $z \in X_2(\Omega)$, $\|z\|_\infty \leqslant 1$, such that*

$$\mathrm{div}(z) = 1 \quad in \ \mathscr{D}'(\Omega),$$

$$[z, \nu] = g \ \mathscr{H}^{N-1}\text{-}a.e. \ on \ \partial\Omega.$$

(ii) *Eq.* (27) *holds.*

Working as above but using the functional

$$\phi(u) := \frac{1}{2}\int_\Omega (|u(x)|+1)^2 \,\mathrm{d}x$$

instead of ψ, we obtain the following result.

Lemma 2. *Let $g \in L^\infty(\partial\Omega)$ with $\|g\|_\infty < 1$. Then the following are equivalent*:

(i) *there exist $z \in X_2(\Omega)$, $\|z\|_\infty \leqslant 1$, and $\tau \in L^\infty(\Omega)$, $\|\tau\|_\infty \leqslant 1$ such that*

$$\mathrm{div}(z) = \tau \quad in\ \mathscr{D}'(\Omega),$$

$$[z,\nu] = g\ \mathscr{H}^{N-1}\text{-}a.e.\ on\ \partial\Omega.$$

(ii) *the following inequality holds*

$$\int_{\partial\Omega} gu \leqslant \int_\Omega |Du| + \int_\Omega |u| \quad \forall u \in BV(\Omega)\cap L^2(\Omega). \tag{28}$$

We state now the main result of this section.

Theorem 3. *Let Ω be a bounded connected open set in $\mathbb{R}^N$, $N \geqslant 2$, with Lipschitz continuous boundary $\partial\Omega$. Assume that $|\Omega|/P(\Omega) \leqslant \lambda < 1$. Then the following are equivalent*:

(i) *there exist $z \in X_2(\Omega)$, $\|z\|_\infty \leqslant 1$, such that*

$$\mathrm{div}(z) = 1 \quad in\ \mathscr{D}'(\Omega),$$

$$[z,\nu] = \lambda\mathscr{H}^{N-1}\text{-}a.e.\ on\ \partial\Omega,$$

(ii)

$$\lambda\int_{\partial\Omega} u \leqslant \int_\Omega |Du| + \int_\Omega u \quad \forall u \in BV(\Omega)\cap L^2(\Omega), \tag{29}$$

(iii)

$$\lambda\int_{\partial\Omega} u \leqslant \int_\Omega |Du| + \int_\Omega |u| \quad \forall u \in BV(\Omega)\cap L^2(\Omega), \tag{30}$$

(iv)

$$\lambda \leqslant \lambda_1(\Omega), \tag{31}$$

(v)

$$\Big||E| - \lambda\mathscr{H}^{N-1}(\partial^* E \cap \partial\Omega)\Big| \leqslant P(E,\Omega) \quad \forall E \in \mathscr{P}_f(\Omega). \tag{32}$$

Proof. By Lemma 1 with $g = \lambda$, (i) and (ii) are equivalent. Obviously, (iii) and (iv) are equivalent, and (ii) implies (iii). Let us see that (iii) implies (v). Taking $u = \chi_E$ in (30), with $E \subset \Omega$ a set of finite perimeter, it follows that

$$-[|E| - \lambda\mathscr{H}^{N-1}(\partial^* E \cap \partial\Omega)] \leqslant P(E,\Omega)$$

and taking $u = \chi_\Omega \backslash \chi_E$, we get

$$\lambda \mathscr{H}^{N-1}(\partial^*(\Omega\backslash E) \cap \partial\Omega) \leqslant P(\Omega\backslash E, \Omega) + |\Omega\backslash E|.$$

Now, since $P(\Omega\backslash E, \Omega) = P(E, \Omega)$ and $\mathscr{H}^{N-1}(\partial^*(\Omega\backslash E) \cap \partial\Omega) = P(\Omega) - \mathscr{H}^{N-1}(\partial^* E \cap \partial\Omega)$, we have

$$\lambda[P(\Omega) - \mathscr{H}^{N-1}(\partial^* E \cap \partial\Omega)] \leqslant P(E, \Omega) + |\Omega| - |E|.$$

