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The Cauchy problem for linear growth functionals

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Dedicated to Ph. Bénilan

1. Introduction and preliminaries

In this paper we are interested in the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div} \mathbf{a}(x, Du) & \text{in} \quad Q = (0, \infty) \times \mathbb{R}^N \\ u(0, x) = u_0(x) & \text{in} \quad x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $u_0 \in L^1_{loc}(\mathbb{R}^N)$ and $\mathbf{a}(x, \xi) = \nabla_\xi f(x, \xi)$, $f : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ being a function with linear growth as $\|\xi\| \to \infty$ satisfying some additional assumptions we shall precise below. An example of function $f(x, \xi)$ covered by our results is the nonparametric area integrand $f(x, \xi) = \sqrt{1 + \|\xi\|^2}$; in this case the right-hand side of the equation in (1.1) is the well-known mean-curvature operator. The case of the total variation, i.e., when $f(\xi) = \|\xi\|$ is not covered by our results. This case has been recently studied by G. Bellettini, V. Caselles and M. Novaga in [8]. The case of a bounded domain for general equations of the form (1.1) has been studied in [3] and [4] (see also [18], [11] and [15]). Our aim here is to introduce a concept of solution of (1.1), for which existence and uniqueness for initial data in $L^1_{loc}(\mathbb{R}^N)$ is proved.

Due to the linear growth condition on the Lagrangian, the natural energy space to study (1.1) is the space of functions of bounded variation. Let Ω be an open subset of $\mathbb{R}^N$. A function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a vector valued Radon measure with finite total variation in Ω is called a function of bounded variation. The class of such functions will be denoted by $BV(\Omega)$. Thus, if $u \in BV(\Omega)$, then Du is a Radon measure that decomposes into its absolutely continuous and singular parts $Du = D^a u + D^s u$. Then $D^a u = \nabla u \, \mathcal{L}^N$ where ∇u is the Radon-Nikodym derivative of the measure Du with respect to the Lebesgue measure $\mathcal{L}^N$. Moreover, we have the polar decomposition $D^s u = \overrightarrow{D^s u} |D^s u|$ where $|D^s u|$ is the total variation measure of $D^s u$. Finally, we denote by $BV_{loc}(\Omega)$ the sspace of functions $u \in L^1_{loc}(\Omega)$ such that $u\varphi \in BV(\Omega)$ for all $\varphi \in C_0^\infty(\Omega)$. For information concerning functions of bounded variation we refer to [1], [13] and [20].

By $L^1_w(0,T;BV(\mathbb{R}^N))$ we denote the space of functions $w:[0,T]\to BV(\mathbb{R}^N)$ such that $w\in L^1(]0,T[\times\mathbb{R}^N)$, the maps

$$t\in[0,T]\mapsto\int_{\mathbb{R}^N}\phi\,dDw(t)$$

are measurable for every $\phi\in C^1_0(\mathbb{R}^N,\mathbb{R}^N)$ and $\int_0^T|Dw(t)|(\mathbb{R}^N)\,dt<\infty$. By $L^1_w(0,T;BV_{loc}(\mathbb{R}^N))$ we denote the space of functions $w:[0,T]\to BV_{loc}(\mathbb{R}^N)$ such that $w\varphi\in L^1_w(0,T,BV(\mathbb{R}^N))$ for all $\varphi\in C_0^\infty(\mathbb{R}^N)$.

Following [5], let

$$X_p(\mathbb{R}^N)=\{z\in L^\infty(\mathbb{R}^N,\mathbb{R}^N)\ :\ \mathrm{div}(z)\in L^p(\mathbb{R}^N)\}. \tag{1.2}$$

If $z\in X_p(\mathbb{R}^N)$ and $w\in BV(\mathbb{R}^N)\cap L^{p'}(\mathbb{R}^N)$ we define the functional $(z,Dw):C_0^\infty(\mathbb{R}^N)\to\mathbb{R}$ by the formula

$$\langle(z,Dw),\varphi\rangle=-\int_{\mathbb{R}^N}w\,\varphi\,\mathrm{div}(z)\,dx-\int_{\mathbb{R}^N}w\,z\cdot\nabla\varphi\,dx. \tag{1.3}$$

