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A strongly degenerate quasilinear elliptic equation

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Abstract

We prove existence and uniqueness of entropy solutions for the quasilinear elliptic equation $u - \operatorname{div} \mathbf{a}(u, Du) = v$, where $0 \leqslant v \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\mathbf{a}(z, \xi) = \nabla_\xi f(z, \xi)$, and f is a convex function of ξ with linear growth as $\|\xi\| \to \infty$, satisfying other additional assumptions. In particular, this class of equations includes the elliptic problems associated to a relativistic heat equation and a flux limited diffusion equation used in the theory of radiation hydrodynamics, respectively. In a second part of this work, using Crandall–Liggett's iteration scheme, this result will permit us to prove existence and uniqueness of entropy solutions for the corresponding parabolic Cauchy problem.

Keywords: Quasilinear elliptic equations; Flux limited diffusion equations

1. Introduction

We are interested in the problem

$$u - \operatorname{div} \mathbf{a}(u, Du) = v, \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $v \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $v \geqslant 0$, $\mathbf{a}(z, \xi) = \nabla_\xi f(z, \xi)$ and f is a function with linear growth as $\|\xi\| \to \infty$.

Our purpose in this paper is to define a notion of entropy solution for (1.1), and to prove existence and uniqueness results when the right-hand side v is a nonnegative function in

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$L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$. Besides the fact that the elliptic problem (1.1) is interesting by itself, this result permits us to associate to the expression $-\operatorname{div}\mathbf{a}(u,Du)$ an accretive operator B in $L^1(\Omega)$ whose domain is contained in $(L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N))^+$ (which amounts to consider the right-hand side v of (1.1) in $(L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N))^+$) and whose closure $\mathscr{B}$ is m-accretive (hence, it generates a nonlinear contraction semigroup $T(t)$) in $L^1(\mathbb{R}^N)^+$ [11,17]. However, we have not been able to characterize $\mathscr{B}$ in distributional terms. In spite of this, the knowledge of the operator B and the fact that, if u is the entropy solution of (1.1), we have that $\|u\|_\infty\leqslant\|v\|_\infty$, permit us to use Crandall–Ligget's iteration scheme and define

$$u(t):=T(t)u_0=\lim_{n\to\infty}\left(I+\frac{t}{n}B\right)^{-n}u_0,\quad u_0\in(L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N))^+,$$

which is a semigroup solution of the Cauchy problem

$$\begin{cases}\dfrac{\partial u}{\partial t}=\operatorname{div}\mathbf{a}(u,Du), & \text{in } Q_T=(0,T)\times\mathbb{R}^N,\\ u(0,x)=u_0(x), & \text{in } x\in\mathbb{R}^N.\end{cases}\tag{1.2}$$

In a subsequent work [6] we shall define the notion of entropy solution for (1.2) and prove that the semigroup solution $u(t)$ is an entropy solution. Moreover, we shall also prove that entropy solutions of (1.2) are unique. As a technical tool we shall use some lower semi-continuity results for energy functionals whose density is a function $g(x,u,Du)$ convex in Du with linear growth rate as $|Du|\to\infty$ (see [18,20]).

Particular instances of problem (1.2) have been studied in [12,19], when $N=1$. In these papers the authors considered the problem

$$\begin{cases}\dfrac{\partial u}{\partial t}=(\varphi(u)\mathbf{b}(u_x))_x, & \text{in } (0,T)\times\mathbb{R},\\ u(0,x)=u_0(x), & \text{in } x\in\mathbb{R},\end{cases}\tag{1.3}$$

corresponding to (1.2) when $N=1$ and $\mathbf{a}(u,u_x)=\varphi(u)\mathbf{b}(u_x)$, where $\varphi:\mathbb{R}\to\mathbb{R}^+$ is smooth and strictly positive, and $\mathbf{b}:\mathbb{R}\to\mathbb{R}$ is a smooth odd function such that $\mathbf{b}'>0$ and $\lim_{s\to\infty}\mathbf{b}(s)=\mathbf{b}_\infty$. Such models appear in the theory of phase transitions where the corresponding free energy functional has a linear growth rate with respect to the gradient [25]. As the authors observed, in general, there are no classical solutions of (1.2). Moreover, they defined the notion of entropy solution and proved existence [12] and uniqueness [19] of entropy solutions of (1.2). Existence was proved for bounded strictly increasing initial conditions $u_0:\mathbb{R}\to\mathbb{R}$ such that $\mathbf{b}(u_0')\in C(\mathbb{R})$ (where $\mathbf{b}(u_0'(x_0))=\mathbf{b}_\infty$ if u_0 is discontinuous at x_0), $\mathbf{b}(u_0'(x))\to 0$ as $x\to\pm\infty$ [12]. The entropy condition was written in Oleinik's form and uniqueness was proved using suitable test functions constructed by regularizing the sign of the difference of two solutions.

In [13], the author considered the following Neumann problem in an interval of $\mathbb{R}$:

$$\begin{cases}\dfrac{\partial u}{\partial t}=(\mathbf{a}(u,u_x))_x, & \text{in } (0,T)\times(0,1),\\ u_x(t,0)=u_x(t,1)=0, & \text{in } t\in(0,T),\\ u(0,x)=u_0(x), & \text{in } x\in(0,1),\end{cases}\tag{1.4}$$

where $\mathbf{a}(u, v)$ is a function of class $C^{1,\alpha}([0,\infty) \times \mathbb{R})$ satisfying other additional assumptions. The author associated an *m*-accretive operator to $-(\mathbf{a}(u, u_x))_x$ with Neumann boundary conditions, and proved the existence and uniqueness of a semigroup solution of (1.4). An example of the equations considered in [13] is the so called *plasma equation* (see [21])

$$\frac{\partial u}{\partial t} = \left(\frac{u^{5/2} u_x}{1 + u|u_x|} \right)_x, \quad \text{in } (0, T) \times (0, 1), \tag{1.5}$$

where the initial condition u_0 is assumed to be positive. In this case u represents the temperature of electrons, and the form of the conductivity $\mathbf{a}(u, u_x) = u^{5/2} u_x / 1 + u|u_x|$ has the effect of limiting the heat flux. But, as far as we know, existence and uniqueness results for higher dimensional problems have not been considered in the literature. This was the purpose of our papers [4,5] in which we studied the Neumann problem for Lagrangians *f* satisfying the following coercivity and linear growth condition:

$$C_0 \|\xi\| - D_0 \leqslant f(z, \xi) \leqslant M_0(1 + \|\xi\|), \tag{1.6}$$

for some positive constants C_0, M_0. Now, there are some relevant cases like the *relativistic heat equation* (see [14,26])

$$u_t = v \operatorname{div} \left(\frac{|u| Du}{\sqrt{u^2 + a^2 |Du|^2}} \right), \tag{1.7}$$

for which the Lagrangian $f(z, \xi) = v/a^2 |z| \sqrt{z^2 + a^2 |\xi|^2}$ does not satisfy (1.6). Observe that, in this case, $f(z, \xi)$ satisfies the following condition:

$$C_0(z) \|\xi\| - D_0(z) \leqslant f(z, \xi) \leqslant M_0(z)(\|\xi\| + 1), \tag{1.8}$$

for any $(z, \xi) \in \mathbb{R} \times \mathbb{R}^N$, and some positive and continuous functions C_0, D_0, M_0, such that $C_0(z) > 0$ for any $z \neq 0$. Eq. (1.7) was introduced by Ph. Rosenau in [26] to overcome the nonphysical dependence of the flux on the gradient as predicted by the classical transport theory. He imposed the acoustic speed as an upper bound of the permitted propagation speed in a medium. This provides the means to control the growth of the flux; flux saturates as the gradients become unbounded. Let us also mention that Eq. (1.7) was recently derived by Y. Brenier by means of Monge–Kantorovich's mass transport theory [14]. As Brenier pointed out in [14], this relativistic heat equation is one among the various *flux limited diffusion equations* used in the theory of radiation hydrodynamics [24]. Indeed, a very similar equation

$$u_t = v \operatorname{div} \left(\frac{u Du}{u + \frac{v}{c} |Du|} \right) \tag{1.9}$$

can be found in [24].

Finally, let us explain the plan of the paper. In Section 2 we recall some basic facts about function spaces, functions of bounded variation, denoted by $BV(\Omega)$, Green's formula, and lower semi-continuity results for energy functionals defined in $BV(\Omega)$. In Section 3 we state the main assumptions on the Lagrangian *f*, recall the meaning of expressions of type

$f(u, Du)$ for functions u in $BV(\mathbb{R}^N)$ and define an associated functional calculus, and finally define the notion of entropy solution for the elliptic problem (1.1). In Section 4 we prove an existence and uniqueness result for (1.1) when the right-hand side v is a nonnegative function in $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. In Section 5 we define an accretive operator associated to $-\mathrm{div}\,\mathbf{a}(u, Du)$ whose closure generates a contraction semigroup in $L^1(\mathbb{R}^N)^+$, providing a solution of (1.2) in the semigroup sense. Finally, in Section 6 we state without proof the analogous results for the Neumann problem.

2. Preliminaries

2.1. Some function spaces. BV functions

Let us start with some notation. We denote by $\mathscr{L}^N$ and $\mathscr{H}^{N-1}$ the N-dimensional Lebesgue measure and the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$, respectively. Given an open set Ω in $\mathbb{R}^N$ we shall denote by $\mathscr{D}(\Omega)$, or $C_0^\infty(\mathbb{R}^N)$, the space of infinitely differentiable functions with compact support in Ω. The space of continuous functions with compact support in $\mathbb{R}^N$ will be denoted by $C_c(\mathbb{R}^N)$.

We shall use several notations borrowed from [10]. Let $M(\mathbb{R}^N)$ the set of Lebesgue's measurable functions from $\mathbb{R}^N$ into $\mathbb{R}$. We denote by $L(\mathbb{R}^N)$ the space $L(\mathbb{R}^N) := L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, which equipped with the norm

$$\|u\|_{1+\infty} := \inf\{\|u_1\|_1 + \|u_2\|_\infty : u = u_1 + u_2,\ u_1 \in L^1(\mathbb{R}^N), u_2 \in L^\infty(\mathbb{R}^N)\}$$

is a Banach space. If we denote

$$L_0(\mathbb{R}^N) := \left\{ u \in M(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|u| - k)^+ < \infty \quad \forall k > 0 \right\},$$

we have that $L_0(\mathbb{R}^N) = \overline{L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)}^{\|\ \|_{1+\infty}}$ [10]. The dual space of $L_0(\mathbb{R}^N)$ is isometrically isomorphic to $L^{1\cap\infty}(\mathbb{R}^N) := L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, when $L^{1\cap\infty}(\mathbb{R}^N)$ is endowed with the norm $\|u\|_{1\cap\infty} := \max\{\|u\|_1, \|u\|_\infty\}$ [10].

Given $u, v \in M(\mathbb{R}^N)$, we shall write

$$u \ll v, \quad \text{if and only if } \int_{\mathbb{R}^N} j(u)\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} j(v)\,\mathrm{d}x,$$

for all $j \in J_0 := \{j : \mathbb{R} \to [0, +\infty], \text{convex, l.s.c}, \ j(0) = 0\}$.

Due to the linear growth condition on the Lagrangian, the natural energy space to study (1.1) is the space of functions of bounded variation. Recall that if Ω is an open subset of $\mathbb{R}^N$, a function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a vector valued Radon measure with finite total variation in Ω is called a *function of bounded variation*. The class of such functions will be denoted by $BV(\Omega)$. For $u \in BV(\Omega)$, the vector measure Du decomposes into its absolutely continuous and singular parts $Du = D^a u + D^s u$. Then $D^a u = \nabla u \ \mathscr{L}^N$ where ∇u is the Radon–Nikodym derivative of the measure Du with respect to the Lebesgue measure $\mathscr{L}^N$. We also split $D^s u$ in two parts: the *jump* part $D^{\mathrm{j}} u$ and the

Cantor part $D^c u$. It is well known (see for instance [1]) that

$$D^j u = (u^+ - u^-)\nu_u \mathscr{H}^{N-1} \llcorner J_u,$$

where J_u denotes the set of approximate jump points of u, and $\nu_u(x) = Du/|Du|(x)$, $Du/|Du|$ being the Radon–Nikodym derivative of Du with respect to its total variation $|Du|$. For further information concerning functions of bounded variation we refer to [1,22] or [27].

2.2. Lower semicontinuity of functionals defined on BV

Let Ω be an open subset of $\mathbb{R}^N$. Given a Borel function $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty[$, we consider the energy functional

$$G(u) := \int_\Omega g(x, u(x), \nabla u(x))\, dx,$$

defined in the Sobolev space $W^{1,1}(\Omega)$. In order to get an integral representation of the relaxed energy associated with G, i.e.,

$$\mathscr{G}(u) := \inf_{\{u_n\}} \left\{ \liminf_{n\to\infty} G(u_n) : u_n \in W^{1,1}(\Omega),\ u_n \to u \in L^1(\Omega) \right\},$$

Dal Maso introduced in [18] the following functional for $u \in BV(\Omega)$:

$$\begin{aligned} \mathscr{R}_g(u) := & \int_\Omega g(x, u(x), \nabla u(x))\, dx + \int_\Omega g^0 \left(x, \tilde{u}(x), \frac{Du}{|Du|}(x) \right) |D^c u| \\ & + \int_{J_u} \left(\int_{u_-(x)}^{u_+(x)} g^0(x, s, \nu_u(x))\, ds \right) d\mathscr{H}^{N-1}(x), \end{aligned} \tag{2.1}$$

where the recession function g^0 of g is defined as

$$g^0(x, z, \xi) = \lim_{t\to 0^+} t g\left(x, z, \frac{\xi}{t} \right). \tag{2.2}$$

In case that Ω is a bounded set, and under standard continuity and coercivity assumptions, Dal Maso proved in [18] that $\mathscr{G}(u) = \mathscr{R}_g(u)$ for all $u \in BV(\Omega)$. Recently, De Cicco et al. [20], have obtained a very general result about the L^1-lower semi-continuity of $\mathscr{R}_g$ in *BV*, which contains, in particular, the following statement.

Theorem 2.1. *Let Ω be an open subset of $\mathbb{R}^N$ and $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty[$ a locally bounded Carathéodory function such that, for every $(z, \xi) \in \mathbb{R} \times \mathbb{R}^N$, the function $g(\cdot, z, \xi)$ is of class C^1. Let us assume that*

(i) $g(x, z, \cdot)$ *is convex in* $\mathbb{R}^N$ *for every* $(x, z) \in \Omega \times \mathbb{R}$,
(ii) $g(x, \cdot, \xi)$ *is continuous in* $\mathbb{R}$ *for every* $(x, \xi) \in \Omega \times \mathbb{R}^N$.