Then, since $\lambda \geqslant |\Omega|/P(\Omega)$, we obtain

$$|E| - \lambda \mathscr{H}^{N-1}(\partial^* E \cap \partial\Omega) \leqslant P(E, \Omega)$$

and (v) holds. Finally, let us see that (v) implies (ii). Given $u \in BV(\Omega)$, since for all $x \in \Omega$,

$$u(x) = \int_0^{+\infty} \chi_{\{u>t\}}(x)\,\mathrm{d}t - \int_{-\infty}^0 \chi_{\{u\leqslant t\}}(x)\,\mathrm{d}t,$$

using (32) and the coarea formula we get

$$\begin{aligned}
\int_\Omega u(x)\,\mathrm{d}x &= \int_0^{+\infty} \int_\Omega \chi_{\{u>t\}}(x)\,\mathrm{d}x\,\mathrm{d}t - \int_{-\infty}^0 \int_\Omega \chi_{\{u\leqslant t\}}(x)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^{+\infty} |\{u>t\}|\,\mathrm{d}t - \int_{-\infty}^0 |\{u\leqslant t\}|\,\mathrm{d}t \\
&\geqslant \int_0^{+\infty} (\lambda \mathscr{H}^{N-1}(\partial^*\{u>t\} \cap \partial\Omega) - P(\{u>t\}, \Omega))\,\mathrm{d}t \\
&\quad - \int_{-\infty}^0 (\lambda \mathscr{H}^{N-1}(\partial^*\{u\leqslant t\} \cap \partial\Omega) + P(\{u\leqslant t\}, \Omega))\,\mathrm{d}t \\
&= \lambda \int_{\partial\Omega} u\,\mathrm{d}\mathscr{H}^{N-1} - \int_{-\infty}^{+\infty} P(\{u>t\}, \Omega)\,\mathrm{d}t \\
&= \lambda \int_{\partial\Omega} u\,\mathrm{d}\mathscr{H}^{N-1} - \int_\Omega |Du|
\end{aligned}$$

and (29) holds. □

Taking $\lambda = |\Omega|/P(\Omega)$ in the above theorem we obtain the following result.

Corollary 1. *Let Ω be a bounded connected open set in $\mathbb{R}^N$, $N \geqslant 2$, with Lipschitz continuous boundary $\partial\Omega$. If $|\Omega|/P(\Omega) < 1$, then the following are equivalent*:

(i)

$$\begin{aligned}
&\exists z \in X_2(\Omega),\ \|z\|_\infty \leqslant 1,\ \textit{such that} \\
&\mathrm{div}(z) = 1 \quad \textit{in } \mathscr{D}'(\Omega), \\
&[z, \nu] = \frac{|\Omega|}{P(\Omega)}\, \mathscr{H}^{N-1}\text{-}\textit{a.e. on } \partial\Omega,
\end{aligned} \tag{33}$$

(ii)

$$\lambda_1(\Omega) = \frac{|\Omega|}{P(\Omega)},$$

(iii)

$$\left| |E| - \frac{|\Omega|}{P(\Omega)} \mathscr{H}^{N-1}(\partial^* E \cap \partial\Omega) \right| \leqslant P(E,\Omega) \quad \forall E \in \mathscr{P}_f(\Omega).$$

We do not know if the assumption Ω connected in Corollary 1 is necessary. So, a natural question is the following: Is there a nonconnected open bounded set Ω such that $|\Omega|/P(\Omega) < 1$, verifying (33) and $\lambda_1(\Omega) < |\Omega|/P(\Omega)$?

There are open sets Ω for which (33) holds and $|\Omega|/P(\Omega) = 1$, as the following examples show.

Example 1. Let $\Omega = B_R(0) \subset \mathbb{R}^N$ the ball in $\mathbb{R}^N$ centered in 0 of radius R. Then, if $z(x) := x/N$, we have

$$\operatorname{div}(z) = 1 \quad \text{in } \mathscr{D}'(\Omega)$$

and

$$[z, \nu] = \frac{R}{N} = \frac{|\Omega|}{P(\Omega)} \mathscr{H}^{N-1}\text{-a.e. on } \partial\Omega.$$

Moreover,

$$\|z\|_\infty = \frac{R}{N} \leqslant 1 \iff \frac{|\Omega|}{P(\Omega)} \leqslant 1.$$

Example 2. Let $\Omega = B_R(0)\backslash\overline{B_r(0)} \subset \mathbb{R}^N$ the annulus in $\mathbb{R}^N$ centered in 0 of radius R and r. Then, it is easy to see that if

$$z(x) := \left[(R^{N-1} + r^{N-1}) - (R+r)\frac{r^{N-1}R^{N-1}}{\|x\|^N} \right] \frac{x}{N(R^{N-1} + r^{N-1})}$$

we have

$$\operatorname{div}(z) = 1 \quad \text{in } \mathscr{D}'(\Omega)$$

and

$$[z, \nu] = \frac{|\Omega|}{P(\Omega)} \mathscr{H}^{N-1}\text{-a.e. on } \partial\Omega.$$

Moreover,

$$\|z\|_\infty \leqslant 1 \iff \frac{|\Omega|}{P(\Omega)} \leqslant 1.$$

Remark 1. Motron in [12] proves that if $\Omega = B_R(0)$ is the ball in $\mathbb{R}^N$ centered in 0 of radius R or $\Omega = B_R(0)\setminus\overline{B_r(0)}$ the annulus in $\mathbb{R}^N$ centered in 0 of radius R and r, then

$$\int_{\partial\Omega} |u| \leqslant \frac{P(\Omega)}{|\Omega|}\int_\Omega |u| + \int_\Omega |\nabla u| \quad \forall u \in W^{1,1}(\Omega) \tag{34}$$

and equality holds in (34) if and only if u is constant.