Then (z,Dw) is a Radon measure in $\mathbb{R}^N$ and

$$\left|\int_B(z,Dw)\right|\le\int_B|(z,Dw)|\le\|z\|_\infty\int_B|Dw| \tag{1.4}$$

for any Borel set $B\subseteq\mathbb{R}^N$. Moreover, we have the following *Green's formula* ([5]), for $z\in X_p(\mathbb{R}^N)$ and $w\in BV(\mathbb{R}^N)\cap L^{p'}(\mathbb{R}^N)$:

$$\int_{\mathbb{R}^N}(z,Dw)+\int_{\mathbb{R}^N}w\,\mathrm{div}(z)\,dx=0. \tag{1.5}$$

We define

$$z\cdot D^s w:=(z,Dw)-(z\cdot\nabla w)\,d\mathcal{L}^N.$$

Then $z\cdot D^s w$ is a bounded measure which is absolutely continuous with respect to $|D^s w|$ [16], hence, it is singular, and we have $|z\cdot D^s w|\le\|z\|_\infty|D^s w|$.

2. The existence and uniqueness result

Our purpose in this section will be to define the notion of solution for the Cauchy problem (1.1) and to state an existence and uniqueness result for initial data in $L^1_{loc}(\mathbb{R}^N)$.

We shall assume that the Lagrangian $f:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ satisfies the following assumptions, which we shall refer collectively as (H):

(H_1) f is continuous on $\mathbb{R}^N \times \mathbb{R}^N$ and is a convex diffentiable function of ξ with continuous gradient for each fixed $x \in \mathbb{R}^N$. Furthermore we require f to satisfy the linear growth condition

$$C_0\|\xi\| - C_1 \leq f(x,\xi) \leq M(\|\xi\| + C_2) \tag{2.1}$$

for some positive constants C_0, C_1, C_2. Moreover, f possesses an asymptotic function, i.e. for almost all $x \in \mathbb{R}^N$ there exists the finite limit

$$\lim_{t\to 0^+} tf\left(x, \frac{\xi}{t}\right) = f^0(x,\xi), \tag{2.2}$$

and $f^0(x,-\xi) = f^0(x,\xi)$ for all $\xi \in \mathbb{R}^N$ and all $x \in \mathbb{R}^N$.

(H_2) Let us consider the function $\tilde{f} : \mathbb{R}^N \times \mathbb{R}^N \times [0,+\infty[\to \mathbb{R}$ defined as

$$\tilde{f}(x,\xi,t) := \begin{cases} f(x,\frac{\xi}{t})t & \text{if} \quad \mathrm{t}>0 \\ f^0(x,\xi) & \text{if} \quad \mathrm{t}=0 \end{cases} \tag{2.3}$$

We assume that $\tilde{f}(x,\xi,t)$ is continuous on $\mathbb{R}^N \times \mathbb{R}^N \times [0,+\infty[$ and convex in (ξ,t) for each fixed $x \in \mathbb{R}^N$.

We consider the function $\mathbf{a}(x,\xi) = \nabla_\xi f(x,\xi)$ associated to the Lagrangian f. By the convexity of f

$$\mathbf{a}(x,\xi)\cdot(\eta-\xi) \leq f(x,\eta) - f(x,\xi), \tag{2.4}$$

and the following monotonicity condition is satisfied

$$(\mathbf{a}(x,\eta) - \mathbf{a}(x,\xi))\cdot(\eta-\xi) \geq 0. \tag{2.5}$$

Moreover, it is easy to see that

$$|\mathbf{a}(x,\xi)| \leq M \quad \forall\ (x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N. \tag{2.6}$$

We consider the function $h : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$h(x,\xi) := \mathbf{a}(x,\xi)\cdot\xi.$$

From (2.4) and (2.1), it follows that

$$C_0\|\xi\| - D_1 \leq h(x,\xi) \leq M\|\xi\| \tag{2.7}$$

for some positive constant D_1.

We assume that

(H_3) $h(x,\xi) \geq 0$ for $x, \xi \in \mathbb{R}^N$, h^0 exists and the function $\tilde{h}$ is continuous on $\mathbb{R}^N \times \mathbb{R}^N \times [0, +\infty[$.

We need to consider the mapping $\mathbf{a}^\infty$ defined by

$$\mathbf{a}^\infty(x,\xi) := \lim_{t\to+\infty} \mathbf{a}(x, t\xi).$$

Observe that

$$h^0(x,\xi) = \mathbf{a}^\infty(x,\xi)\cdot\xi \quad \text{and} \quad C_0\|\xi\| \leq h^0(x,\xi) \leq M\|\xi\|.$$

(H_4) $\mathbf{a}^\infty(x,\xi) = \nabla_\xi f^0(x,\xi)$ for all $\xi \neq 0$ and all $x \in \mathbb{R}^N$.