Then, the functional $\mathscr{R}_g(u)$ is lower semi-continuous respect to the $L^1(\Omega)$-convergence.

Let $f : \mathbb{R} \times \mathbb{R}^N \to [0, \infty[$ a continuous function, such that there exists f^0 and $|f^0(z, \xi)| \leqslant M\|\xi\|$ for any $z \in \mathbb{R}$, $\xi \in \mathbb{R}^N$. Given a function $u \in BV(\mathbb{R}^N)$, we define the Radon measure $f(u, Du)$ in $\mathbb{R}^N$ as

$$\langle f(u, Du), \phi\rangle := \mathscr{R}_{\phi f}(u), \quad \phi \in C_c(\mathbb{R}^N). \tag{2.3}$$

Let us observe that if $f^0(z, \xi) = \varphi(z)\psi^0(\xi)$, where φ is Lipschitz continuous and ψ^0 is an homogeneous function of degree 1, by applying chain's rule for *BV*-functions (see [1]), we have

$$\mathscr{R}_{\phi f}(u) = \int_{\mathbb{R}^N} \phi(x) f(u, \nabla u)\,\mathrm{d}x + \int_{\mathbb{R}^N} \phi(x)\psi^0\left(\frac{Du}{|Du|}\right)|D^s J_\varphi(u)|, \tag{2.4}$$

where $J_\varphi(r) = \int_0^r \varphi(s)\,\mathrm{d}s$. Then, under these conditions, we have

$$f(u, Du)^s = \psi^0\left(\frac{Du}{|Du|}\right)|D^s J_\varphi(u)|. \tag{2.5}$$

2.3. A generalized Green's formula

We shall need several results from [8] (see also [3]) in order to give a sense to integrals of bounded vector fields with divergence in L^1 integrated with respect to the gradient of a *BV* function. Following [8], we denote

$$X_1(\mathbb{R}^N) = \{\mathbf{z} \in L^\infty(\mathbb{R}^N, \mathbb{R}^N) : \mathrm{div}(\mathbf{z}) \in L^1(\mathbb{R}^N)\}. \tag{2.6}$$

If $\mathbf{z} \in X_1(\mathbb{R}^N)$ and $w \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ we define the functional $(\mathbf{z}, Dw) : C_c^\infty(\mathbb{R}^N) \to \mathbb{R}$ by the formula

$$\langle(\mathbf{z}, Dw), \varphi\rangle := -\int_{\mathbb{R}^N} w\varphi\,\mathrm{div}(\mathbf{z})\,\mathrm{d}x - \int_{\mathbb{R}^N} w\,\mathbf{z}\cdot\nabla\varphi\,\mathrm{d}x. \tag{2.7}$$

Then $(\mathbf{z}, Dw)$ is a Radon measure in $\mathbb{R}^N$, and

$$\int_{\mathbb{R}^N} (\mathbf{z}, Dw) = \int_{\mathbb{R}^N} \mathbf{z}\cdot\nabla w\,\mathrm{d}x, \quad \forall w \in W^{1,1}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \tag{2.8}$$

Moreover, $(\mathbf{z}, Dw)$ is absolutely continuous with respect to $|Dw|$. Its Radon–Nikodym derivative, denoted by $\theta(\mathbf{z}, Dw, x)$, is a $|Dw|$ measurable function from $\mathbb{R}^N$ to $\mathbb{R}$ such that

$$\int_B (\mathbf{z}, Dw) = \int_B \theta(\mathbf{z}, Dw, x)|Dw|, \quad \text{for any Borel set } B \subseteq \mathbb{R}^N. \tag{2.9}$$

By writing

$$\mathbf{z}\cdot D^s u := (\mathbf{z}, Du) - (\mathbf{z}\cdot\nabla u)\,\mathrm{d}\mathscr{L}^N,$$

we see that $\mathbf{z}\cdot D^s u$ is a bounded measure.

We have the following *Green's formula* for $\mathbf{z} \in X_1(\mathbb{R}^N)$ and $w \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ [8]:

$$\int_{\mathbb{R}^N} w\,\mathrm{div}(\mathbf{z})\,\mathrm{d}x + \int_{\mathbb{R}^N} (\mathbf{z}, Dw) = 0. \tag{2.10}$$

3. The notion of entropy solution for the elliptic problem

3.1. Assumptions on the Lagrangian f

Our purpose in this section is to introduce the main assumptions on the Lagrangian f and to give a sense to the expression

$$v = -\mathrm{div}\, \mathbf{a}(u, Du), \quad \text{in } \mathbb{R}^N. \tag{3.1}$$

We assume that the Lagrangian $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^+$ satisfies the following assumptions, to which we refer collectively as (H):

(H_1) f is continuous on $\mathbb{R} \times \mathbb{R}^N$ and is a convex differentiable function of ξ such that $\nabla_\xi f(z, \xi) \in C(\mathbb{R} \times \mathbb{R}^N)$. Further we require f to satisfy the linear growth condition

$$C_0(z)\|\xi\| - D_0(z) \leqslant f(z, \xi) \leqslant M_0(z)(\|\xi\| + 1), \tag{3.2}$$

for any $(z, \xi) \in \mathbb{R} \times \mathbb{R}^N$, and some positive and continuous functions C_0, D_0, M_0, such that $C_0(z) > 0$ for any $z \neq 0$. Moreover, we assume f^0 exists.

We consider the function $\mathbf{a}(z, \xi) = \nabla_\xi f(z, \xi)$ associated to the Lagrangian f. By the convexity of f

$$\mathbf{a}(z, \xi) \cdot (\eta - \xi) \leqslant f(z, \eta) - f(z, \xi) \tag{3.3}$$

and the following monotonicity condition is satisfied

$$(\mathbf{a}(z, \eta) - \mathbf{a}(z, \xi)) \cdot (\eta - \xi) \geqslant 0. \tag{3.4}$$

Moreover, it is easy to see that for each $R > 0$, there is a constant $M = M(R) > 0$, such that

$$|\mathbf{a}(z, \xi)| \leqslant M, \quad \forall (z, \xi) \in \mathbb{R} \times \mathbb{R}^N, \ |z| \leqslant R. \tag{3.5}$$

We also assume that $\mathbf{a}(z, 0) = 0$, for all $z \in \mathbb{R}$ and $\mathbf{a}(z, \xi) = z\mathbf{b}(z, \xi)$ with

$$|\mathbf{b}(z, \xi)| \leqslant M_0, \quad \forall (z, \xi) \in \mathbb{R} \times \mathbb{R}^N, |z| \leqslant R. \tag{3.6}$$

We consider the function $h : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$h(z, \xi) := \mathbf{a}(z, \xi) \cdot \xi.$$

By (3.4), we have

$$h(z, \xi) \geqslant 0, \quad \forall \xi \in \mathbb{R}^N, \ z \in \mathbb{R}. \tag{3.7}$$

Moreover, from (3.3) and (3.2), it follows that

$$C_0(z)\|\xi\| - D_1(z) \leqslant h(z, \xi) \leqslant M\|\xi\|, \tag{3.8}$$

for any $(z, \xi) \in \mathbb{R} \times \mathbb{R}^N$, $|z| \leqslant R$, where $D_1(z) = D_0(z) + f(z, 0)$. We note that the left inequality holds for any $(z, \xi) \in \mathbb{R} \times \mathbb{R}^N$. Moreover, we assume that there exist constants A, $B > 0$ and $\alpha, \beta \geqslant 1$, such that

$$|D_1(z)| \leqslant A|z|^\alpha + B|z|^\beta, \quad \text{for any } z \in \mathbb{R}^N. \tag{3.9}$$

This condition will be used to prove some estimates during the proof of existence, and we assume it for simplicity, since a more general condition could be used.

(H_2) We assume that $\partial\mathbf{a}/\partial\xi_i(z,\xi) \in C(\mathbb{R}\times\mathbb{R}^N)$ for any $i=1,\dots,N$.

We assume that

(H_3) $h(z,\xi)=h(z,-\xi)$, for all $z\in\mathbb{R}$ and $\xi\in\mathbb{R}^N$ and h^0 exists.

Observe that we have

$$C_0(z)\|\xi\| \leqslant h^0(z,\xi) \leqslant M\|\xi\|, \quad \text{for any } (z,\xi)\in\mathbb{R}\times\mathbb{R}^N,\ |z|\leqslant R.$$

(H_4) $f^0(z,\xi)=h^0(z,\xi)$, for all $\xi\in\mathbb{R}^N$ and all $z\in\mathbb{R}$.

(H_5) $\mathbf{a}(z,\xi)\cdot\eta \leqslant h^0(z,\eta)$, for all $\xi,\eta\in\mathbb{R}^N$ and all $z\in\mathbb{R}$.

(H_6) We assume that $h^0(z,\xi)$ can be written in the form $h^0(z,\xi)=\varphi(z)\psi^0(\xi)$, where φ is a Lipschitz continuous function such that $\varphi(z)>0$ for any $z\neq 0$, and ψ^0 is a convex function which homogeneous of degree 1.

(H_7) For any $R>0$, there is a constant $C>0$ such that

$$|(\mathbf{a}(z,\xi)-\mathbf{a}(\hat{z},\xi))\cdot(\xi-\hat{\xi})| \leqslant C|z-\hat{z}|\,\|\xi-\hat{\xi}\|, \tag{3.10}$$

for any $(z,\xi),(\hat{z},\hat{\xi})\in\mathbb{R}\times\mathbb{R}^N$, $|z|,|\hat{z}|\leqslant R$.

Observe that, by the monotonicity condition (3.4) and using (3.10), it follows that

$$(\mathbf{a}(z,\xi)-\mathbf{a}(\hat{z},\hat{\xi}))\cdot(\xi-\hat{\xi}) \geqslant -C|z-\hat{z}|\,\|\xi-\hat{\xi}\|, \tag{3.11}$$

for any $(z,\xi),(\hat{z},\xi)\in\mathbb{R}\times\mathbb{R}^N$, $|z|,|\hat{z}|\leqslant R$.

Let us observe that under assumptions (H_4) and (H_6), applying (2.5), we have

$$f(u,Du)^s = h(u,Du)^s = \psi^0\left(\frac{Du}{|Du|}\right)|D^s J_\varphi(u)|. \tag{3.12}$$

Remark 3.1. There are physical models for plasma fusion by inertial confinement in which the temperature evolution of the electrons satisfies an equation of type (1.2), where $\mathbf{a}(z,\xi)=|z|^{5/2}\xi/1+|z||\xi|$ which corresponds to $f(z,\xi)=|z|^{3/2}|\xi|-|z|^{1/2}\ \ln(1+|z||\xi|)$ [21], (see also [13] for a mathematical study in the one-dimensional case). It is easy to check that (H_1) (in particular (3.2) and (3.8)) holds for any $(z,\xi)\in\mathbb{R}\times\mathbb{R}^N$. Notice that condition (H_2) holds. We observe that $h^0(z,\xi)=|z|^{3/2}|\xi|$ and (H_3)–(H_6) hold. Finally, to check (H_7) we observe that

$$\frac{\partial\mathbf{a}}{\partial z}(z,\xi) = \frac{5}{2}\frac{z^{3/2}\xi}{1+z|\xi|} - \frac{z^{5/2}|\xi|\xi}{(1+z|\xi|)^2}$$

and therefore

$$\left|\frac{\partial\mathbf{a}}{\partial z}(z,\xi)\right| \leqslant \frac{7}{2}\,|z|^{1/2},$$

for any $(z,\xi)\in\mathbb{R}\times\mathbb{R}^N$. From this, it follows that

$$|\mathbf{a}(z,\xi)-\mathbf{a}(\hat{z},\xi)| \leqslant \tfrac{7}{2}\,R^{1/2}|z-\hat{z}|,$$

for any $(z, \xi) \in \mathbb{R} \times \mathbb{R}^N$, $|z| \leqslant R$. Thus also (H_7) holds. In this case, the results below will prove existence and uniqueness of entropy solutions of (1.2) for any initial condition $u_0 \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, $u_0 \geqslant 0$.

Remark 3.2. The function $f(z, \xi) = \nu/a^2|z|\sqrt{z^2 + a^2|\xi|^2}$ satisfies the assumptions (H_1)–(H_7), with $a(z, \xi) = \nu \frac{|\mathbf{z}|\xi}{\sqrt{z^2+a^2|\xi|^2}}$. This particular case is related to the so-called *relativistic heat equation* (see [14,26])

$$u_t = \nu \operatorname{div}\left(\frac{|u|Du}{\sqrt{u^2 + a^2|Du|^2}}\right), \tag{3.13}$$

with $a = \nu/c$, c being a bound of the propagation speed, and ν being a constant representing a kinematic viscosity.

Let us mention that, as pointed out by Brenier in [14], this relativistic heat equation can be derived using Monge–Kantorovich's mass transport theory. On the other hand it is one among the various *flux limited diffusion equations* used in the theory of radiation hydrodynamics [24]. Indeed, a very similar equation

$$u_t = \nu \operatorname{div}\left(\frac{uDu}{u + \frac{\nu}{c}|Du|}\right) \tag{3.14}$$

can be found in [24]. In this case, the Lagrangian f associated with the above equation is

$$f(z, \xi) := cz\left(|\xi| - \frac{cz}{\nu}\log\left(1 + \frac{\nu}{cz}|\xi|\right)\right)$$

and satisfies the assumptions (H_1)–(H_7).

3.2. A functional calculus

We need to consider the following truncature functions. For $a < b$, let $T_{a,b}(r) := \max(\min(b, r), a)$. As usual, we denote $T_k = T_{-k,k}$. We also consider the truncature functions of the form $T^l_{a,b}(r) := T_{a,b}(r) - l$ ($l \in \mathbb{R}$). We denote

$$\mathcal{T}_r := \{T_{a,b} : 0 < a < b\}, \quad \mathcal{T}^+ := \{T^l_{a,b} : 0 < a < b,\ l \in \mathbb{R},\ T^l_{a,b} \geqslant 0\}.$$

We need to consider the function space

$$TBV^+(\mathbb{R}^N) := \{u \in L^1(\mathbb{R}^N)^+ : T(u) \in BV_{\mathrm{loc}}(\mathbb{R}^N), \quad \forall T \in \mathcal{T}_r\}$$

and to give a sense to the Radon–Nikodym derivative ∇u of a function $u \in TBV^+(\mathbb{R}^N)$. Using chain's rule for *BV*-functions (see, for instance, [1]), and with a similar proof to the one given in Lemma 2.1 of [9], we obtain the following result:

Lemma 3.3. *For every $u \in TBV^+(\mathbb{R}^N)$ there exists a unique measurable function $v : \mathbb{R}^N \to \mathbb{R}^N$ such that*

$$\nabla T_{a,b}(u) = v\chi_{[a<u<b]}, \quad \mathcal{L}^N\text{-a.e.}, \ \forall\ T_{a,b} \in \mathcal{T}_r. \tag{3.15}$$

Thanks to this result we define ∇u for a function $u \in TBV^+(\mathbb{R}^N)$ as the unique function v which satisfies (3.15). This notation will be used throughout in the sequel.