From (10) and (34), it follows that if $\Omega = B_R(0) \subset \mathbb{R}^N$ or $\Omega = B_R(0)\setminus\overline{B_r(0)} \subset \mathbb{R}^N$, then

$$\lambda_1(\Omega) = \begin{cases} \dfrac{|\Omega|}{P(\Omega)} & \text{if } \dfrac{|\Omega|}{P(\Omega)} \leqslant 1, \\ 1 & \text{if } \dfrac{|\Omega|}{P(\Omega)} \geqslant 1. \end{cases}$$

Moreover, if $|\Omega|/P(\Omega) \leqslant 1$, then $u = (1/P(\Omega))\chi_\Omega$ is a minimizer of the variational problem (1), being the only minimizer in the case $|\Omega|/P(\Omega) = 1$, and if $|\Omega|/P(\Omega) > 1$, the variational problem (1) does not have minimizer.

In the following example we show that there exists bounded connected open sets Ω, with $|\Omega|/P(\Omega) < 1$, for which $\lambda_1(\Omega) < |\Omega|/P(\Omega)$.

Example 3. For $\delta > 0$ and $0 < \alpha \leqslant \pi/2$, let

$$\begin{aligned}\Omega_{\delta,\alpha} := {} & B_1(0) \cup (B_{2+\delta}(0)\setminus\overline{B_2(0)}) \\ & \cup \left\{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ :\ x^2 + y^2 \leqslant 2,\ \operatorname{arctg}\left(\frac{y}{x}\right) < \alpha\right\}.\end{aligned}$$

We have

$$\frac{|\Omega_{\delta,\alpha}|}{P(\Omega_{\delta,\alpha})} = \frac{\pi + \pi(\delta^2 + 4\delta) + \frac{3}{2}\alpha}{10\pi + 2\pi\delta + 2 - 3\alpha}.$$

Thus, for $0 < \delta \leqslant 1$ and $0 < \alpha \leqslant \pi/2$, we have

$$\frac{|\Omega_{\delta,\alpha}|}{P(\Omega_{\delta,\alpha})} < 1.$$

Now, if we take $u := \chi_{B_{2+\delta}(0)\setminus\overline{B_2(0)}}$,

$$\lambda_1(\Omega_{\delta,\alpha}) \leqslant \frac{\int_{\Omega_{\delta,\alpha}} |Du| + \int_{\Omega_{\delta,\alpha}} |u|}{\int_{\partial\Omega_{\delta,\alpha}} |u|} = \frac{2\alpha + \pi(\delta^2 + 4\delta)}{8\pi + 2\delta\pi - 2\alpha}.$$

Then, it is easy to see that for δ and α small enough, we get

$$\lambda_1(\Omega_{\delta,\alpha}) < \frac{|\Omega_{\delta,\alpha}|}{P(\Omega_{\delta,\alpha})} < 1.$$

In the next example we will see that even we can take Ω convex.

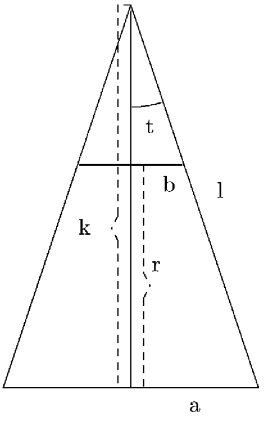

Fig. 1. Triangle.

Example 4. Let Ω be the set in $\mathbb{R}^2$ with the boundary isosceles triangle with height k, base of length $2a$ and the two equal sides of length l. Let t the angle between the height and one of the equal side (see Fig. 1). Then,

$$\frac{|\Omega|}{P(\Omega)} = \frac{ak}{2(a+l)} = \frac{ak}{a2(a+a/\sin t)} = \frac{k \sin t}{2(1+\sin t)}.$$

Let $E \subset \Omega$ the set with boundary the isosceles triangle with height $k-r$, base of length $2b$ and the two equal sides of length $\tilde{l}$. Then, if $u := \chi_E$, we have

$$\lambda_1(\Omega) \leqslant \frac{\int_\Omega |Du| + \int_\Omega |u|}{\int_{\partial\Omega} |u|} = \frac{2b + b(k-r)}{2\tilde{l}} = \frac{b(k+2-r)}{2b/\sin t} = \frac{\sin t}{2}(k+2-r).$$