In particular, as a consequence of Euler's Theorem, we have

$$f^0(x,\xi) = \mathbf{a}^\infty(x,\xi)\cdot\xi = h^0(x,\xi),$$

for all $\xi \in \mathbb{R}^N$ and all $x \in \mathbb{R}^N$, and consequently,

$$C_0\|\xi\| \leq f^0(x,\xi) \leq M\|\xi\| \qquad \forall \xi \in \mathbb{R}^N, \ \forall x \in \mathbb{R}^N. \tag{2.8}$$

(H_5) $\mathbf{a}(x,\xi)\cdot\eta \leq h^0(x,\eta)$ for all $x, \xi, \eta \in \mathbb{R}^N$.

Either from (H_4) or (H_5) it follows that $\mathbf{a}^\infty(x,\xi)\cdot\eta \leq h^0(x,\eta)$ for all $\xi, \eta \in \mathbb{R}^N$, $\xi \neq 0$, and all $x \in \mathbb{R}^N$. Indeed, it suffices to replace ξ by $t\xi$ in (H_5) and let $t \to +\infty$.

(H_6) $\mathbf{a}(x, 0) = 0$.

(H_7) We assume that

$$|a(x,\xi) - a(y,\xi)| \leq \omega(\|x-y\|) \tag{2.9}$$

for all $x, y \in \mathbb{R}^N$, and all $\xi \in \mathbb{R}^N$, where $\omega(r)$ is a modulus of continuity.

REMARK 2.1. Assumption (H_7) is only needed to prove uniqueness. The Lipschitz continuity in x of the flux is a common assumption to prove uniqueness of Kruzkov's solutions of scalar conservation laws ([17]).

We need to consider the space $BV(\mathbb{R}^N)_2$, defined as $BV(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ endowed with the norm

$$\|w\|_{BV(\mathbb{R}^N)_2} := \|w\|_{L^2(\mathbb{R}^N)} + |Dw|(\mathbb{R}^N).$$

As usual, we denote by $BV(\mathbb{R}^N)_2^*$ the topological dual of $BV(\mathbb{R}^N)_2$. It is easy to see that $L^2(\mathbb{R}^N) \subset BV(\mathbb{R}^N)_2^*$ and

$$\|w\|_{BV(\mathbb{R}^N)_2^*} \leq \|w\|_{L^2(\mathbb{R}^N)} \quad \forall\, w \in L^2(\mathbb{R}^N). \tag{2.10}$$

We define the space

$$Z(\mathbb{R}^N) := \{(z, \xi) \in L^\infty(\mathbb{R}^N, \mathbb{R}^N) \times BV(\mathbb{R}^N)^* \ : \ \operatorname{div}(z) = \xi \ \ in \ \ \mathcal{D}'(\mathbb{R}^N)\}.$$

To make precise our notion of solution we need the following definitions.

DEFINITION 2.2. Let $\Psi \in L^1(0, T; BV(\mathbb{R}^N)_2)$ and $Q_T = (0, T) \times \mathbb{R}^N$ for $T > 0$. We say Ψ admits a *weak derivative* in the space $L^1_w(0, T; BV(\mathbb{R}^N)) \cap L^\infty(Q_T)$ if there is a function $\Theta \in L^1_w(0, T; BV(\mathbb{R}^N)) \cap L^\infty(Q_T)$ such that $\Psi(t) = \int_0^t \Theta(s) ds$, the integral being taken as a Pettis integral.

DEFINITION 2.3. Let $\xi \in (L^1(0, T; BV(\mathbb{R}^N)_2))^*$. We say that ξ is *the time derivative* in the space $(L^1(0, T; BV(\mathbb{R}^N)_2))^*$ of a function $u \in L^1(0, T; L^1_{loc}(\mathbb{R}^N))$ if

$$\int_0^T < \xi(t), \Psi(t) > dt = -\int_0^T \int_{\mathbb{R}^N} u(t, x)\Theta(t, x) \, dxdt$$

for all test functions $\Psi \in L^1(0, T; BV(\mathbb{R}^N)_2)$ with compact support in time, which admit a weak derivative $\Theta \in L^1_w(0, T; BV(\mathbb{R}^N)) \cap L^\infty(Q_T)$ which is a function of compact support.