We denote by $\mathscr{P}$ the set of Lipschitz continuous function $p : [0, +\infty[\to \mathbb{R}$ satisfying $p'(s) = 0$ for s large enough. We write $\mathscr{P}^+ := \{p \in \mathscr{P} : p \geqslant 0\}$. We recall the following result ([2], Lemma 2).

Lemma 3.4. *If $u \in TBV^+(\mathbb{R}^N)$, then $p(u) \in BV(\mathbb{R}^N)$ for every $p \in \mathscr{P}$ such that there exists $a > 0$ with $p(r) = 0$ for all $0 \leqslant r \leqslant a$. Moreover, $\nabla p(u) = p'(u)\nabla u$ $\mathscr{L}^N$*-a.e.

For any function q, let $J_q(r)$ denote the primitive of q, i.e., $J_q(r) = \int_0^r q(s)\,\mathrm{d}s$. Let $S \in \mathscr{P}$ and $T = T^a_{a,b}$. Given a function $u \in TBV^+(\mathbb{R}^N)$, by Lemma 3.4, we have $S(u)T(u)$, $J_{T'S}(u)$, $J_{TS'}(u) \in BV(\mathbb{R}^N)$. Moreover, it is easy to see that

$$D(S(u)T(u)) = DJ_{T'S}(u) + DJ_{TS'}(u). \tag{3.16}$$

Hence, if $\mathbf{z} \in X_1(\mathbb{R}^N)$, we have

$$(\mathbf{z}, D(S(u)T(u))) = (\mathbf{z}, DJ_{T'S}(u)) + (\mathbf{z}, DJ_{TS'}(u)). \tag{3.17}$$

Let $g : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to [0, \infty[$ be a function satisfying the assumption of Theorem 2.1, and $T \in \mathscr{T}^+$. Then there is some $T_{a,b} \in \mathscr{T}_r$ and a constant $c \in \mathbb{R}$ such that $T = T_{a,b} - c$. Observe that

$$r = T(r) + c \quad \text{at the values of } r \in \mathbb{R}, \text{ where } T'(r) = 1. \tag{3.18}$$

Let us consider the functional

$$R(g, T)(u) := \int_{\mathbb{R}^N} g(x, u(x), \nabla T(u(x)))\,\mathrm{d}x, \quad u \in W^{1,1}(\mathbb{R}^N).$$

For $u \in TBV^+(\mathbb{R}^N)$, if we define

$$\begin{aligned} \mathscr{R}(g, T)(u) := \mathscr{R}_g(T_{a,b}(u)) &+ \int_{[u \leqslant a]} (g(x, u(x), 0) - g(x, a, 0))\,\mathrm{d}x \\ &+ \int_{[u \geqslant b]} (g(x, u(x), 0) - g(x, b, 0))\,\mathrm{d}x, \end{aligned} \tag{3.19}$$

by Theorem 2.1, we have that $\mathscr{R}(g, T)$ is lower semi-continuous in $TBV^+(\mathbb{R}^N)$ with respect to $L^1(\mathbb{R}^N)$-convergence. Observe that, with this notation, we have

$$\mathscr{R}(g, T)(u) = \mathscr{R}(g, T_{a,b})(u).$$

Moreover, if $u \in W^{1,1}(\mathbb{R}^N)$, using (3.18) we have

$$R(g, T)(u) = \mathscr{R}(g, T)(u).$$

Since it will be sufficient for our purposes, let us assume that g does not depend on x. If $u \in TBV^+(\mathbb{R}^N)$ and $T \in \mathscr{T}^+$, we define the Radon measure $g(u, DT(u))$ in $\mathbb{R}^N$ by

$$\langle g(u, DT(u)), \phi\rangle := \mathscr{R}(\phi g, T)(u), \quad \phi \in C_{\mathrm{c}}(\mathbb{R}^N). \tag{3.20}$$

If $T \in \mathcal{T}_r$, then $T(r) = r$ for any $r \in \mathbb{R}$ such that $T'(r) = 1$, and, using (3.19), (3.20), and (2.3), we have that

$$\langle g(u, DT(u)), \phi\rangle = \langle g(T(u), DT(u)), \phi\rangle + \int_{[u \leqslant a]} \phi(g(x, u(x), 0) - g(x, a, 0))\,\mathrm{d}x + \int_{[u \geqslant b]} \phi\,(g(x, u(x), 0) - g(x, b, 0))\,\mathrm{d}x.$$

Let $S \in \mathcal{P}^+$ and $T \in \mathcal{T}^+$. We denote by $f_S(u, DT(u)), h_S(u, DT(u))$, the Radon measures defined by (3.20) with $g(z, \xi) = S(z) f(z, \xi)$, and $g(z, \xi) = S(z)h(z, \xi)$, respectively.

Since $h(z, 0) = 0$, for all $z \in \mathbb{R}$, if $S, T \in \mathcal{T}^+$, with $T = T_{a,b} - c$, we have

$$h_S(u, DT(u)) = h_S(T_{a,b}(u), DT(u)) = h_S(T_{a,b}(u), DT_{a,b}(u)) \tag{3.21}$$

and, by (2.5),

$$(f_S(u, DT(u)))^s = (f_S(T_{a,b}(u), DT_{a,b}(u)))^s = \psi^0\left(\frac{DT_{a,b}(u)}{|DT_{a,b}(u)|}\right)|D^s J_{S\varphi}(T_{a,b}(u))|. \tag{3.22}$$

Similarly, we have

$$(h_S(u, DT(u)))^s = (h_S(u, DT_{a,b}(u)))^s = \psi^0\left(\frac{DT_{a,b}(u)}{|DT_{a,b}(u)|}\right)|D^s J_{S\varphi}(T_{a,b}(u))|. \tag{3.23}$$

Note that both singular parts are identical. By the representation formulas in Section 2.2, the absolutely continuous part of $h_S(u, DT(u))$ is $S(u)h(u, \nabla T(u))$. Similar identities are true when $S = 1$.

3.3. *The notion of entropy solution*

We introduce the following concept of solution for problem (3.1):

Definition 3.5. Given $v \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$, $v \geqslant 0$, we say that $u \geqslant 0$ is an *entropy solution* of (3.1) if $u \in TBV^+(\mathbb{R}^N)$, and $\mathbf{a}(u, \nabla u) \in X_1(\mathbb{R}^N)$ satisfies

$$v = -\mathrm{div}\,(\mathbf{a}(u, \nabla u)), \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \tag{3.24}$$

$$h_S(u, DT(u)) \leqslant (\mathbf{a}(u, \nabla u), DJ_{T'S}(u)) \quad \text{as measures } \forall S \in \mathcal{P}^+,\ T \in \mathcal{T}^+, \tag{3.25}$$

$$h(u, DT(u)) \leqslant (\mathbf{a}(u, \nabla u), DT(u)) \quad \text{as measures } \forall T \in \mathcal{T}^+. \tag{3.26}$$

Note that (3.25) and (3.26) are equivalent to

$$h_S(u, DT(u))^s \leqslant (\mathbf{a}(u, \nabla u), DJ_{T'S}(u))^s \quad \text{as measures } \forall S \in \mathcal{P}^+,\ T \in \mathcal{T}^+ \tag{3.27}$$

and

$$h(u, DT(u))^s \leqslant (\mathbf{a}(u, \nabla u), DT(u))^s \quad \text{as measures } \forall\, T \in \mathcal{T}^+, \tag{3.28}$$

respectively. The inequalities in (3.25) will be useful to prove uniqueness and the ones in (3.26) are convenient to prove Lemma 5.2. We could have restricted the inequalities in (3.25) to hold only for $S, T \in \mathcal{T}^+$, but using $S \in \mathcal{P}^+$ turns out to be convenient to simplify the proof of uniqueness.

4. Existence and uniqueness of entropy solution

This section is devoted to prove the following existence and uniqueness result.

Theorem 4.1. *Assume that assumptions* (H) *hold. Then, for any* $0 \leqslant v \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ *there exists a unique entropy solution* $u \in TBV^+(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ *of the problem*

$$u - \operatorname{div} \mathbf{a}(u, Du) = v, \quad in\ \mathbb{R}^N. \tag{4.1}$$

Moreover, given $v, \overline{v} \in L^\infty(\mathbb{R}^N)^+$, *if* $u, \overline{u}$ *are bounded entropy solutions of the problems*

$$u - \operatorname{div} \mathbf{a}(u, Du) = v, \quad in\ \mathbb{R}^N$$

and

$$\overline{u} - \operatorname{div} \mathbf{a}(\overline{u}, D\overline{u}) = \overline{v}, \quad in\ \mathbb{R}^N,$$

respectively, then

$$\int_{\mathbb{R}^N} (u - \overline{u})^+ \leqslant \int_{\mathbb{R}^N} (v - \overline{v})^+.$$

Proof. *Existence of entropy solutions*. We divide the proof in different steps.

Step 1: *Approximation and basic estimates.* Let $0 \leqslant v \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. For every $n \in \mathbb{N}$, consider $\mathbf{a}_n(z, \xi) := \mathbf{a}(z, \xi) + \frac{1}{n}\,\xi$. As a consequence of the results about pseudomonotone operators given in [16] we know that for any $n \in \mathbb{N}$ there exists $u_n \in W^{1,2}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} (w - u_n)(v - u_n)\,\mathrm{d}x \leqslant \int_{\mathbb{R}^N} \mathbf{a}_n(u_n, \nabla u_n) \cdot \nabla(w - u_n)\,\mathrm{d}x, \quad \forall\, w \in W^{1,2}(\mathbb{R}^N). \tag{4.2}$$

Let

$$\mathcal{P}_0 := \{p \in C^\infty(\mathbb{R}) \,:\, 0 \leqslant p' \leqslant 1,\ \operatorname{supp}(p')\ \text{compact},\ 0 \notin \operatorname{supp}(p)\}.$$

Given $p \in \mathscr{P}_0$, taking $w = u_n - p(u_n)$ as test function in (4.2), we obtain

$$\int_{\mathbb{R}^N} u_n p(u_n) \leqslant \int_{\mathbb{R}^N} v p(u_n).$$

Then, by results in [10], it follows that $u_n \ll v$, for all $n \in \mathbb{N}$ and consequently, we have

$$\|u_n\|_p \leqslant \|v\|_p, \quad \text{for all } n \in \mathbb{N}, \quad \text{for all } p \in [1, \infty] \tag{4.3}$$

and

$$\|u_n\|_{1\cap\infty} \leqslant \|v\|_{1\cap\infty}, \quad \text{for all } n \in \mathbb{N}. \tag{4.4}$$

Moreover, $u_n \geqslant 0$ and

$$\{u_n : n \in \mathbb{N}\} \quad \text{is a weakly sequentially compact subset of } L_0(\mathbb{R}^N). \tag{4.5}$$

Taking $w = 0$ in (4.2), applying Young's inequality and using (4.3) we get

$$\int_{\mathbb{R}^N} \mathbf{a}(u_n, \nabla u_n) \cdot \nabla u_n \, \mathrm{d}x + \frac{1}{n} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, \mathrm{d}x \leqslant \int_{\mathbb{R}^N} u_n (v - u_n) \, \mathrm{d}x \leqslant C,$$

for some constant $C > 0$ depending on $\|v\|_2$. Since, by (3.8), we have

$$a(u_n, \nabla u_n) \cdot \nabla u_n \geqslant C_0(u_n) |\nabla u_n| - D_1(u_n),$$

using (3.9), we obtain

$$\int_{\mathbb{R}^N} |\nabla Q(u_n)| \, \mathrm{d}x \leqslant C + \int_{\mathbb{R}^N} D_1(u_n) \leqslant M_1, \quad \forall n \in \mathbb{N} \tag{4.6}$$

and

$$\frac{1}{n} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, \mathrm{d}x \leqslant C, \quad \forall n \in \mathbb{N}, \tag{4.7}$$

where $Q(r)$ is a primitive of C_0. By (4.5) and (4.3), by extracting a subsequence if is necessary, we may assume that u_n converges weakly in $L_0(\mathbb{R}^N)$ and in $L^2(\mathbb{R}^N)$ to some nonnegative function u as $n \to +\infty$. Moreover, by (4.4), we have that $0 \leqslant u \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. On the other hand, if $0 < a < b$, by the coarea formula and (4.6), we have

$$\begin{aligned}
\int_{\mathbb{R}^N} |\nabla T_{a,b}(u_n)| &= \int_a^b |D\chi_{[u_n \leqslant t]}|(\mathbb{R}^N) \, \mathrm{d}t = \int_a^b |D\chi_{[Q(u_n) \leqslant Q(t)]}|(\mathbb{R}^N) \, \mathrm{d}t \\
&= \int_{Q(a)}^{Q(b)} |D\chi_{[Q(u_n) \leqslant s]}|(\mathbb{R}^N) \, \frac{\mathrm{d}s}{Q'(Q^{-1}(s))} \\
&\leqslant \frac{1}{\inf_{[a,b]} C_0} \int_{-\infty}^{+\infty} |D\chi_{[Q(u_n) \leqslant s]}|(\mathbb{R}^N) \, \mathrm{d}s \leqslant \frac{M_1}{\inf_{[a,b]} C_0}.
\end{aligned}$$

Thus, $T_{a,b}(u_n) \to T_{a,b}(u)$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Consequently, we may assume that u_n converges almost everywhere to u. Then, by Vitali's Convergence Theorem, we get that $u_n \to u$ in $L^1(\mathbb{R}^N)$, and using the above estimate on the gradients we obtain that $u \in TBV^+(\mathbb{R}^N)$.