Hence,

$$\lambda_1(\Omega) < \frac{|\Omega|}{P(\Omega)} < 1$$

if

$$k < \min\left\{ (r-2)\,\frac{1+\sin t}{\sin t}, 2\,\frac{1+\sin t}{\sin t} \right\}.$$

Now, obviously, we can find k, r, and t satisfying the above inequality, and consequently, we can obtain a convex, bounded open set Ω satisfying

$$\lambda_1(\Omega) < \frac{|\Omega|}{P(\Omega)} < 1.$$

The next example show the necessity of the assumption Ω connected in Lemma 2.

Example 5. For $0 < \rho < r$ and $\delta > 0$, let

$$\Omega_{\rho,r,\delta} := B_\rho(0) \cup (B_{r+\delta}(0) \backslash B_r(0)) \subset \mathbb{R}^2.$$

We have

$$\frac{|\Omega_{\rho,r,\delta}|}{P(\Omega_{\rho,r,\delta})} = \frac{\delta^2+\rho^2+2r\delta}{2(2r+\delta+\rho)}.$$

If we take $u := \chi_{B_\rho(0)}$ and $v := \chi_{B_{r+\delta}(0)\setminus\overline{B_r(0)}}$, then

$$\Phi(u) := \frac{\int_{\Omega_{\rho,r,\delta}} |Du| + \int_{\Omega_{\rho,r,\delta}} |u|}{\int_{\partial\Omega_{\rho,r,\delta}} |u|} = \frac{\rho}{2}$$

and

$$\Phi(v) := \frac{\int_{\Omega_{\rho,r,\delta}} |Dv| + \int_{\Omega_{\rho,r,\delta}} |v|}{\int_{\partial\Omega_{\rho,r,\delta}} |v|} = \frac{\delta}{2}.$$

Suppose that $0<\rho<\delta\leqslant 2$. If we consider the vector field z in $\Omega_{\rho,r,\delta}$ defined by

$$z(x,y) := \begin{cases} \dfrac{\delta(x,y)}{2\rho} & \text{if } (x,y)\in B_\rho(0), \\ \left[\delta-(\delta+2r)\dfrac{r(r+\delta)}{\|(x,y)\|}\right]\dfrac{(x,y)}{2(2r+\delta)} & \text{if } (x,y)\in B_{r+\delta}(0)\setminus B_r(0), \end{cases}$$

we have $\|z\|_\infty\leqslant 1$,

$$\mathrm{div}(z)=\tau \quad \text{in } \mathscr{D}'(\Omega_{\rho,r,\delta})$$

with

$$\tau = \frac{\delta}{\rho}\chi_{\partial B_\rho(0)} + \chi_{B_{r+\rho}(0)\setminus B_r(0)}$$

and

$$[z,v] = \frac{\delta}{2}\ \mathscr{H}^1\text{-a.e. on } \partial\Omega_{\rho,r,\rho}.$$

Now,

$$\lambda_1(\Omega)\leqslant\Phi(u)=\frac{\rho}{2}.$$

Hence,

$$\lambda_1(\Omega)<\frac{\delta}{2}.$$

Consequently, in general, Lemma 2 it is not true if Ω is not connected.

If $\delta=\rho$, we have

$$\lambda_1(\Omega_{\rho,r,\rho})\leqslant\Phi(u)=\Phi(v)=\Phi(\chi_{\Omega_{\rho,r,\rho}})=\frac{|\Omega_{\rho,r,\rho}|}{P(\Omega_{\rho,r,\rho})}=\frac{\rho}{2}.$$

Suppose that $\rho \leqslant 1$. Then, if we consider the vector field z in $\Omega_{\rho,r,\rho}$ defined by

$$z(x,y) := \begin{cases} \dfrac{(x,y)}{2} & \text{if } (x,y) \in B_\rho(0), \\ 0 & \text{if } (x,y) \in B_{r+\rho}(0) \setminus B_r(0), \end{cases}$$

we have

$$\operatorname{div}(z) = \chi_{B_\rho(0)} \quad \text{in } \mathscr{D}'(\Omega_{\rho,r,\rho})$$

and

$$[z, \nu] = \frac{\rho}{2} \chi_{\partial B_\rho(0)} \, \mathscr{H}^1\text{-a.e. on } \partial\Omega_{\rho,r,\rho}.$$