Observe that if $w \in L^1(0, T, BV(\mathbb{R}^N)_2) \cap L^\infty(Q_T)$ and $z \in L^\infty(Q_T, \mathbb{R}^N)$ is such that there exists $\xi \in (L^1(0, T; BV(\mathbb{R}^N)_2))^*$ with $\operatorname{div}(z) = \xi$ in $\mathcal{D}'(Q_T)$, associated to the pair (z, ξ), we define the distribution (z, Dw) in Q_T by

$$\begin{aligned} \langle (z, Dw), \phi \rangle := & -\int_0^T \langle \xi(t), w(t)\phi(t) \rangle \, dt \\ & -\int_0^T \int_{\mathbb{R}^N} z(t, x) w(t, x) \nabla_x \phi(t, x) \, dxdt \end{aligned} \tag{2.11}$$

for all $\phi \in \mathcal{D}(Q_T)$.

DEFINITION 2.4. Let $\xi \in (L^1(0, T; BV(\mathbb{R}^N)_2))^*$, $z \in L^\infty(Q_T, \mathbb{R}^N)$. We say that $\xi = \operatorname{div}(z)$ in $(L^1(0, T; BV(\mathbb{R}^N)_2))^*$ if (z, Dw) is a Radon measure in Q_T such that

$$\int_{Q_T} (z, Dw) + \int_0^T \langle \xi(t), w(t) \rangle \, dt = 0,$$

for all $w \in L^1(0, T; BV(\mathbb{R}^N)_2) \cap L^\infty(Q_T)$.

We also need the following set of truncatures

$$\mathcal{P} := \{p \in W^{1,\infty}(\mathbb{R}) : \ p' \geq 0, \ \operatorname{supp}(p') \text{ compact}\}.$$

Our concept of solution for the Dirichlet problem (1.1) is the following.

DEFINITION 2.5. A measurable function $u : (0, T) \times \mathbb{R}^N \to \mathbb{R}$ is an *entropy solution* of (1.1) in Q_T if $u \in C([0, T]; L^1_{loc}(\mathbb{R}^N))$, $u(t)$ converges to u_0 in $L^1_{\rm loc}(\mathbb{R}^N)$ as $t \to 0^+$, $p(u(\cdot)) \in L^1_w(0, T; BV_{\rm loc}(\mathbb{R}^N))$ for all $p \in \mathcal{P}$, and there exists $\xi \in (L^1(0, T; BV(\mathbb{R}^N)_2))^*$ such that:

(i) $(\mathbf{a}(x, \nabla u(t)), \xi(t)) \in Z(\mathbb{R}^N)$ a.e. $t \in [0, T]$,
(ii) ξ is the time derivative of u in $(L^1(0, T; BV(\mathbb{R}^N)_2))^*$ in the sense of Definition 2.3,
(iii) $\xi = \mathrm{div}(\mathbf{a}(x, \nabla u(t)))$ in $(L^1(0, T; BV(\mathbb{R}^N)_2))^*$ in the sense of Definition 2.4,
(iv) the following inequality is satisfied

$$- \int_0^T \int_{\mathbb{R}^N} j(u(t) - l)\eta_t \, dxdt + \int_0^T \int_{\mathbb{R}^N} \eta(t) h(x, Dp(u(t) - l)) \, dt$$
$$+ \int_0^T \int_{\mathbb{R}^N} \mathbf{a}(x, \nabla u(t)) \cdot \nabla \eta(t) p(u(t) - l) \, dxdt \leq 0$$

for all $l \in \mathbb{R}$, all $\eta \in C^\infty\left(]0, T[\times \mathbb{R}^N\right)$, with $\eta \geq 0$, $\eta(t, x) = \phi(t)\psi(x)$, being $\phi \in C_0^\infty(]0, T[)$, $\psi \in C_0^\infty(\mathbb{R}^N)$, and all $p \in \mathcal{P}$, where $j(r) := \int_0^r p(s)\, ds$.

Our main result is the following existence and uniqueness theorem.

THEOREM 2.6. *Assume we are under assumptions (H). Let* $u_0 \in L^1_{\rm loc}(\mathbb{R}^N)$. *Then there exists a unique entropy solution of* (1.1) *in* $[0, T] \times \mathbb{R}^N$ *for all* $T > 0$.