Observe that by (3.5) we may assume that

$$\mathbf{a}(u_n, \nabla u_n) \rightharpoonup \mathbf{z} \quad \text{as } n \to \infty, \text{ weakly}^* \text{ in } L^\infty(\mathbb{R}^N, \mathbb{R}^N). \tag{4.8}$$

Since, by assumption we have that $\mathbf{a}(u_n, \nabla u_n) = |u_n|\mathbf{b}(u_n, \nabla u_n)$ with $|\mathbf{b}(u_n, \nabla u_n)| \leqslant M_0$ (where the constant M_0 is independent of n), $\|u_n\|_\infty \leqslant \|v\|_\infty$, and $u_n \to u$ a.e. as $n \to \infty$, we may assume that

$$\mathbf{b}(u_n, \nabla u_n) \rightharpoonup \mathbf{z}_b \quad \text{as } n \to \infty, \text{ weakly}^* \text{ in } L^\infty(\mathbb{R}^N, \mathbb{R}^N)$$

and

$$\mathbf{z} = u\mathbf{z}_b. \tag{4.9}$$

On the other hand, by (4.7),

$$\frac{1}{n}|\nabla u_n| \to 0, \quad \text{in } L^2(\mathbb{R}^N). \tag{4.10}$$

Given $\phi \in \mathscr{D}(\mathbb{R}^N)$, taking $w = u_n \pm \phi$ in (4.2) we obtain

$$\int_{\mathbb{R}^N} \phi(v - u_n)\,\mathrm{d}x = \int_{\mathbb{R}^N} \mathbf{a}(u_n, \nabla u_n) \cdot \nabla\phi\,\mathrm{d}x + \frac{1}{n}\int_{\mathbb{R}^N} \nabla u_n \cdot \nabla\phi\,\mathrm{d}x.$$

Letting $n \to +\infty$, having in mind (4.8) and (4.10), we obtain

$$\int_{\mathbb{R}^N} (v - u)\phi\,\mathrm{d}x = \int_{\mathbb{R}^N} \mathbf{z} \cdot \nabla\phi\,\mathrm{d}x,$$

that is,

$$v - u = -\mathrm{div}(\mathbf{z}), \quad \text{in } \mathscr{D}'(\mathbb{R}^N) \tag{4.11}$$

and

$$\mathrm{div}\,\mathbf{a}_n(u_n, \nabla u_n) \rightharpoonup \mathrm{div}(\mathbf{z}), \quad \text{weakly in } L^2(\mathbb{R}^N). \tag{4.12}$$

Note that by (4.11), we have $\mathbf{z} \in X_1(\mathbb{R}^N)$.

Step 2: *Identification of* $\mathbf{z}(x)$.

Lemma 4.2. *We have*

$$\mathbf{z}(x) = \mathbf{a}(u(x), \nabla u(x)), \quad \text{a.e. } x \in \mathbb{R}^N. \tag{4.13}$$

Proof. We use Minty–Browder's technique. Let $0 < a < b$, let $0 \leqslant \phi \in C_c^1(\mathbb{R}^N)$ and $g \in C^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$. By (3.4), we have

$$\int_{\mathbb{R}^N} \phi[\mathbf{a}(u_n, \nabla u_n) - \mathbf{a}(u_n, \nabla g) \cdot \nabla(u_n - g)]T'_{a,b}(u_n)\,\mathrm{d}x \geqslant 0.$$

Now, since

$$\begin{aligned}
&\int_{\mathbb{R}^N} \phi \mathbf{a}(u_n, \nabla u_n) \cdot \nabla(u_n - g) T'_{a,b}(u_n)\, \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^N} \phi \mathbf{a}(u_n, \nabla u_n) \cdot \nabla(T_{a,b}(u_n) - g)\, \mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^N} \phi \mathbf{a}(u_n, \nabla u_n) \cdot \nabla g\, (1 - T'_{a,b}(u_n))\, \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^N} \phi \mathbf{a}_n(u_n, \nabla u_n) \cdot \nabla(T_{a,b}(u_n) - g)\, \mathrm{d}x \\
&\qquad - \frac{1}{n} \int_{\mathbb{R}^N} \phi \nabla u_n \cdot \nabla(T_{a,b}(u_n) - g)\, \mathrm{d}x \\
&\qquad + \int_{\mathbb{R}^N} \phi \mathbf{a}(u_n, \nabla u_n) \cdot \nabla g\, (1 - T'_{a,b}(u_n))\, \mathrm{d}x \\
&\quad \leqslant - \int_{\mathbb{R}^N} \mathrm{div}(\mathbf{a}(u_n, \nabla u_n)) \phi (T_{a,b}(u_n) - g)\, \mathrm{d}x \\
&\qquad - \int_{\mathbb{R}^N} (T_{a,b}(u_n) - g) \mathbf{a}(u_n, \nabla u_n) \cdot \nabla \phi\, \mathrm{d}x \\
&\qquad + \frac{1}{n} \int_{\mathbb{R}^N} \phi \nabla u_n \cdot \nabla g\, \mathrm{d}x + \int_{\mathbb{R}^N} \phi \mathbf{a}(u_n, \nabla u_n) \\
&\qquad \times \nabla g\, (1 - T'_{a,b}(u_n))(T'_{a,b}(u) + (1 - T'_{a,b}(u)))\, \mathrm{d}x,
\end{aligned}$$

we get

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{R}^N} \phi \mathbf{a}(u_n, \nabla u_n) \cdot \nabla(u_n - g)\, T'_{a,b}(u_n)\, \mathrm{d}x \\
&\quad \leqslant - \int_{\mathbb{R}^N} \mathrm{div}(\mathbf{z}) \phi (T_{a,b}(u) - g)\, \mathrm{d}x - \int_{\mathbb{R}^N} (T_{a,b}(u) - g) \mathbf{z} \cdot \nabla \phi\, \mathrm{d}x \\
&\qquad + M \|\nabla g\|_\infty \int_{\mathbb{R}^N} \phi\, (1 - T'_{a,b}(u))\, \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^N} \phi (\mathbf{z}, D(T_{a,b}(u) - g)) + M \|\nabla g\|_\infty \int_{\mathbb{R}^N} \phi\, (1 - T'_{a,b}(u))\, \mathrm{d}x.
\end{aligned}$$

On the other hand, let us denote by

$$J_{\mathbf{a}_i}(x, r) := \int_0^r \mathbf{a}_i(s, \nabla g(x))\, \mathrm{d}s \quad \text{and} \quad J_{\frac{\partial \mathbf{a}_i}{\partial x_j}}(x, r) := \int_0^r \frac{\partial}{\partial x_j} \mathbf{a}_i(s, \nabla g(x))\, \mathrm{d}s,$$

$i, j \in \{1, \ldots, N\}$ and observe that

$$\frac{\partial}{\partial x_j} J_{\mathbf{a}_i}(x, T_{a,b}(u_n(x))) = \mathbf{a}_i(u_n(x), \nabla g(x)) \frac{\partial u_n}{\partial x_j}(x) T'_{a,b}(u_n) + J_{\frac{\partial \mathbf{a}_i}{\partial x_j}}(x, T_{a,b}(u_n(x))).$$

Now, since

$$\frac{\partial}{\partial x_j} J_{\mathbf{a}_i}(x, T_{a,b}(u_n)) \rightharpoonup \frac{\partial}{\partial x_j} J_{\mathbf{a}_i}(x, T_{a,b}(u)),$$

weakly as measures, and $J_{\frac{\partial \mathbf{a}_i}{\partial x_j}}(x, T_{a,b}(u_n(x))) \to J_{\frac{\partial \mathbf{a}_i}{\partial x_j}}(x, T_{a,b}(u(x)))$ a.e., we have

$$\begin{aligned}
&\lim_{n\to\infty} \int_{\mathbb{R}^N} \phi\, \mathbf{a}(u_n, \nabla g) \cdot \nabla(u_n - g)\, T'_{a,b}(u_n)\, \mathrm{d}x \\
&\quad \geqslant \lim_{n\to\infty} \int_{\mathbb{R}^N} \phi \sum_{i=1}^N \left[\frac{\partial}{\partial x_i} J_{\mathbf{a}_i}(x, T_{a,b}(u_n(x))) - J_{\frac{\partial \mathbf{a}_i}{\partial x_i}}(x, T_{a,b}(u_n(x))) \right] \\
&\qquad - \lim_{n\to\infty} \int_{\mathbb{R}^N} \phi\, \mathbf{a}(u_n, \nabla g) \cdot \nabla g\, T'_{a,b}(u_n)(T'_{a,b}(u) + (1 - T'_{a,b}(u)))\, \mathrm{d}x \\
&\quad \geqslant \int_{\mathbb{R}^N} \phi \sum_{i=1}^N \left[\frac{\partial}{\partial x_i} J_{\mathbf{a}_i}(x, T_{a,b}(u)) - J_{\frac{\partial \mathbf{a}_i}{\partial x_i}}(x, T_{a,b}(u(x))) \right] \\
&\qquad - \int_{\mathbb{R}^N} \phi\, \mathbf{a}(u, \nabla g) \cdot \nabla g\, T'_{a,b}(u)\, \mathrm{d}x.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \phi(\mathbf{z}, D(T_{a,b}(u) - g)) + M\|\nabla g\|_\infty \int_{\mathbb{R}^N} \phi\, (1 - T'_{a,b}(u))\, \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \phi \mathbf{a}(u, \nabla g) \cdot \nabla g\, T'_{a,b}(u) \\
&\quad - \int_{\mathbb{R}^N} \phi \left(\sum_{i=1}^N \left[\frac{\partial}{\partial x_i} J_{\mathbf{a}_i}(x, T_{a,b}(u(x))) - J_{\frac{\partial \mathbf{a}_i}{\partial x_i}}(x, T_{a,b}(u(x))) \right] \right) \geqslant 0, \qquad (4.14)
\end{aligned}$$

for all $0 \leqslant \phi \in C_0^1(\mathbb{R}^N)$. Thus the measure

$$\begin{aligned}
&(\mathbf{z}, D(T_{a,b}(u) - g)) - \sum_{i=1}^N \left[\frac{\partial}{\partial x_i} J_{\mathbf{a}_i}(x, T_{a,b}(u(x))) - J_{\frac{\partial \mathbf{a}_i}{\partial x_i}}(x, T_{a,b}(u(x))) \right] \\
&\quad + \mathbf{a}(u, \nabla g) \cdot \nabla g\, T'_{a,b}(u)\, \mathscr{L}^N + M\|\nabla g\|_\infty (1 - T'_{a,b}(u))\, \mathscr{L}^N \geqslant 0.
\end{aligned}$$

Then using chain's rule for *BV* functions ([1], Theorem 3.96) applied to $J_{\mathbf{a}_i}(u_1, u_2)$ with $u_1(x) = x$, $u_2(x) = T_{a,b}(u(x))$, $x \in \mathbb{R}^N$, we deduce that the absolutely continuous part of

$$\sum_{i=1}^N \left[\frac{\partial}{\partial x_i} J_{\mathbf{a}_i}(x, T_{a,b}(u(x))) - J_{\frac{\partial \mathbf{a}_i}{\partial x_i}}(x, T_{a,b}(u(x))) \right]$$

is $\mathbf{a}(u, \nabla g) \cdot \nabla T_{a,b}(u)\, \mathscr{L}^N$, and we obtain

$$\begin{aligned}
&\mathbf{z} \cdot \nabla(T_{a,b}(u) - g) - \mathbf{a}(u, \nabla g) \cdot \nabla T_{a,b}(u) \\
&\quad + \mathbf{a}(u, \nabla g) \cdot \nabla g T'_{a,b}(u) + M\|\nabla g\|_\infty (1 - T'_{a,b}(u)) \geqslant 0.
\end{aligned}$$

In particular, for $x \in [a < u < b]$, we have

$$(\mathbf{z} - \mathbf{a}(u, \nabla g)) \cdot \nabla(u - g) \geqslant 0, \quad \text{a.e.}$$

Since we may take a countable set of functions $g \in C^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ which is dense in $C^1(\mathbb{R}^N)$, we have that the above inequality holds for all $x \in \Omega \cap [a < u < b]$, where

$\Omega \subset \mathbb{R}^N$ is such that $\mathscr{L}^N(\mathbb{R}^N \setminus \Omega) = 0$, and all $g \in C^1(\mathbb{R}^N)$. Now, fixed $x \in \Omega \cap [a < u < b]$ and given $\xi \in \mathbb{R}^N$, there is $g \in C^1(\mathbb{R}^N)$ such that $\nabla g(x) = \xi$. Then

$$(\mathbf{z}(x) - \mathbf{a}(u(x), \xi)) \cdot (\nabla u(x) - \xi) \geqslant 0, \quad \forall \xi \in \mathbb{R}^N.$$

By an application of Minty–Browder's method in $\mathbb{R}^N$, these inequalities imply that

$$\mathbf{z}(x) = \mathbf{a}(u(x), \nabla u(x)) \quad \text{a.e. on } [a < u < b].$$

Since this holds for any $0 < a < b$, we obtain (4.13) a.e. on the points x of $\mathbb{R}^N$ such that $u(x) \neq 0$. Now, by our assumptions on $\mathbf{a}$ and (4.9) we deduce that $\mathbf{z}(x) = \mathbf{a}(u(x), \nabla u(x)) = 0$ a.e. on $[u = 0]$. We have proved (4.13). $\square$

From (4.13) and (4.11), it follows that

$$v - u = -\operatorname{div} \mathbf{a}(u, \nabla u), \quad \text{in } \mathscr{D}'(\mathbb{R}^N). \tag{4.15}$$

Step 3: To finish the existence part of the proof we only need to prove that

$$h_S(u, DT(u)) \leqslant (\mathbf{a}(u, \nabla u), DJ_{T'S}(u)) \quad \text{as measures } \forall S \in \mathscr{P}^+,\ T \in \mathscr{T}^+ \tag{4.16}$$

and

$$h(u, DT(u)) \leqslant (\mathbf{a}(u, \nabla u), DT(u)) \quad \text{as measures } \forall T \in \mathscr{T}^+. \tag{4.17}$$

To prove (4.16) we require some intermediate inequalities summarized in next Lemma.