Therefore, u is a solution of the problem

$$\begin{cases} \Delta_1 w := \operatorname{div}\left(\dfrac{Dw}{|Du|}\right) = \dfrac{w}{|w|} & \text{in } \Omega_{\rho,r,\rho}, \\ \dfrac{Dw}{|Dw|} \cdot \nu = \dfrac{\rho}{2} \dfrac{w}{|w|} & \text{on } \partial\Omega_{\rho,r,\rho}. \end{cases} \tag{35}$$

Now, if we consider the vector field z in $\Omega_{\rho,r,\rho}$ defined by

$$z(x,y) := \begin{cases} 0 \quad \text{if } (x,y) \in B_\rho(0), \\ \left[\rho - (\rho + 2r)\dfrac{r(r+\rho)}{\|(x,y)\|}\right] \dfrac{(x,y)}{2(2r+\rho)} & \text{if } (x,y) \in B_{r+\rho}(0) \setminus B_r(0), \end{cases}$$

we have

$$\operatorname{div}(z) = \chi_{B_{r+\rho}(0) \setminus B_r(0)} \quad \text{in } \mathscr{D}'(\Omega_{\rho,r,\rho})$$

and

$$[z, \nu] = \frac{\rho}{2} \chi_{\partial(B_{r+\rho}(0) \setminus B_r(0))} \mathscr{H}^1\text{-a.e. on } \partial\Omega_{\rho,r,\rho}.$$

Therefore, v is also a solution of problem (35). Moreover, in this case also $\chi_{\Omega_{\rho,r,\rho}}$ is a solution of problem (35).

Problem. Is $\lambda_1(\Omega_{\rho,r,\rho}) = \rho/2$?

The next example shows that there are bounded connected open sets, Ω for which $\lambda_1(\Omega) < 1$ and $|\Omega|/P(\Omega) > 1$.

Example 6. Let $\Omega :=]-k, 0[\times]0, k[\cup\{0\}\times]0, \delta[\cup]0, 1[\times]0, \delta[\subset \mathbb{R}^2$ be. Then

$$\frac{|\Omega|}{P(\Omega)} = \frac{k^2 + \delta}{4k + 2} > 1 \quad \text{if } k > 2 + \sqrt{6 - \delta}.$$

Now, if we take $u := \chi_{]0,1[\times]0,\delta[}$, we have

$$\lambda_1(\Omega) \leqslant \frac{\int_\Omega |Du| + \int_\Omega |u|}{\int_{\partial\Omega} |u|} = \frac{2\delta}{2 + \delta} < 1 \iff 0 < \delta < 2.$$

Therefore, for instance, if $\delta = 1$ and $k = 5$, we have $|\Omega|/P(\Omega) > 1$ and $\lambda_1(\Omega) < 1$.

4. Proof of Theorem 2

Let Ω be a bounded connected open set in $\mathbb{R}^N$, $N \geqslant 2$, with Lipschitz continuous boundary $\partial\Omega$. By Lemma 2, if we assume that $\lambda_1(\Omega) < 1$, there exist $\overline{z} \in X_2(\Omega)$, $\|\overline{z}\|_\infty \leqslant 1$, and $\|\mathrm{div}(\overline{z})\|_\infty \leqslant 1$ such that

$$[\overline{z}, v] = \lambda_1(\Omega)\mathscr{H}^{N-1}\text{-a.e. on } \partial\Omega.$$

We recall the following definition.

Definition 2. Let Ω be a bounded open set in $\mathbb{R}^N$ with Lipschitz continuous boundary $\partial\Omega$. We shall say that Ω has the *trace-property* if there exists a vector field $z_\Omega \in L^\infty(\Omega, \mathbb{R}^N)$, with $\|z_\Omega\|_\infty \leqslant 1$ such that $\mathrm{div}(z_\Omega) \in L^\infty(\Omega)$ and

$$[z_\Omega, v] = \lambda_1(\Omega)\mathscr{H}^{N-1}\text{-a.e.on } \partial\Omega.$$

By the above, we have that every bounded connected open set Ω in $\mathbb{R}^N$, $N \geqslant 2$, with Lipschitz continuous boundary and $\lambda_1(\Omega) < 1$, has the trace-property. Also, as a consequence of Examples 1, 2, and Remark 1, we have that if $\Omega = B_R(0) \subset \mathbb{R}^N$ or $\Omega = B_R(0)\setminus\overline{B_r(0)} \subset \mathbb{R}^N$, and $|\Omega|/P(\Omega) \leqslant 1$, then Ω has the trace-property. Therefore there exists Ω with $\lambda_1(\Omega) = 1$ satisfying the trace-property.

Let us present some examples of planar domains that verify $\lambda_1(\Omega) < 1$.