3. The approximation problem with finite energy

To prove the existence part of Theorem 2.6. we approximate (1.1) by problems of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathrm{div}\,(\varphi \mathbf{a}(x, Du)) & \text{in} \quad Q = (0, \infty) \times \mathbb{R}^N \\ u(0, x) = u_0(x) & \text{in} \quad x \in \mathbb{R}^N, \end{cases} \tag{3.1}$$

where $u_0 \in L^2(\mathbb{R}^N)$ and $\varphi \in \mathcal{S}(\mathbb{R}^N)$ where $\mathcal{S}(\mathbb{R}^N)$ denotes the space of rapidly decreasing C^∞ functions in $\mathbb{R}^N$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^N)$, $\varphi(x) > 0$ for every $x \in \mathbb{R}^N$. We define the space $BV(\mathbb{R}^N, \varphi\, dx)$ as the space of functions in $L^1_{loc}(\mathbb{R}^N)$ such that the distributional derivative Du is locally a Radon measure such that

$$\int_{\mathbb{R}^N} \varphi \, d|Du| < \infty.$$

By $W^{1,1}(\mathbb{R}^N, \varphi\, dx)$ we denote the space of functions in $BV(\mathbb{R}^N, \varphi\, dx)$ such that $Du \in L^1_{loc}(\mathbb{R}^N)$.

For simplicity, in what follows we shall assume that $\varphi \in \mathcal{S}(\mathbb{R}^N)$, $\varphi(x) > 0$ for all $x \in \mathbb{R}^N$, and satisfies the following property

$$|\varphi(x) - \varphi(y)| \leq C\varphi(y)\|x - y\| \tag{3.2}$$

for all $x, y \in \mathbb{R}^N$ such that $\|x - y\| \leq 1$ for some constant $C > 0$. It is easy to construct a function $\varphi(x) = \tilde{\varphi}(\|x\|)$ satisfying (3.2) if we take $\tilde{\varphi}$ a decreasing function such that $\tilde{\varphi}(r) = e^{-r}$ for $r \geq 1$ and such that φ is C^∞ in $\mathbb{R}^N$. Condition (3.2) enables us to prove the following Lemma.

LEMMA 3.1. *Assume that φ satisfies* (3.2). *Let $v \in BV(\mathbb{R}^N, \varphi\, dx) \cap L^p(\mathbb{R}^N)$, $1 \leq p < \infty$. Let $\eta \in C_0^\infty(\mathbb{R}^N)$, $\eta \geq 0$, with $supp(\eta) \subseteq B(0,1)$, $\int_{\mathbb{R}^N} \eta(x)\, dx = 1$ and let $\tau_j \downarrow 0+$, $\eta_j = \frac{1}{\tau_j^N}\eta(\frac{x}{\tau_j})$. Then $v_j = \eta_j * v \in W^{1,p}(\mathbb{R}^N)$ satisfy*

$$v_j \to v \quad \text{in } L^p(\mathbb{R}^N) \text{ and } \quad \int_{\mathbb{R}^N} |Dv_j|\varphi\, dx \to \int_{\mathbb{R}^N} \varphi\, d|Dv| \quad \text{as } j \to \infty.$$

Proof. We only have to check that

$$\limsup_j \int_{\mathbb{R}^N} |Dv_j|\varphi\, dx \leq \int_{\mathbb{R}^N} \varphi\, d|Dv|. \tag{3.3}$$

For that, we write

$$\begin{aligned}
\int_{\mathbb{R}^N} |Dv_j|\varphi\, dx &\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \eta_j(x-y) d|Dv|(y)\varphi(x)\, dx \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \eta_j(x-y)(\varphi(x) - \varphi(y)) d|Dv|(y)\, dx \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \eta_j(x-y)\varphi(y) d|Dv|(y)\, dx = (\mathrm{I}) + (\mathrm{II}).
\end{aligned}$$

Observe that the second of the integrals above is convergent since η_j have compact support. By interchanging the order of integration in (II) we have

$$(\mathrm{II}) = \int_{\mathbb{R}^N} \varphi\, d|Dv|.$$

Using (3.2) we have

$$\begin{aligned}
(\mathrm{I}) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \eta_j(x-y)(\varphi(x) - \varphi(y)) d|Dv|(y)\, dx \\
&\leq C \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \eta_j(x-y)\|x-y\|\varphi(y) d|Dv|(y)\, dx \\
&= C \int_{\mathbb{R}^N} \eta_j(z)\|z\|\, dz \int_{\mathbb{R}^N} \varphi\, d|Dv|.
\end{aligned}$$