Lemma 4.3. *We have the inequalities*

$$\limsup_n \int_{\mathbb{R}^N} \mathbf{a}(u_n, \nabla u_n) \cdot \nabla (J_{T'S}(u_n)) \phi(x) \, \mathrm{d}x \leqslant \int_\Omega \phi (\mathbf{a}(u, \nabla u), D(J_{T'S}(u))) \tag{4.18}$$

for any $0 \leqslant \phi \in C_c(\mathbb{R}^N)$ *and*

$$f_S(u, DT(u)) \leqslant (\mathbf{a}(u, \nabla u), D(J_{T'S}(u))) + S(u) f(u, 0)\, \mathscr{L}^N. \tag{4.19}$$

Before proving the Lemma, let us give the proof of (4.16). Using (4.19), we have

$$\begin{aligned}
(\mathbf{a}(u, \nabla u), D(J_{T'S}(u))) &= (\mathbf{a}(u, \nabla u), D(J_{T'S}(u)))^{ac} + (\mathbf{a}(u, \nabla u), D(J_{T'S}(u)))^s \\
&\geqslant \mathbf{a}(u, \nabla u) \cdot \nabla (J_{T'S}(u)) + (f_S(u, DT(u)))^s \\
&= \mathbf{a}(u, \nabla u) \cdot \nabla (J_{T'S}(u)) + (h_S(u, DT(u)))^s \\
&= h_S(u, DT(u)
\end{aligned}$$

and (4.16) holds.

Proof. Let us prove (4.18). By (4.2), we have

$$\int_{\mathbb{R}^N} w(v - u_n) \, \mathrm{d}x = \int_\Omega \mathbf{a}_n(u_n, \nabla u_n) \cdot \nabla w \, \mathrm{d}x, \quad \forall w \in W^{1,2}(\mathbb{R}^N). \tag{4.20}$$

Then, taking $w = J_{T'S}(u_n)\phi$ as test function in (4.20), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} \phi\mathbf{a}(u_n,\nabla u_n)\cdot\nabla(J_{T'S}(u_n))\,\mathrm{d}x + \frac{1}{n}\int_{\mathbb{R}^N} \phi\nabla u_n\cdot\nabla(J_{T'S}(u_n))\,\mathrm{d}x\\ &\quad= \int_{\mathbb{R}^N}(v-u_n)J_{T'S}(u_n)\phi\,\mathrm{d}x - \int_{\mathbb{R}^N} J_{T'S}(u_n)\mathbf{a}(u_n,\nabla u_n)\cdot\nabla\phi\,\mathrm{d}x\\ &\qquad - \frac{1}{n}\int_{\mathbb{R}^N} J_{T'S}(u_n)\nabla u_n\cdot\nabla\phi\,\mathrm{d}x.\end{aligned}$$

Letting $n\to\infty$ we get

$$\begin{aligned}&\limsup_n \int_{\mathbb{R}^N} \phi\mathbf{a}(u_n,\nabla u_n)\cdot\nabla(J_{T'S}(u_n))\,\mathrm{d}x\\ &\quad\leqslant \int_{\mathbb{R}^N} \phi(v-u)J_{T'S}(u)\,\mathrm{d}x - \int_{\mathbb{R}^N} J_{T'S}(u)\mathbf{a}(u,\nabla u)\cdot\nabla\phi\,\mathrm{d}x\\ &\quad= -\int_{\mathbb{R}^N} \operatorname{div}\mathbf{a}(u,\nabla u)J_{T'S}(u)\phi\,\mathrm{d}x - \int_{\mathbb{R}^N} J_{T'S}(u)\mathbf{a}(u,\nabla u)\cdot\nabla\phi\\ &\quad= \int_{\mathbb{R}^N} \phi(\mathbf{a}(u,\nabla u), D(J_{T'S}(u))).\end{aligned}$$

Let us prove (4.19). Using the convexity of f, and using that

$$\mathbf{a}(u_n,\nabla T(u_n))\cdot\nabla T(u_n) = \mathbf{a}(u_n,\nabla u_n)\cdot\nabla T(u_n),$$

we have

$$\begin{aligned}&\int_{\mathbb{R}^N} \phi S(u_n)f(u_n,\nabla T(u_n))\,\mathrm{d}x\\ &\quad\leqslant \int_{\mathbb{R}^N} \phi S(u_n)\mathbf{a}(u_n,\nabla T(u_n))\cdot\nabla T(u_n)\,\mathrm{d}x + \int_{\mathbb{R}^N} \phi S(u_n)f(u_n,0)\,\mathrm{d}x\\ &\quad= \int_{\mathbb{R}^N} \phi\mathbf{a}(u_n,\nabla u_n)\cdot\nabla(J_{T'S}(u_n))\,\mathrm{d}x + \int_{\mathbb{R}^N} \phi S(u_n)f(u_n,0)\,\mathrm{d}x.\end{aligned}$$

Then, since $\mathscr{R}(\phi Sf, T)$ is lower semi-continuous respect to the L^1-convergence and (4.18), letting $n\to\infty$ we obtain

$$\begin{aligned}&\langle f_S(u, DT(u)),\phi\rangle\\ &\quad= \mathscr{R}(\phi Sf, T)(u) \leqslant \liminf_n \mathscr{R}(\phi Sf, T)(u_n)\,\mathrm{d}x\\ &\quad= \liminf_n \int_{\mathbb{R}^N} \phi S(u_n)f(u_n,\nabla T(u_n))\,\mathrm{d}x\\ &\quad\leqslant \liminf_n \int_{\mathbb{R}^N} \phi\mathbf{a}(u_n,\nabla u_n)\cdot\nabla(J_{T'S}(u_n))\,\mathrm{d}x + \int_{\mathbb{R}^N} \phi S(u_n)f(u_n,0)\,\mathrm{d}x\\ &\quad\leqslant \limsup_n \int_{\mathbb{R}^N} \mathbf{a}(u_n,\nabla u_n)\cdot\nabla(J_{T'S}(u_n))\phi(x)\,\mathrm{d}x + \int_{\mathbb{R}^N} \phi S(u)f(u,0)\,\mathrm{d}x\\ &\quad\leqslant \int_{\mathbb{R}^N} \phi(\mathbf{a}(u,\nabla u), D(J_{T'S}(u))) + \int_{\mathbb{R}^N} \phi S(u)f(u,0)\,\mathrm{d}x\end{aligned}$$

and (4.19) holds. □

In a similar way (4.17) follows from the following Lemma:

Lemma 4.4. *We have the following inequalities*

$$\limsup_n \int_{\mathbb{R}^N} \phi \mathbf{a}(u_n, \nabla u_n) \cdot \nabla T(u_n)) \, \mathrm{d}x \leqslant \int_{\mathbb{R}^N} \phi(\mathbf{a}(u, \nabla u), D(T(u))), \tag{4.21}$$

for any $0 \leqslant \phi \in C_{\mathrm{c}}(\mathbb{R}^N)$ *and*

$$f(u, DT(u)) \leqslant (\mathbf{a}(u, \nabla u), D(T(u))) + f(u, 0)\mathscr{L}^N. \tag{4.22}$$

Before going into the proof, let us prove (4.17). From (4.22) it follows that

$$(h(u, DT(u)))^s(u) = (f(u, DT(u)))^s(u) \leqslant \mathbf{a}(u, \nabla u) \cdot D^s(T(u))).$$

Hence,

$$\begin{aligned}(\mathbf{a}(u, \nabla u), D(T(u))) &= \mathbf{a}(u, \nabla u) \cdot \nabla T(u) + \mathbf{a}(u, \nabla u) \cdot D^s(T(u))) \\ &\geqslant \mathbf{a}(u, \nabla u) \cdot \nabla T(u) + (h(u, DT(u)))^s = h(u, DT(u)).\end{aligned}$$

Proof. To prove (4.21) let $0 \leqslant \phi \in C_{\mathrm{c}}(\mathbb{R}^N)$, taking $w = T(u_n)\phi$ as test function in (4.20), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} \phi \mathbf{a}(u_n, \nabla u_n) \cdot \nabla T(u_n)) \, \mathrm{d}x + \frac{1}{n} \int_{\mathbb{R}^N} \phi \nabla u_n \cdot \nabla T(u_n)) \, \mathrm{d}x \\ &\quad = \int_{\mathbb{R}^N} (v - u_n) T(u_n) \phi \, \mathrm{d}x - \int_{\mathbb{R}^N} T(u_n) \mathbf{a}(u_n, \nabla u_n) \cdot \nabla \phi \, \mathrm{d}x \\ &\qquad - \frac{1}{n} \int_{\mathbb{R}^N} T(u_n) \nabla u_n \cdot \nabla \phi \, \mathrm{d}x.\end{aligned}$$

Letting $n \to \infty$ we get

$$\begin{aligned}&\limsup_n \int_{\mathbb{R}^N} \phi \mathbf{a}(u_n, \nabla u_n) \cdot \nabla T(u_n)) \, \mathrm{d}x \\ &\quad \leqslant \int_{\mathbb{R}^N} \phi(v - u) T(u) \, \mathrm{d}x - \int_{\mathbb{R}^N} T(u) \mathbf{a}(u, \nabla u) \cdot \nabla \phi \, \mathrm{d}x \\ &\quad = - \int_{\mathbb{R}^N} \operatorname{div} \mathbf{a}(u, \nabla u) T(u) \phi \, \mathrm{d}x - \int_{\mathbb{R}^N} T(u) \mathbf{a}(u, \nabla u) \cdot \nabla \phi \\ &\quad = \int_{\mathbb{R}^N} \phi(\mathbf{a}(u, \nabla u), D(T(u))).\end{aligned}$$

Let us prove (4.22). Using the convexity of *f*, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \phi f(u_n, \nabla T(u_n)) \, \mathrm{d}x \leqslant &\int_{\mathbb{R}^N} \phi \mathbf{a}(u_n, \nabla T(u_n)) \cdot \nabla T(u_n) \, \mathrm{d}x \\ &+ \int_{\mathbb{R}^N} \phi f(u_n, 0) \, \mathrm{d}x.\end{aligned}$$

Then, since $\mathscr{R}(\phi f, T)$ is lower-semi-continuous in $BV(\mathbb{R}^N)$ respect to the L^1-convergence, letting $n \to \infty$ we obtain

$$\begin{aligned}
&\langle f(u, DT(u)), \phi\rangle \\
&\quad = \mathscr{R}(\phi f, T)(u) \leqslant \liminf_n \int_{\mathbb{R}^N} \phi f(u_n, \nabla T(u_n))\, \mathrm{d}x \\
&\quad \leqslant \liminf_n \int_{\mathbb{R}^N} \phi \mathbf{a}(u_n, \nabla T(u_n)) \cdot \nabla T(u_n)\, \mathrm{d}x + \int_{\mathbb{R}^N} \phi f(u, 0)\, \mathrm{d}x \\
&\quad \leqslant \limsup_n \int_{\mathbb{R}^N} \mathbf{a}(u_n, \nabla u_n) \cdot \nabla T(u_n))\phi(x)\, \mathrm{d}x + \int_{\mathbb{R}^N} \phi f(u, 0)\, \mathrm{d}x \\
&\quad \leqslant \int_{\mathbb{R}^N} \phi(\mathbf{a}(u, \nabla u), D(T(u))) + \int_{\mathbb{R}^N} \phi f(u, 0)\, \mathrm{d}x.
\end{aligned}$$

Hence, (4.22) follows. □

Uniqueness of entropy solutions. Given $v, \overline{v} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $v \geqslant 0$, $\overline{v} \geqslant 0$, let $u, \overline{u} \geqslant 0$ be two bounded entropy solutions of the problems

$$u - \operatorname{div} \mathbf{a}(u, Du) = v, \quad \text{in } \mathbb{R}^N \tag{4.23}$$

and

$$\overline{u} - \operatorname{div} \mathbf{a}(\overline{u}, D\overline{u}) = \overline{v}, \quad \text{in } \mathbb{R}^N, \tag{4.24}$$

respectively.

Let ρ_n be a classical mollifiers in $\mathbb{R}^N$, $b > a > 2\varepsilon > 0$. Let us write

$$\xi_n(x, y) = \rho_n(x - y) \quad \text{and} \quad T = T^a_{a,b}.$$

We need to consider truncature functions of the form

$$S_{\varepsilon,l}(r) := T_\varepsilon(r - l)^+ = T_{l,l+\varepsilon}(r) - l \in \mathscr{P}^+$$

and

$$S^l_\varepsilon(r) := T_\varepsilon(r - l)^- + \varepsilon = T_{l-\varepsilon,l}(r) + \varepsilon - l \in \mathscr{P}^+,$$

where $l \geqslant 0$. Observe that

$$S^l_\varepsilon(r) = -T_\varepsilon(l - r)^+ + \varepsilon.$$

If we denote $\mathbf{z}(y) = \mathbf{a}(u(y), \nabla u(y))$ and $\overline{\mathbf{z}}(x) = \mathbf{a}(\overline{u}(x), \nabla \overline{u}(x))$, we have

$$u - \operatorname{div}(\mathbf{z}) = v \quad \text{and} \quad \overline{u} - \operatorname{div}(\overline{\mathbf{z}}) = \overline{v}, \quad \text{in } \mathscr{D}'(\mathbb{R}^N).$$

Then, multiplying the first equation by $T(u(y))S_{\varepsilon,\overline{u}(x)}(u(y))\xi_n(x,y)$, the second by $T(\overline{u}(x))S_\varepsilon^{u(y)}(\overline{u}(x))\xi_n(x,y)$ and integrating by parts, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} u(y)T(u(y))\,T_\varepsilon(u(y)-\overline{u}(x))^+\xi_n(x,y)\,\mathrm{d}y\\
&\qquad+\int_{\mathbb{R}^N}\xi_n(x,y)(\mathbf{z},D_y(T(u)S_{\varepsilon,\overline{u}(x)}(u(y))))\\
&\qquad+\int_{\mathbb{R}^N}T(u(y))S_{\varepsilon,\overline{u}(x)}(u(y))\mathbf{z}(y)\cdot\nabla_y\xi_n(x,y)\,\mathrm{d}y\\
&\quad=\int_{\mathbb{R}^N}v(y)T(u(y))T_\varepsilon(u(y)-\overline{u}(x))^+\xi_n(x,y)\,\mathrm{d}y
\end{aligned}\tag{4.25}$$

and

$$\begin{aligned}
&\quad-\int_{\mathbb{R}^N}\overline{u}(x)T(\overline{u}(x))(T_\varepsilon(u(y)-\overline{u}(x))^+-\varepsilon)\xi_n(x,y)\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^N}\xi_n(x,y)(\overline{\mathbf{z}},D_x(T(\overline{u})S_\varepsilon^{u(y)}(\overline{u})))\\
&\quad+\int_{\mathbb{R}^N}T(\overline{u})S_\varepsilon^{u(y)}(\overline{u}(x))\overline{\mathbf{z}}(x)\cdot\nabla_x\xi_n(x,y)\,\mathrm{d}x\\
&=-\int_{\mathbb{R}^N}\overline{v}(x)T(\overline{u}(x))(T_\varepsilon(u(y)-\overline{u}(x))^+-\varepsilon)\xi_n(x,y)\,\mathrm{d}x.
\end{aligned}\tag{4.26}$$

Integrating (4.25) in x and (4.26) in y, and adding both identities we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(u(y)T(u(y))-\overline{u}(x)T(\overline{u}(x)))T_\varepsilon(u(y)-\overline{u}(x))^+\xi_n(x,y)\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad+\varepsilon\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(\overline{u}(x)-\overline{v}(x))T(\overline{u}(x))\xi_n(x,y)\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad+\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\xi_n(x,y)(\mathbf{z},D_y(T(u)S_{\varepsilon,\overline{u}(x)}(u))\right)\mathrm{d}x\\
&\qquad+\int_{\mathbb{R}^N\times\mathbb{R}^N}T(u(y))S_{\varepsilon,\overline{u}(x)}(u(y))\mathbf{z}(y)\cdot\nabla_y\xi_n(x,y)\,\mathrm{d}y\,\mathrm{d}x\\
&\qquad+\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\xi_n(x,y)(\overline{\mathbf{z}},D_x(T(\overline{u})S_\varepsilon^{u(y)}(\overline{u})))\right)\mathrm{d}y\\
&\qquad+\int_{\mathbb{R}^N\times\mathbb{R}^N}T(\overline{u}(x))S_\varepsilon^{u(y)}(\overline{u}(x))\overline{\mathbf{z}}(x)\cdot\nabla_x\xi_n(x,y)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(v(y)T(u(y))-\overline{v}(x)T(\overline{u}(x)))T_\varepsilon(u(y)-\overline{u}(x))^+\xi_n(x,y)\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}\tag{4.27}$$

Let I_1, I_2 be, respectively, the first term and the rest of the terms at the left-hand side of the above identity, and let I_3 be the right-hand side term. From now on, since u, $\mathbf{z}$ are always functions of y, and $\overline{u}$, $\overline{\mathbf{z}}$ are always functions of x, to make our expressions shorter, we shall omit the arguments except on sub- and super-indices and in some additional cases where we find useful to remind them.