Example 7. Assume that $\Omega \subset \mathbb{R}^2$ is a bounded open set such that there exists some point, $x_0 \in \partial\Omega$ (we may assume $x_0 = 0$), with curvature of the boundary at that point greater than 2, we will show that in this case $\lambda_1(\Omega) < 1$. So, let us assume that locally near the origin Ω can be described as $\Omega \cap B_r(0) = \{(x, y):\ y > ax^2\}$. As we are assuming that the curvature at the origin is greater than 2 we have $a > 1$. Let us consider the function $u_\varepsilon = \chi_{\Omega\cap\{y<\varepsilon\}}$ as a test function to estimate λ_1. We have

$$\begin{aligned}\lambda_1(\Omega) &\leqslant \frac{\int_\Omega |Du_\varepsilon| + \int_\Omega |u_\varepsilon|}{\int_{\partial\Omega} |u_\varepsilon|} = \frac{\sqrt{\varepsilon/a} + \int_0^{\sqrt{\varepsilon/a}} (\varepsilon - as^2)\,\mathrm{d}s}{\int_0^{\sqrt{\varepsilon/a}} \sqrt{1 + (2a)^2 s^2}\,\mathrm{d}s} \\ &= \frac{\sqrt{\varepsilon/a} + \frac{2}{3}\,\varepsilon\sqrt{\varepsilon/a}}{\int_0^{\sqrt{\varepsilon/a}} (1 + 2a^2s^2 + O(s^3))\,\mathrm{d}s} < 1\end{aligned}$$

if ε is small enough.

Remark that if $\Omega = B_R(0)$ we have that the curvature is $1/R$ and, by Example 1, we have $\lambda_1(B_R(0)) < 1$ if and only if $R < 2$. Hence, some restriction on the curvature must be imposed.

Next we prove that on every domain that enjoys the trace-property the best Sobolev trace constant is attained, Theorem 2.

Proof of Theorem 2. First let us see that the function u obtained in (17) is a minimizer of the variational problem (1). Let z_Ω be the vector field given in Definition 2. Then, by (18), we have

$$\int_\Omega (z_\Omega, Du) = \lim_{p\searrow 1} \int_\Omega (z_\Omega, \nabla u_p) = \lim_{p\searrow 1} \left(- \int_\Omega \operatorname{div}(z_\Omega) u_p + \int_{\partial\Omega} [z_\Omega, \nu] u_p \right)$$
$$= - \int_\Omega \operatorname{div}(z_\Omega) u + \lambda_1(\Omega) = \int_\Omega (z_\Omega, Du) - \lambda_1(\Omega) \int_{\partial\Omega} u + \lambda_1(\Omega)$$

from where it follows that

$$\int_{\partial\Omega} |u| = 1. \tag{36}$$

Therefore, by (20) we get that u is a minimizer of (1). Moreover, by (19), it follows that

$$\lim_{p\searrow 1} \int_\Omega |\nabla u_p| = \lambda_1(\Omega) - \int_\Omega |u| = \int_\Omega |Du|. \tag{37}$$

Hence $u_p \to u$ respect to the strict convergence, and consequently

$$u_p \to u \quad \text{in } L^1(\partial\Omega) \text{ as } p \searrow 1. \tag{38}$$

Let us see now that the function u is a solution of problem (21). By Hölder's inequality, we have

$$\int_\Omega |u_p|^{p-1} \leqslant |\Omega|^{1/p} \left(\int_\Omega |u_p|^p \right)^{(p-1)/p} \leqslant |\Omega|^{1/p} \lambda_p(\Omega)^{(p-1)/p} \leqslant M_1. \tag{39}$$

On the other hand, if E is a measurable subset of Ω with $|E| < 1$, we have

$$\left| \int_E |u_p|^{p-2} u_p \right| \leqslant \int_E |u_p|^{p-1} \leqslant M_2 |E|^{1/p}. \tag{40}$$

By (39) and (40), it follows that $\{|u_p|^{p-2}u_p\colon\ 1 < p \leqslant 2\}$ is a weakly relatively compact subset of $L^1(\Omega)$. Hence, we can assume that there exists $\tau \in L^1(\Omega)$ such that

$$|u_p|^{p-2} u_p \to \tau \quad \text{weakly in } L^1(\Omega) \text{ as } p \searrow 1. \tag{41}$$

In a similar way, it is easy to see that there exists $z \in L^1(\Omega, \mathbb{R}^N)$ such that

$$|\nabla u_p|^{p-2} \nabla u_p \to z \quad \text{weakly in } L^1(\Omega, \mathbb{R}^N) \text{ as } p \searrow 1. \tag{42}$$