Letting $j \to \infty$, we obtain (3.3). □

We denote by

$$X_\varphi(\mathbb{R}^N) := \{z \in X_2(\mathbb{R}^N) \ : \ z = \varphi z_1, \ \text{with } z_1 \in L^\infty(\mathbb{R}^N)\}.$$

If $z \in X_\varphi(\mathbb{R}^N)$ and $w \in BV(\mathbb{R}^N, \varphi\, dx) \cap L^2(\mathbb{R}^N)$ we define the functional $(z, Dw) : C_0^\infty(\mathbb{R}^N) \to \mathbb{R}$ by

$$\langle (z, Dw), \phi\rangle = -\int_{\mathbb{R}^N} w\, \phi \operatorname{div}(z)\, dx - \int_{\mathbb{R}^N} w\, z \cdot \nabla\phi\, dx. \tag{3.4}$$

Using Lemma 3.1, instead of the one used by Anzellotti in [5], and some small modifications of the proofs given in [5], we can show that (z, Dw) is a Radon measure in $\mathbb{R}^N$ and

$$\left|\int_B (z, Dw)\right| \leq \|z_1\|_\infty \int_B \varphi\, d|Dw| \tag{3.5}$$

for all Borel set $B \subset \mathbb{R}^N$. Moreover, we also have the following *Green's formula* for $z \in X_\varphi(\mathbb{R}^N)$ and $w \in BV(\mathbb{R}^N, \varphi\, dx) \cap L^2(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} (z, Dw) + \int_{\mathbb{R}^N} w \operatorname{div}(z)\, dx = 0. \tag{3.6}$$

Our notion of solution for problem (3.1) when $u_0 \in L^2(\mathbb{R}^N)$ is the following:

DEFINITION 3.2. Let $u_0 \in L^2(\mathbb{R}^N)$. A measurable function $u : (0, T) \times \mathbb{R}^N \to \mathbb{R}$ is a *solution* of (3.1) in Q_T if $u \in C([0, T], L^2(\mathbb{R}^N))$, $u(0) = u_0$, $u'(t) \in L^2(\mathbb{R}^N)$, $u(t) \in BV(\mathbb{R}^N, \varphi\, dx) \cap L^2(\mathbb{R}^N)$, $\varphi\mathbf{a}(x, \nabla u(t)) \in X_2(\mathbb{R}^N)$ a.e. $t \in [0, T]$, and for almost all $t \in [0, T]$ $u(t)$ satisfies:

$$u'(t) = \operatorname{div}(\varphi\mathbf{a}(x, \nabla u(t))) \quad \text{in } \mathcal{D}'(\mathbb{R}^N) \tag{3.7}$$

$$\varphi(x)\mathbf{a}(x, \nabla u(t)) \cdot D^s u(t) = \varphi(x) f^0(x, D^s u(t)) = \varphi(x) f^0(x, \overrightarrow{D^s u})|D^s u|. \tag{3.8}$$

THEOREM 3.3. *Assume we are under assumptions (H) and φ satisfies* (3.2). *Given $u_0 \in L^2(\mathbb{R}^N)$, there exists a unique solution u of* (3.1) *in Q_T for every $T > 0$ such that $u(0) = u_0$.*

To prove Theorem 3.3 we shall use the nonlinear semigroup theory ([9]). For that we need to study the energy functional associated with the problem (1.1). In order to consider the relaxed energy we recall the definition of function of a measure ([6], [11]). Let $g : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that

$$|g(x, \xi)| \leq M(1 + \|\xi\|) \quad \forall\, (x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N, \tag{3.9}$$

for some constant $M \geq 0$. Furthermore, we assume that g possesses an asymptotic function g^0. It is clear that the function $g^0(x, \xi)$ is positively homogeneous of degree one in ξ.

We denote by $\mathcal{M}(\mathbb{R}^N, \mathbb{R}^N)$ the set of all $\mathbb{R}^N$-valued bounded Radon measures on $\mathbb{R}^N$. Given $\mu \in \mathcal{M}(\mathbb{R}^N, \mathbb{R}^N)$, we consider its Lebesgue decomposition $\mu = \mu^a + \mu^s$, where μ^a is the absolutely continuous part of μ with respect to the Lebesgue measure $\mathcal{L}^N$ of $\mathbb{R}^N$ and μ^s is singular with respect to $\mathcal{L}^N$. We denote by $\mu^a(x)$ the density of the measure μ^a with respect to $\mathcal{L}^N$ and by $(d\mu^s/d|\mu|^s)(x)$ the density of μ^s with respect to $|\mu|^s$.