Now, since $\overline{u} - \overline{v} = \operatorname{div} \overline{\mathbf{z}}$ and $\nabla_y \xi_n(x, y) + \nabla_x \xi_n(x, y) = 0$, we have

$$
\begin{aligned}
I_2 = \varepsilon & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \operatorname{div}(\overline{\mathbf{z}})\, T(\overline{u}) \xi_n \, \mathrm{d}x \, \mathrm{d}y \\
& + \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \xi_n (\mathbf{z}, D_y(T(u) S_{\varepsilon, \overline{u}(x)}(u))) \right) \mathrm{d}x \\
& - \int_{\mathbb{R}^N \times \mathbb{R}^N} T(\overline{u}) S_\varepsilon^{u(y)}(\overline{u}) \overline{\mathbf{z}} \cdot \nabla_y \xi_n \, \mathrm{d}x \, \mathrm{d}y \\
& + \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \xi_n (\overline{\mathbf{z}}, D_x(T(\overline{u}) S_\varepsilon^{u(y)}(\overline{u}))) \right) \mathrm{d}y \\
& - \int_{\mathbb{R}^N \times \mathbb{R}^N} T(u) S_{\varepsilon, \overline{u}(x)}(u)) \mathbf{z} \cdot \nabla_x \xi_n \, \mathrm{d}y \, \mathrm{d}x \\
= \varepsilon & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \operatorname{div}(\overline{\mathbf{z}})\, T(\overline{u}) \xi_n \, \mathrm{d}x \, \mathrm{d}y \\
& + \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \xi_n (\mathbf{z}, D_y(T(u) S_{\varepsilon, \overline{u}(x)}(u))) \right) \mathrm{d}x \\
& + \int_{\mathbb{R}^N \times \mathbb{R}^N} \xi_n \overline{\mathbf{z}} \cdot D_y(T(\overline{u}) T_\varepsilon(\overline{u} - u)^-) \, \mathrm{d}x \, \mathrm{d}y \\
& + \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \xi_n (\overline{\mathbf{z}}, D_x(T(\overline{u}) S_\varepsilon^{u(y)}(\overline{u}))) \right) \mathrm{d}y \\
& + \int_{\mathbb{R}^N \times \mathbb{R}^N} \xi_n \mathbf{z} \cdot D_x(T(u) T_\varepsilon(u - \overline{u})^+)) \, \mathrm{d}y \, \mathrm{d}x \\
= \varepsilon & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \operatorname{div}(\overline{\mathbf{z}})\, T(\overline{u}) \xi_n \, \mathrm{d}x \, \mathrm{d}y, \\
& + \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \xi_n (\mathbf{z}, D_y J_{T' S_{\varepsilon, \overline{u}(x)}}(u)) \right) \mathrm{d}x \\
& + \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \xi_n (\overline{\mathbf{z}}, D_x J_{T' S_\varepsilon^{u(y)}}(\overline{u})) \right) \mathrm{d}y \\
& + \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \xi_n (\mathbf{z}, D_y J_{T S'_{\varepsilon, \overline{u}(x)}}(u)) \right) \mathrm{d}x \\
& - \int_{\mathbb{R}^N} T(\overline{u}) \left(\int_{\mathbb{R}^N} \xi_n \overline{\mathbf{z}} \cdot D_y \, T_\varepsilon(u - \overline{u})^+ \right) \mathrm{d}x \\
& + \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \xi_n (\overline{\mathbf{z}}, D_x J_{T S_\varepsilon^{u(y)\prime}}(\overline{u})) \right) \mathrm{d}y \\
& + \int_{\mathbb{R}^N} T(u(y)) \left(\int_{\mathbb{R}^N} \xi_n \mathbf{z} \cdot D_x T_\varepsilon(u - \overline{u})^+ \right) \mathrm{d}y \\
= I_2^1 & + I_2^2,
\end{aligned}
$$

where I_2^1 denotes the sum of the first three terms and I_2^2 denotes the sum from the fourth to the seventh terms.

Let us consider the second and third terms in I_2^1. Since

$$h_{S_{\varepsilon,\overline{u}(x)}}(u, DT(u)) \leqslant (\mathbf{z}, D_y J_{T' S_{\varepsilon,\overline{u}(x)}}(u))$$

and

$$h_{S_\varepsilon^{u(y)}}(\overline{u}, DT(\overline{u})) \leqslant (\overline{\mathbf{z}}, D_x J_{T' S_\varepsilon^{u(y)}}(\overline{u}))$$

as measures in $\mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \xi_n (\mathbf{z}, D_y J_{T' S_{\varepsilon,\overline{u}(x)}}(u)) \right) \mathrm{d}x \geqslant 0$$

and

$$\int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \xi_n (\overline{\mathbf{z}}, D_x J_{T' S_\varepsilon^{u(y)}}(\overline{u})) \right) \mathrm{d}y \geqslant 0.$$

Hence,

$$I_2^1 \geqslant \varepsilon \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \mathrm{div}(\overline{\mathbf{z}}) T(\overline{u}) \xi_n \, \mathrm{d}x \, \mathrm{d}y. \tag{4.28}$$

Let us write the term

$$I_2^2 = I_2^2(ac) + I_2^2(s),$$

where $I_2^2(ac)$ contains the absolutely continuous parts of I_2^2 and $I_2^2(s)$ contains its singular parts. Now,

$$\begin{aligned}
I_2^2(ac) = & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \xi_n T(u)\, \mathbf{z} \cdot \nabla_y T_\varepsilon(u - \overline{u})^+ \, \mathrm{d}y \, \mathrm{d}x \\
& - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \xi_n T(\overline{u})\, \overline{\mathbf{z}} \cdot \nabla_y T_\varepsilon(u - \overline{u})^+ \, \mathrm{d}y \, \mathrm{d}x \\
& - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \xi_n T(\overline{u})\, \overline{\mathbf{z}} \cdot \nabla_x T_\varepsilon(u - \overline{u})^+ \, \mathrm{d}x \, \mathrm{d}y \\
& + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \xi_n T(u)\, \mathbf{z} \cdot \nabla_x T_\varepsilon(u - \overline{u})^+ \, \mathrm{d}x \, \mathrm{d}y \\
= & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \xi_n (\mathbf{z} T(u) - \overline{\mathbf{z}} T(\overline{u}))(\nabla_y T_\varepsilon(u - \overline{u})^+ + \nabla_x T_\varepsilon(u - \overline{u})^+) \, \mathrm{d}x \, \mathrm{d}y \\
= & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \xi_n (\mathbf{z} - \overline{\mathbf{z}}) T(u)(\nabla_y T_\varepsilon(u - \overline{u})^+ + \nabla_x T_\varepsilon(u - \overline{u})^+) \, \mathrm{d}x \, \mathrm{d}y \\
& + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \xi_n \overline{\mathbf{z}} (T(u) - T(\overline{u}))(\nabla_y T_\varepsilon(u - \overline{u})^+ + \nabla_x T_\varepsilon(u - \overline{u})^+) \, \mathrm{d}x \, \mathrm{d}y \\
=: & \, A^1 + A^2.
\end{aligned}$$

Let us estimate A^1. First, observe that

$$\nabla_y T_\varepsilon(u - \overline{u}(x))^+(y) = \chi_{[\overline{u}(x), \overline{u}(x)+\varepsilon]}(u(y)) \nabla_y u(y),$$

$$\nabla_x T_\varepsilon(u(y) - \overline{u})^+(x) = -\chi_{[u(y)-\varepsilon, u(y)]}(\overline{u}(x)) \nabla_x \overline{u}(x) = -\chi_{[\overline{u}(x), \overline{u}(x)+\varepsilon]}(u(y)) \nabla_x \overline{u}(x).$$

By (3.11), we have

$$A^1 = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \xi_n(\mathbf{z}-\overline{\mathbf{z}})T(u)(\nabla_y u - \nabla_x \overline{u})\chi_{[\overline{u}(x),\overline{u}(x)+\varepsilon]}(u)\,\mathrm{d}x\,\mathrm{d}y$$
$$\geqslant -C\|T(u)\|_\infty \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \chi_{[u\geqslant a]}\xi_n\chi_{[\overline{u}(x),\overline{u}(x)+\varepsilon]}(u)$$
$$\times |u-\overline{u}|\,\|\nabla_y u - \nabla_x\overline{u}\|\,\mathrm{d}x\,\mathrm{d}y.$$

Now, observe that, if $0\leqslant u(y)-\overline{u}(x)\leqslant\varepsilon$ and $u(y)\geqslant a$, then $\overline{u}(x)\geqslant a-\varepsilon$. Thus $\chi_{[\overline{u}\geqslant a-\varepsilon]}\nabla_x\overline{u}$, $\chi_{[u\geqslant a]}\nabla_y u \in L^1(\mathbb{R}^N)$. Let us remind here that the argument of $\chi_{[u\geqslant a]}$ is x, the argument of $\chi_{[\overline{u}\geqslant a-\varepsilon]}$ is y, and $\chi_{[0\leqslant u-\overline{u}\leqslant\varepsilon]}$ denotes $\chi_{[0\leqslant u-\overline{u}\leqslant\varepsilon]}(x,y) = \chi_{[(x,y):0\leqslant u(y)-\overline{u}(x)\leqslant\varepsilon]}(x,y)$. Hence

$$A^1 \geqslant -C\|T(u)\|_\infty\varepsilon \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \chi_{[u\geqslant a]}\,\chi_{[\overline{u}\geqslant a-\varepsilon]}\xi_n\,\chi_{[0\leqslant u-\overline{u}\leqslant\varepsilon]}$$
$$\times \|\nabla_y u - \nabla_x\overline{u}\|\,\mathrm{d}x\,\mathrm{d}y.$$

Similarly

$$|A^2| = \left|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \xi_n\overline{\mathbf{z}}(T(u)-T(\overline{u}))(\nabla_y u - \nabla_x\overline{u})\chi_{[\overline{u}(x),\overline{u}(x)+\varepsilon]}(u)\,\mathrm{d}x\,\mathrm{d}y\right|$$
$$\leqslant M\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \chi_{[u\geqslant a-\varepsilon]}\chi_{[\overline{u}\geqslant a-\varepsilon]}\chi_{[0\leqslant u-\overline{u}\leqslant\varepsilon]}\xi_n|u-\overline{u}|\,\|\nabla_y u-\nabla_x\overline{u}\|\,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant M\varepsilon\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \chi_{[u\geqslant a-\varepsilon]}\chi_{[\overline{u}\geqslant a-\varepsilon]}\xi_n\chi_{[0\leqslant u-\overline{u}\leqslant\varepsilon]}\,\|\nabla_y u-\nabla_x\overline{u}\|\,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant M\varepsilon\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \chi_{[u\geqslant a-\varepsilon]}\chi_{[\overline{u}\geqslant a-\varepsilon]}\xi_n\chi_{[0\leqslant u-\overline{u}\leqslant\varepsilon]}(\|\nabla_y u\|+\|\nabla_x\overline{u}\|)\,\mathrm{d}x\,\mathrm{d}y,$$

where $M>0$ denotes the Lipschitz constant of T. Now observe that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \chi_{[u\geqslant a-\varepsilon]}\chi_{[\overline{u}\geqslant a-\varepsilon]}\xi_n\chi_{[0\leqslant u-\overline{u}\leqslant\varepsilon]}\,\|\nabla_y u\|\,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant \int_{\mathbb{R}^N} \chi_{[\overline{u}\geqslant a-\varepsilon]}\rho_n(x)\left(\int_{\overline{u}(x)}^{\overline{u}(x)+\varepsilon} P([u\geqslant\lambda])\,\mathrm{d}\lambda\right)\mathrm{d}x \leqslant o(\varepsilon),$$

where $o(\varepsilon)$ denotes an expression such that $o(\varepsilon)\to 0$ as $\varepsilon\to 0$. Thus

$$\frac{1}{\varepsilon}A^1 \geqslant -Co(\varepsilon)$$

and

$$\frac{1}{\varepsilon}|A^2| \leqslant o(\varepsilon).$$

Hence,

$$\frac{1}{\varepsilon}\,I_2^2(ac) \geqslant o(\varepsilon). \tag{4.29}$$

Finally, let us compute $I_2^2(s)$.