Now, given $\varphi \in \mathscr{D}(\Omega)$, from (41) and (42), it follows that

$$\langle \operatorname{div}(z), \varphi \rangle = - \int_\Omega z \cdot \nabla\varphi = - \lim_{p\searrow 1} \int_\Omega |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla\varphi$$
$$= \lim_{p\searrow 1} \int_\Omega \operatorname{div}(|\nabla u_p|^{p-2} \nabla u_p)\varphi = \lim_{p\searrow 1} \int_\Omega |u_p|^{p-2} u_p \varphi = \int_\Omega \tau\varphi.$$

Thus,

$$\operatorname{div}(z) = \tau \quad \text{in } \mathscr{D}'(\Omega). \tag{43}$$

We claim now that

$$\|z\|_\infty \leqslant 1. \tag{44}$$

In fact, for any $k>0$, let

$$B_{p,k} := \{x \in \Omega\colon\ |\nabla u_p(x)| > k\}.$$

As above, there exists some $g_k \in L^1(\Omega, \mathbb{R}^N)$ such that

$$|\nabla u_p|^{p-2}\nabla u_p \chi_{B_{p,k}} \to g_k \quad \text{weakly in } L^1(\Omega, \mathbb{R}^N) \text{ as } p \searrow 1. \tag{45}$$

Now, since

$$|B_{p,k}| = \int_\Omega \chi_{B_{p,k}}(x)\,\mathrm{d}x \leqslant \int_\Omega \frac{|\nabla u_p(x)|^p}{k^p}\,\mathrm{d}x \leqslant \frac{\lambda_p(\Omega)}{k^p}$$

for any $\phi \in L^\infty(\Omega)$ with $\|\phi\|_\infty \leqslant 1$, we have

$$\begin{aligned}\left|\int_\Omega |\nabla u_p|^{p-2}\nabla u_p \cdot \phi\chi_{B_{p,k}}\right| &\leqslant \left(\int_\Omega |\nabla u_p|^p\right)^{(p-1)/p} |B_{p,k}|^{1/p}\\ &\leqslant \lambda_p(\Omega)^{(p-1)/p}\left(\frac{\lambda_p(\Omega)}{k^p}\right)^{1/p} = \frac{\lambda_p(\Omega)}{k}.\end{aligned}$$

Letting $p \searrow 1$, we get that

$$\int_\Omega |g_k| \leqslant \frac{\lambda_1(\Omega)}{k} \quad \text{for every } k>0. \tag{46}$$

On the other hand, since we have

$$||\nabla u_p|^{p-2}\nabla u_p \chi_{\Omega\setminus B_{p,k}}| \leqslant k^{p-1} \quad \text{for any } p>1,$$

letting $p \searrow 1$, we obtain that there exists some $f_k \in L^1(\Omega, \mathbb{R}^N)$ with $\|f_k\|_\infty \leqslant 1$ such that

$$|\nabla u_p|^{p-2}\nabla u_p \chi_{\Omega\setminus B_{p,k}} \to f_k \quad \text{weakly in } L^1(\Omega, \mathbb{R}^N) \text{ as } p \searrow 1. \tag{47}$$

Hence, for any $k>0$, we may write $z = f_k + g_k$, with $\|f_k\|_\infty \leqslant 1$ and g_k satisfying (46). From where (44) follows.

Since $u_p \to u$ a.e. in Ω, by (41) it follows that

$$\tau u = |u| \text{ a.e. in } \Omega \quad \text{and} \quad \|\tau\|_\infty \leqslant 1.$$

On the other hand, given a measurable subset $E \subset \partial\Omega$, by Hölder's inequality we have

$$\int_E u_p^{p-1}\,\mathrm{d}\mathscr{H}^{N-1} \leqslant \left(\int_{\partial\Omega} u_p\,\mathrm{d}\mathscr{H}^{N-1}\right)^{p-1} \mathscr{H}^{N-1}(E)^{2-p} \leqslant \mathscr{H}^{N-1}(E)^{2-p}$$

from where it follows that $\{u_p^{p-1}\colon\ 1<p\leqslant 2\}$ is a weakly relatively compact subset of $L^1(\partial\Omega)$. Hence, we can assume that there exists $\theta \in L^1(\partial\Omega)$, such that

$$u_p^{p-1} \to \theta \quad \text{weakly in } L^1(\partial\Omega). \tag{48}$$

Moreover, by (48), (14) and applying Fatou's Lemma, it is easy to see that

$$\|\theta\|_\infty \leqslant 1. \tag{49}$$

On the other hand, by (38) and (48), we get

$$\theta u = |u| \mathscr{H}^{N-1}\text{-a.e. on } \partial\Omega. \tag{50}$$

Now, since u_p is a weak solution of (15), having in mind (41), (9) and (48), if $w \in W^{1,1}(\Omega) \cap C(\Omega) \cap L^\infty(\Omega)$, we have