$$\begin{aligned}
I_2^2(s) = & \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \xi_n \mathbf{z} \cdot D_y^s J_{TS'_{\varepsilon,\overline{u}(x)}}(u) \right) \mathrm{d}x \\
& - \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \xi_n T(\overline{u})\, \overline{\mathbf{z}} \cdot D_y^s\, T_\varepsilon(u-\overline{u})^+ \right) \mathrm{d}x \\
& + \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \xi_n \overline{\mathbf{z}} \cdot D_x^s J_{TS_\varepsilon^{u(y)\prime}}(\overline{u}) \right) \mathrm{d}y \\
& + \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \xi_n T(u)\, \mathbf{z} \cdot D_x^s T_\varepsilon(u-\overline{u})^+ \right) \mathrm{d}y \\
=: & \; I_2^2(1,s) + I_2^2(2,s).
\end{aligned}$$

Note that, if $\overline{u}(x) > 0$, we have

$$\mathbf{z} \cdot D_y^s J_{TS'_{\varepsilon,\overline{u}(x)}}(u) \geqslant h_T(u, D_y T_\varepsilon(u-\overline{u}(x))^+)^s = h_T(u_\varepsilon, D_y u_\varepsilon)^s \geqslant 0,$$

where $u_\varepsilon(x,y) = T_{\overline{u}(x),\overline{u}(x)+\varepsilon}(u(y))$ and by (H_5) and (H_6), we have

$$\overline{\mathbf{z}}(x) \cdot D_y^s T_\varepsilon(u-\overline{u}(x))^+ \leqslant \varphi(\overline{u}(x))\psi^0(\overrightarrow{D_y^s u_\varepsilon})|D_y^s u_\varepsilon|.$$

Since the integrand of the first term is positive and the support of $T(\overline{u})$ is contained in $[\overline{u} \geqslant a]$, we have

$$\begin{aligned}
I_2^2(1,s) \geqslant & \int_{[\overline{u}\geqslant a]} \left(\int_{\mathbb{R}^N} \xi_n \mathbf{z} \cdot D_y^s J_{TS'_{\varepsilon,\overline{u}(x)}}(u) \right) \mathrm{d}x \\
& - \int_{[\overline{u}\geqslant a]} \left(\int_{\mathbb{R}^N} \xi_n T(\overline{u})\, \overline{\mathbf{z}} \cdot D_y^s\, T_\varepsilon(u-\overline{u}) \right) \mathrm{d}x \\
\geqslant & \int_{[\overline{u}\geqslant a]} \left(\int_{\mathbb{R}^N} \xi_n h_T(u_\varepsilon, D_y u_\varepsilon)^s \right) \mathrm{d}x \\
& - \int_{[\overline{u}\geqslant a]} \left(\int_{\mathbb{R}^N} \xi_n T(\overline{u})\varphi(\overline{u})\psi^0(\overrightarrow{D_y^s u_\varepsilon})|D_y^s u_\varepsilon| \right) \mathrm{d}x \\
= & \int_{[\overline{u}\geqslant a]} \left(\int_{\mathbb{R}^N} \xi_n T(u_\varepsilon)\varphi(u_\varepsilon)\psi^0(\overrightarrow{D_y^s u_\varepsilon})|D_y^c u_\varepsilon| \right) \mathrm{d}x \\
& - \int_{[\overline{u}\geqslant a]} \left(\int_{\mathbb{R}^N} \xi_n T(\overline{u})\varphi(\overline{u})\psi^0(\overrightarrow{D_y^s u_\varepsilon})|D_y^c u_\varepsilon| \right) \mathrm{d}x \\
& + \int_{[\overline{u}\geqslant a]} \left(\int_{J_{u_\varepsilon}} \xi_n \frac{1}{(u_\varepsilon)^+(y) - (u_\varepsilon)^-(y)} \right. \\
& \times \left. \left(\int_{(u_\varepsilon)^-(y)}^{(u_\varepsilon)^+(y)} T(s)\varphi(s)\,\mathrm{d}s \right) \psi^0(\overrightarrow{D_y^s u_\varepsilon})|D_y^j u_\varepsilon| \right) \mathrm{d}x \\
& - \int_{[\overline{u}\geqslant a]} \left(\int_{\mathbb{R}^N} \xi_n T(\overline{u})\varphi(\overline{u})\psi^0(\overrightarrow{D_y^s u_\varepsilon})|D_y^j u_\varepsilon| \right) \mathrm{d}x =: J_1 + J_2,
\end{aligned}$$

where J_1 denotes the sum of the first and second terms of the above expression, and J_2 the sum of the third and fourth terms.

Now, since T and φ are Lipschitz continuous, we have

$$
\begin{aligned}
|J_1| &\leqslant \int_{\mathbb{R}^N} \left(\int_{[\overline{u} \geqslant a]} \xi_n |T(u_\varepsilon)\varphi(u_\varepsilon) - T(\overline{u})\varphi(\overline{u})| \psi^0(\overrightarrow{D_y^s u_\varepsilon}) |D_y^c u_\varepsilon| \right) \mathrm{d}x \\
&\leqslant M \int_{[\overline{u} \geqslant a]} \rho_n(x) \left(\int_{\mathbb{R}^N} |u_\varepsilon - \overline{u}| \chi_{[\overline{u}(x), \overline{u}(x)+\varepsilon]}(u) |D_y^c u| \right) \mathrm{d}x \\
&\leqslant \varepsilon M \int_{[\overline{u} \geqslant a]} \rho_n(x) \left(\int_{\{y \in \mathbb{R}^N \,:\, \overline{u}(x) < u(y) < \varepsilon + \overline{u}(x)\}} |D_y^c u| \right) \mathrm{d}x,
\end{aligned}
$$

where $M > 0$ denotes the Lipschitz constant of $T(\cdot)\varphi(\cdot)$.

Using the coarea formula, we obtain

$$
|J_1| \leqslant \varepsilon M \int_{\mathbb{R}^N} \chi_{[\overline{u} \geqslant a]} \rho_n(x) \left(\int_{\overline{u}(x)}^{\overline{u}(x)+\varepsilon} \mathrm{Per}(\{u(y) \geqslant \lambda\}) \, \mathrm{d}\lambda \right) \mathrm{d}x,
$$

which yields

$$
\frac{1}{\varepsilon} |J_1| \leqslant M o(\varepsilon). \tag{4.30}
$$

For convenience, let us write

$$
J(u_\varepsilon, y) = \frac{1}{(u_\varepsilon)^+(y) - (u_\varepsilon)^-(y)}.
$$

Working in a similar way as before, we have

$$
\begin{aligned}
|J_2| &\leqslant \int_{[\overline{u} \geqslant a]} \left(\int_{J_{u_\varepsilon}} \xi_n J(u_\varepsilon, y) \left(\int_{(u_\varepsilon)^-(y)}^{(u_\varepsilon)^+(y)} |T(s)\varphi(s) - T(\overline{u}(x))\varphi(\overline{u}(x))| \, \mathrm{d}s \right) \right. \\
&\qquad \left. \times \ \psi^0(\overrightarrow{D_y^s u_\varepsilon}) |D_y^j u_\varepsilon| \right) \mathrm{d}x \\
&\leqslant \varepsilon M \int_{\mathbb{R}^N} \chi_{[\overline{u} \geqslant a]} \rho_n(x) \left(\int_{\overline{u}(x)}^{\overline{u}(x)+\varepsilon} \mathrm{Per}(\{u(y) \geqslant \lambda\}) \, \mathrm{d}\lambda \right) \mathrm{d}x
\end{aligned}
$$

and we obtain that

$$
\frac{1}{\varepsilon} J_2 \geqslant o(\varepsilon). \tag{4.31}
$$

Collecting all these facts, we obtain

$$
\frac{1}{\varepsilon} I_2^2(1, s) \geqslant o(\varepsilon).
$$

In a similar way we prove that

$$
\frac{1}{\varepsilon} I_2^2(2, s) \geqslant o(\varepsilon).
$$

Hence,

$$\frac{1}{\varepsilon} I_2^2(s) \geqslant o(\varepsilon).$$

Then, by (4.29), it follows that

$$\frac{1}{\varepsilon} I_2^2 \geqslant o(\varepsilon).$$

Hence, with the estimates of all terms of I_2, we have

$$\frac{1}{\varepsilon}\, I_2 \geqslant o(\varepsilon) + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \operatorname{div}(\overline{\mathbf{z}}) T(\overline{u}) \xi_n \,\mathrm{d}x\,\mathrm{d}y.$$

Therefore, dividing (4.27) by ε, and letting $\varepsilon \to 0$ and $n \to \infty$ in this order we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} (u(x)T(u(x)) - \overline{u}(x)T(\overline{u}(x)))\operatorname{sign}_0^+(u(x) - \overline{u}(x))\,\mathrm{d}x\\ &\quad\leqslant \int_{\mathbb{R}^N} (v(x)T(u(x)) - \overline{v}(x)T(\overline{u}(x)))\operatorname{sign}_0^+(u(x) - \overline{u}(x))\,\mathrm{d}x\\ &\qquad - \int_{\mathbb{R}^N} \operatorname{div}(\overline{\mathbf{z}})\, T(\overline{u}(x))\,\mathrm{d}x.\end{aligned}$$

As above, let us skip the argument x in the expressions below. Letting $a \to 0^+$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} (uT_{0,b}(u) - \overline{u}T_{0,b}(\overline{u}))\operatorname{sign}_0^+(u - \overline{u})\,\mathrm{d}x\\ &\quad\leqslant \int_{\mathbb{R}^N} (vT_{0,b}(u) - \overline{v}T_{0,b}(\overline{u}))\operatorname{sign}_0^+(u - \overline{u})\,\mathrm{d}x - \int_{\mathbb{R}^N} \operatorname{div}(\overline{\mathbf{z}})\, T_{0,b}(\overline{u})\,\mathrm{d}x.\end{aligned}$$

Dividing by $b > 0$, and letting $b \to 0^+$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} (u\chi_{[u>0]} - \overline{u}\chi_{[\overline{u}>0]})\operatorname{sign}_0^+(u - \overline{u})\,\mathrm{d}x\\ &\quad\leqslant \int_{\mathbb{R}^N} (v\chi_{[u>0]} - \overline{v}\chi_{[\overline{u}>0]})\operatorname{sign}_0^+(u - \overline{u})\,\mathrm{d}x - \int_{\mathbb{R}^N} \operatorname{div}(\overline{\mathbf{z}})\, \chi_{[\overline{u}>0]}\,\mathrm{d}x. \qquad (4.32)\end{aligned}$$

We claim now that

$$v = 0 \text{ a.e. on } [u = 0] \quad \text{and} \quad \overline{v} = 0 \text{ a.e on } [\overline{u} = 0]. \qquad (4.33)$$

Let $0 \leqslant \phi \in \mathscr{D}(\mathbb{R}^N)$ be and $a > 0, \varepsilon > 0$. Multiplying $v - u = -\operatorname{div}(\mathbf{z})$ in $\mathscr{D}'(\mathbb{R}^N)$ by $T^a_{a,a+\varepsilon}(u)\phi$ and integrating by parts, we have

$$\begin{aligned}&\int_{\mathbb{R}^N} (v - u)T^a_{a,a+\varepsilon}(u)\phi\,\mathrm{d}x\\ &\quad= \int_{\mathbb{R}^N} \phi(\mathbf{z}, DT^a_{a,a+\varepsilon}(u)) + \int_{\mathbb{R}^N} \mathbf{z}\cdot\nabla\phi T^a_{a,a+\varepsilon}(u)\,\mathrm{d}x\\ &\quad\geqslant \int_{\mathbb{R}^N} \mathbf{z}\cdot\nabla\phi T^a_{a,a+\varepsilon}(u)\,\mathrm{d}x.\end{aligned}$$

Dividing by ε and letting $\varepsilon \to 0^+$, we get

$$\int_{\mathbb{R}^N} (v-u)\chi_{[u>a]}\phi \, dx \geqslant \int_{\mathbb{R}^N} \mathbf{z} \cdot \nabla\phi\chi_{[u>a]} \, dx.$$

Hence,

$$\begin{aligned} \int_{\mathbb{R}^N} (v-u)\chi_{[u\leqslant a]}\phi \, dx &= \int_{\mathbb{R}^N} (v-u)\phi \, dx - \int_{\mathbb{R}^N} (v-u)\chi_{[u>a]}(x)\phi \, dx \\ &\leqslant \int_{\mathbb{R}^N} (v-u)\phi \, dx - \int_{\mathbb{R}^N} \mathbf{z} \cdot \nabla\phi\chi_{[u>a]} \, dx \\ &= \int_{\mathbb{R}^N} \mathbf{z} \cdot \nabla\phi\chi_{[u\leqslant a]} \, dx. \end{aligned}$$

Then, letting $a \to 0^+$, since $\mathbf{z} = 0$ in $[u = 0]$, we have

$$\int_{\mathbb{R}^N} v\chi_{[u=0]}\phi \, dx = \int_{\mathbb{R}^N} (v-u)\chi_{[u=0]}\phi \, dx \leqslant 0,$$

for all $0 \leqslant \phi \in \mathscr{D}(\mathbb{R}^N)$, hence $v\chi_{[u=0]} = 0$ a.e. in $\mathbb{R}^N$. Similarly, $\overline{v}\chi_{[\overline{u}=0]} = 0$ a.e. in $\mathbb{R}^N$ and (4.33) holds.

On the other hand, by (4.33), we have

$$\int_{\mathbb{R}^N} \operatorname{div}(\overline{\mathbf{z}}) \, \chi_{[\overline{u}>0]} \, dx = \int_{\mathbb{R}^N} (\overline{u} - \overline{v}) \, \chi_{[\overline{u}>0]} \, dx = \int_{\mathbb{R}^N} (\overline{u} - \overline{v}) \, dx = \int_{\mathbb{R}^N} \operatorname{div}(\overline{\mathbf{z}}) \, dx = 0.$$

Then, from (4.32), it follows that

$$\int_{\mathbb{R}^N} (u\chi_{[u>0]} - \overline{u}\chi_{[\overline{u}>0]})\operatorname{sign}_0^+(u - \overline{u}) \, dx \leqslant \int_{\mathbb{R}^N} (v\chi_{[u>0]} - \overline{v}\chi_{[\overline{u}>0]})\operatorname{sign}_0^+(u - \overline{u}) \, dx.$$

Hence, using (4.33), we obtain

$$\int_{\mathbb{R}^N} (u - \overline{u})^+ \, dx \leqslant \int_{\mathbb{R}^N} (v - \overline{v})\operatorname{sign}_0^+(u - \overline{u}) \, dx \leqslant \int_{\mathbb{R}^N} (v - \overline{v})^+ \, dx.$$

This concludes the proof of the Theorem. □

5. Semigroup solution

In this section we shall associate an accretive operator in $L^1(\mathbb{R}^N)$ to the formal differential expression $-\operatorname{div} \mathbf{a}(u, \nabla u)$.