$$\begin{aligned}
\langle z, w\rangle_{\partial\Omega} := & \int_\Omega \operatorname{div}(z) w + \int_\Omega z \cdot \nabla w = \int_\Omega \tau w + \int_\Omega z \cdot \nabla w \\
&= \lim_{p \searrow 1} \int_\Omega u_p^{p-1} w + \int_\Omega |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla w \\
&= \lim_{p \searrow 1} \int_\Omega u_p^{p-1} w - \int_\Omega \operatorname{div}(|\nabla u_p|^{p-2} \nabla u_p) w + \int_{\partial\Omega} |\nabla u_p|^{p-2} \nabla u_p \cdot \nu w \\
&= \lim_{p \searrow 1} \lambda_p(\Omega) \int_{\partial\Omega} u_p^{p-1} w = \lambda_1(\Omega) \int_{\partial\Omega} \theta w.
\end{aligned}$$

Thus, having in mind the definition of the weak trace on $\partial\Omega$ of the normal component of z given in [5], we get

$$[z, \nu] = \lambda_1(\Omega)\theta. \tag{51}$$

Finally, since

$$\begin{aligned}
\int_\Omega |Du| &= \lambda_1(\Omega) - \int_\Omega |u| = \lambda_1(\Omega) - \int_\Omega \tau u \\
&= \lambda_1(\Omega) - \int_\Omega \operatorname{div}(z) u = \lambda_1(\Omega) + \int_\Omega (z, Du) - \int_{\partial\Omega} [z, \nu] u \\
&= \lambda_1(\Omega) + \int_\Omega (z, Du) - \lambda_1(\Omega) \int_{\partial\Omega} \theta u = \int_\Omega (z, Du),
\end{aligned}$$

we have $(z, Du) = |Du|$ as measures. □

Remark 2. Let us remark that as a consequence of Theorem 1 in [8] it is obtained the above theorem in the particular case that Ω is a bounded open subset of $\mathbb{R}^N$, whose boundary $\partial\Omega$ is at least piecewise $\mathscr{C}^2$ and $\lambda_1(\Omega) < 1$.

Remark 3. Note that in the above theorem we have proved that if u is the limit as $p \searrow 1$ of the minimizers u_p of the variational problem

$$\inf \left\{ \int_\Omega |u|^p + \int_\Omega |\nabla u|^p :\ u \in W^{1,p}(\Omega), \int_{\partial\Omega} |u| = 1, \right\} \tag{52}$$

then, if $\int_{\partial\Omega} |u| = 1$, we have that u is a minimizer of the variational problem (1) and a solution of problem (21).

Acknowledgements

The first and second authors have been partially supported EC through the RTN Programme *Nonlinear Partial Differential Equations Describing Front propagation and other Singular Phenomena*, HPRN-CT-2002-00274, and by PNPGC project, reference BFM2002-01145. The third author acknowledges partial support by ANPCyT PICT No. 03-05009, CONICET PIP0660/98, UBA X431 and Fundación Antorchas.

References

[1] L. Ambrosio, N. Fusco, D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, in: Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.
[2] F. Andreu, C. Ballester, V. Caselles, J.M. Mazón, Minimizing total variational flow, Differential Integral Equations 14 (2001) 321–360.
[4] F. Andreu, V. Caselles, J.M. Mazón, Parabolic Quasilinear Equations Minimizing Linear Growth Functionals, Progress in Mathematics, vol. 223, Birkhauser, Basel, 2004.
[5] G. Anzellotti, Pairings between measures and bounded functions and compensated compactness, Ann. Mat. Pura Appl. IV (135) (1983) 293–318.
[6] G. Ballettini, V. Caselles, M. Novaga, Explicit solutions of the eigenvalue problem $-\mathrm{div}(\frac{Du}{|Du|}) = u$, preprint.
[8] F. Demengel, On some nonlinear equation involving the 1-Laplacian and trace map inequalities, Nonlinear Anal. TMA 48 (2002) 1151–1163 (1983) 293–318.
[9] L.C. Evans, R.F. Gariepy, Measure Theory and Fine Properties of Functions, in: Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.
[10] J. Fernández Bonder, J.D. Rossi, Existence results for the p-Laplacian with nonlinear boundary conditions, J. Math. Anal. Appl. 263 (2001) 195–223.
[11] S. Martínez, J. Rossi, Isolation and simplicity for the first eigenvalue of the p-Laplacian with a nonlinear boundary condition, Abstract Appl. Anal. 7 (2002) 287–293.
[12] M. Motron, Around the best constants for the Sobolev trace map from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$, Asymptotic Anal. 29 (2002) 69–90.
[13] W.P. Ziemer, Weakly Differentiable Functions, GTM 120, Springer, Berlin, 1989.