Definition 5.1. $(u, v) \in B$ if and only if $0 \leqslant u \in TBV^+(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $0 \leqslant v \in L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and u is the entropy solution of problem (3.1).

For $0 < \varepsilon \leqslant \|u\|_\infty$, let $S_\varepsilon := T_{\varepsilon, \|u\|_\infty}$ be. If $(u, v) \in B$, by Green's formula (2.10), we obtain

$$\int_{\mathbb{R}^N} (w - S_\varepsilon(u))v \, \mathrm{d}x \leqslant \int_{\mathbb{R}^N} (\mathbf{a}(u, \nabla u), Dw) - h(u, DS_\varepsilon(u)), \tag{5.1}$$

for all $w \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and $\varepsilon > 0$.

Lemma 5.2. *Given $\lambda > 0$, and $v \in L^1(\mathbb{R}^N)^+$, if $u = (I + \lambda B)^{-1} v$, then*

$$u \leqslant v. \tag{5.2}$$

Proof. Since $u = (I + \lambda B)^{-1} v$, we have $\left(u, \frac{1}{\lambda}(v - u)\right) \in B$. Then, $\mathbf{a}(u, \nabla u) \in X_1(\mathbb{R}^N)$ and

$$\frac{1}{\lambda}(v - u) = -\mathrm{div}\, \mathbf{a}(u, \nabla u) \quad \text{in } \mathscr{D}'(\mathbb{R}^N).$$

Given $p \in \mathscr{P}_0$ and $\varepsilon > 0$, we denote by

$$p_\varepsilon(r) := \begin{cases} p(\varepsilon), & \text{if } 0 \leqslant r \leqslant \varepsilon, \\ p(r), & \text{if } r \geqslant \varepsilon. \end{cases}$$

By Green's formula, we have

$$\int_{\mathbb{R}^N} p_\varepsilon(u)(u - v) \, \mathrm{d}x = \lambda \int_{\mathbb{R}^N} p_\varepsilon(u) \mathrm{div}\, \mathbf{a}(u, \nabla u) \, \mathrm{d}x = -\lambda \int_{\mathbb{R}^N} (\mathbf{a}(u, \nabla u), Dp_\varepsilon(u)).$$

If $S_\varepsilon := T_{\varepsilon, \|p_\varepsilon\|_\infty}$, we have $p_\varepsilon(u) = p_\varepsilon(S_\varepsilon(u))$. On the other hand, by chain's rule for BV-functions (see [1]), we have $D(p_\varepsilon(S_\varepsilon(u))) = \overline{(p_\varepsilon)}_{S_\varepsilon(u)} DS_\varepsilon(u)$ with $\overline{(p_\varepsilon)}_{S_\varepsilon(u)} \geqslant 0$, $\overline{(p_\varepsilon)}_{S_\varepsilon(u)}$ being the Vol'pert's averaged superposition. Moreover, by [8],

$$\theta(\mathbf{a}(u, \nabla u), D(p_\varepsilon(S_\varepsilon(u))), \cdot) = \theta(\mathbf{a}(u, \nabla u), DS_\varepsilon(u), \cdot)|Dp_\varepsilon(S_\varepsilon(u))|\text{-a.e.}$$

Then,

$$\begin{aligned} (\mathbf{a}(u, \nabla u), D(p_\varepsilon(u))) &= (\mathbf{a}(u, \nabla u), D(p_\varepsilon(S_\varepsilon(u))) \\ &= \theta(\mathbf{a}(u, \nabla u), DS_\varepsilon(u), \cdot)|Dp_\varepsilon(S_\varepsilon(u))| \\ &= \overline{(p_\varepsilon)}_{S_\varepsilon(u)} \theta(\mathbf{a}(u, \nabla u), DS_\varepsilon(u), \cdot)|DS_\varepsilon(u)| \\ &= \overline{(p_\varepsilon)}_{S_\varepsilon(u)} (\mathbf{a}(u, \nabla u), DS_\varepsilon(u)) \geqslant \overline{(p_\varepsilon)}_{S_\varepsilon(u)} h(u, DS_\varepsilon(u)) \geqslant 0. \end{aligned}$$

Therefore, we get

$$\int_{\mathbb{R}^N} p_\varepsilon(u)(u - v) \, \mathrm{d}x \leqslant 0$$

and consequently, letting $\varepsilon \to 0^+$, we get

$$\int_{\mathbb{R}^N} p(u) u \, \mathrm{d}x \leqslant \int_\Omega p(u) v \, \mathrm{d}x, \quad \forall p \in \mathscr{P}_0$$

and, by results in [10], this implies (5.2). □

Proposition 5.3. *Assume we are under assumptions* (H). *Then B is accretive in $L^1(\mathbb{R}^N)$,*

$$(L^\infty(\mathbb{R}^N) \cap L^1(\mathbb{R}^N))^+ \subset R(I+B)$$

and $D(B)$ is dense in $L^1(\mathbb{R}^N)^+$.

Proof. The accretivity of the operator B in $L^1(\mathbb{R}^N)$ and the range condition follow from Theorem 4.1. To prove the density of $D(B)$ in $L^1(\mathbb{R}^N)^+$, we prove that $\mathscr{D}(\mathbb{R}^N)^+ \subseteq \overline{B}^{L^1(\mathbb{R}^N)}$. Let $0 \leqslant v \in \mathscr{D}(\mathbb{R}^N)$. By Theorem 4.1, $v \in R(I + \frac{1}{n} B)$ for all $n \in \mathbb{N}$. Thus, for each $n \in \mathbb{N}$, there exists $u_n \in D(B)$, $\|u_n\|_\infty \leqslant \|v\|_\infty$, such that $(u_n, n(v-u_n)) \in B$. Consequently, by (5.1) with $S_\varepsilon := T_{\varepsilon, \|v\|_\infty}$, we get

$$\int_{\mathbb{R}^N} (w - S_\varepsilon(u_n)) n(v-u_n)\, \mathrm{d}x \leqslant \int_{\mathbb{R}^N} (\mathbf{a}(u_n, \nabla u_n), Dw) - h(u_n, DS_\varepsilon(u_n)),$$

for all $w \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Taking $w = v$, we get

$$\begin{aligned} \int_{\mathbb{R}^N} (v - S_\varepsilon(u_n))(v-u_n)\, \mathrm{d}x &\leqslant \frac{1}{n} \left(\int_{\mathbb{R}^N} \mathbf{a}(u_n, \nabla u_n) \cdot \nabla v \, \mathrm{d}x - h(u_n, DS_\varepsilon(u_n)) \right) \\ &\leqslant \frac{1}{n} \int_{\mathbb{R}^N} \mathbf{a}(u_n, \nabla u_n) \cdot \nabla v \, \mathrm{d}x \leqslant \frac{M}{n} \int_{\mathbb{R}^N} |\nabla v| \, \mathrm{d}x. \end{aligned}$$

Letting $\varepsilon \to 0^+$, we get

$$\int_{\mathbb{R}^N} (v - u_n)^2 \, \mathrm{d}x \leqslant \frac{M}{n} \int_{\mathbb{R}^N} |\nabla v| \, \mathrm{d}x$$

and we obtain that $u_n \to v$ in $L^2(\mathbb{R}^N)$, as $n \to \infty$. Moreover, by Lemma 5.2, $u_n \ll v$ for all $n \in \mathbb{N}$. Hence, by results in [10], we have $u_n \to v$ in $L^1(\mathbb{R}^N)$, as $n \to \infty$. Therefore $v \in \overline{D(B)}^{L^1(\Omega)}$ and the proof is complete. □

From Proposition 5.3, if we denote by $\mathscr{B}$ the closure in $L^1(\mathbb{R}^N)$ of the operator B, it follows that $\mathscr{B}$ is accretive in $L^1(\mathbb{R}^N)$, it satisfies the comparison principle, and verifies the range condition $\overline{D(\mathscr{B})}^{L^1(\mathbb{R}^N)} = L^1(\mathbb{R}^N)^+ \subset R(I + \lambda\mathscr{B})$ for all $\lambda > 0$. Therefore, according to Crandall–Liggett's Theorem (c.f., e.g., [11]), for any $0 \leqslant u_0 \in L^1(\mathbb{R}^N)$ there exists a unique mild solution $u \in C([0,T]; L^1(\mathbb{R}^N))$ of the abstract Cauchy problem

$$u'(t) + \mathscr{B}u(t) \ni 0, \quad u(0) = u_0. \tag{5.3}$$

Moreover, $u(t) = T(t)u_0$ for all $t \geqslant 0$, where $(T(t))_{t \geqslant 0}$ is the semigroup in $L^1(\mathbb{R}^N)^+$ generated by Crandall–Liggett's exponential formula, i.e.,

$$T(t)u_0 = \lim_{n \to \infty} \left(I + \frac{t}{n} \mathscr{B} \right)^{-n} u_0.$$

On the other hand, by Lemma 5.2, and using the results in [10], we have that the comparison principle also holds for $T(t)$, i.e., if $u_0, \overline{u}_0 \in L^1(\mathbb{R}^N)^+$, we have the estimate

$$\|(T(t)u_0 - T(t)\overline{u}_0)^+\|_1 \leqslant \|(u_0 - \overline{u}_0)^+\|_1. \tag{5.4}$$

Remark 5.4. Since, by Proposition 5.3, $(L^\infty(\mathbb{R}^N)\cap L^1(\mathbb{R}^N))^+ \subset R(I+B)$, using Lemma 5.2, we have that

$$T(t)u_0 \in (L^\infty(\mathbb{R}^N)\cap L^1(\mathbb{R}^N))^+, \quad \forall t\geqslant 0, \quad \forall u_0 \in (L^\infty(\mathbb{R}^N)\cap L^1(\mathbb{R}^N))^+. \tag{5.5}$$

Remark 5.5. In the proof of the existence part of Theorem 4.1, we have proved that the resolvent of the operator B_n associated to $-\mathrm{div}\,\mathbf{a}(u, Du) - \frac{1}{n}\Delta u$ converges to the resolvent of B, i.e., if $v \in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$, $v\geqslant 0$, and u_n are solutions of $(I + B_n)u = v$ in the sense defined by the inequalities (4.2), then $u_n \to u$ in $L^1(\mathbb{R}^N)$ (and in $L^p(\mathbb{R}^N)$ for all $1\leqslant p<\infty$), where $u = (I + B)^{-1}v$.

6. The Neumann problem

Using similar techniques as above we may prove an existence and uniqueness result for the following Neumann problem:

$$\begin{cases} u - \mathrm{div}\,\mathbf{a}(u, Du) = v, & \text{in } \Omega,\\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

where Ω is a bounded set in $\mathbb{R}^N$ with boundary $\partial\Omega$ of class C^1, $v \in L^\infty(\Omega)^+$, $\mathbf{a}(z, \xi) = \nabla_\xi f(z, \xi)$, and f satisfies similar assumptions to the ones considered in the Cauchy problem. We use the notation $\partial/\partial\eta$ for the Neumann boundary operator associated to $\mathbf{a}(u, Du)$, i.e.,

$$\frac{\partial u}{\partial \eta} := \mathbf{a}(u, Du)\cdot \nu,$$

where ν is the outward unit normal on $\partial\Omega$.

We introduce the concept of entropy solution for problem (6.1).

Definition 6.1. Given $v \in L^\infty(\Omega)$, $v\geqslant 0$, we say that u is an *entropy solution* of

$$\begin{cases} -\mathrm{div}\,\mathbf{a}(u, Du) = v, & \text{in } \Omega,\\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{6.2}$$

if $u \in TBV^+(\Omega)$ and $\mathbf{a}(u, \nabla u) \in X_1(\Omega)$ satisfies

$$-\mathrm{div}\,\mathbf{a}(u, \nabla u)) = v, \quad \text{in } \mathscr{D}'(\Omega), \tag{6.3}$$

$$h_S(u, DT(u)) \leqslant (\mathbf{a}(u, \nabla u), DJ_{T'S}(u)) \quad \text{as measures } \forall S \in \mathscr{P}^+, T \in \mathscr{T}^+, \tag{6.4}$$

$$h_T(u, DT(u)) \leqslant (\mathbf{a}(u, \nabla u), DT(u)) \quad \text{as measures } \forall T \in \mathscr{T}^+, \tag{6.5}$$

$$[\mathbf{a}(u, \nabla u), \nu] = 0, \quad H^{N-1}\text{-a.e. on } \partial\Omega. \tag{6.6}$$

Working as in the proof of Theorem 4.1, we prove the following result:

Theorem 6.2. *Assume that assumptions* (H) *hold. Then, for any* $v \in L^\infty(\Omega)$, $v \geqslant 0$, *there exists a unique entropy solution* $u \in TBV^+(\Omega) \cap L^\infty(\Omega)$ *of* (6.1).

As in Section 5, we can associate an accretive operator in $L^1(\Omega)$ to the formal differential expression $-\mathrm{div}\, \mathbf{a}(u, \nabla u)$ together with the Neumann boundary condition. More precisely, we define the operator B in $L^1(\Omega)$ by

$(u, v) \in B$, if and only if $u \in TBV^+(\Omega) \cap L^\infty(\Omega)$, $0 \leqslant v \in L^\infty(\Omega)$ and u is the entropy solution of problem (6.2).

Then, assuming that assumptions (H) hold, we have that B is accretive in $L^1(\Omega)$, $L^\infty(\Omega)^+ \subset R(I + B)$ and $D(B)$ is dense in $L^1(\Omega)^+$. Therefore, if we denote by $\mathscr{B}$ the closure of B in $L^1(\Omega)$, it follows that $\mathscr{B}$ is accretive in $L^1(\Omega)$ and verifies the range condition

$$\overline{D(\mathscr{B})}^{L^1(\Omega)} = L^1(\Omega)^+ \subset R(I + \lambda\mathscr{B}), \quad \text{for all } \lambda > 0.$$

Therefore, according to Crandall–Liggett's Theorem (c.f., e.g., [11]), for any $0 \leqslant u_0 \in L^1(\Omega)$ there exists a unique mild solution $u \in C([0, T]; L^1(\Omega))$ of the abstract Cauchy problem

$$u'(t) + \mathscr{B}u(t) \ni 0, \quad u(0) = u_0. \tag{6.7}$$

Moreover, $u(t) = T(t)u_0$ for all $t \geqslant 0$, being $(T(t))_{t \geqslant 0}$ the semigroup in $L^1(\Omega)^+$ generated by the Crandall–Liggett's exponential formula.

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