For $\mu \in \mathcal{M}(\mathbb{R}^N, \mathbb{R}^N)$ and g satisfying the above conditions, we define the measure $g(x, \mu)$ on $\mathbb{R}^N$ as

$$\int_B g(x, \mu) := \int_B g(x, \mu^a(x))\, dx + \int_B g^0\left(x, \frac{d\mu^s}{d|\mu|^s}(x)\right) d|\mu|^s \tag{3.10}$$

for all Borel set $B \subset \mathbb{R}^N$. In formula (3.10) we may write $(d\mu/d|\mu|)(x)$ instead of $(d\mu^s/d|\mu|^s)(x)$, because the two functions are equal $|\mu|^s$-a.e.

As it is proved in [6], if g is a Carathéodory function satisfying (3.9), then another way of writing the measure $g(x, \mu)$ is the following:

$$\int_B g(x, \mu) = \int_B \tilde{g}\left(x, \frac{d\mu}{d\alpha}(x), \frac{d\mathcal{L}^N}{d\alpha}(x)\right) d\alpha, \tag{3.11}$$

where α is any positive Borel measure such that $|\mu| + \mathcal{L}^N \ll \alpha$.

Let g be a function satisfying (3.9). Then for every $u \in BV(\mathbb{R}^N, \varphi\, dx)$ we have the measure $g(x, Du)\varphi$ defined by

$$\int_B g(x, Du)\varphi := \int_B g(x, \nabla u(x))\, \varphi\, dx + \int_B g^0(x, \overrightarrow{D^s u}(x))\, \varphi\, d|D^s u|$$

for all Borel set $B \subset \mathbb{R}^N$. Observe that if $\lambda = (\varphi Du, \varphi\mathcal{L}^N)$ and $\alpha = (|Du|, \mathcal{L}^N)$, then by Lemma 2.2 of [6] we have

$$g(x, Du)\varphi = \tilde{g}\left(x, \frac{d\lambda}{d\alpha}\right) = \tilde{g}(x, \lambda). \tag{3.12}$$

We define the energy functional

$$G_\varphi(u) := \int_{\mathbb{R}^N} g(x, Du)\, \varphi. \tag{3.13}$$

In [6], G. Anzellotti proves the lower semicontinuity of the functional G_φ in case of a bounded domain and $\varphi = 1$. Adapting the results in [6] or [4] we can also prove the lower semicontinuity of G_φ in our case. Indeed we have the following Lemma.

LEMMA 3.4. *Assume that $\tilde{g}(x, \xi, t)$ is lower-semicontinuous on $\mathbb{R}^N \times \mathbb{R}^N \times [0, +\infty[$, convex in (ξ, t) for each fixed $x \in \mathbb{R}^N$, and $g(x, \xi) \geq a\|\xi\| - b$ for all x and ξ. Then, for any sequence $u_n \in BV(\mathbb{R}^N, \varphi\, dx)$ such that $u_n \to u$ in $L^1_{loc}(\mathbb{R}^N)$ one has*

$$\liminf_{n\to\infty} G_\varphi(u_n) \geq G_\varphi(u).$$

We consider the energy functional associated with the problem (1.1) $\Phi_\varphi : L^2(\mathbb{R}^N) \to [0, +\infty]$ defined by

$$\Phi_\varphi(u) := \begin{cases} \displaystyle\int_{\mathbb{R}^N} f(x, Du)\, \varphi, & \text{if } u \in BV(\mathbb{R}^N, \varphi\, dx) \cap L^2(\mathbb{R}^N) \\ +\infty & \text{if } u \in L^2(\mathbb{R}^N) \setminus BV(\mathbb{R}^N, \varphi\, dx). \end{cases}$$

Functional Φ_φ is clearly convex and has the form given in (3.13). Then, as a consequence of Lemma 3.4, we have that Φ_φ is lower-semicontinuous. Therefore, the subdifferential $\partial\Phi_\varphi$ of Φ_φ, is a maximal monotone operator in $L^2(\mathbb{R}^N)$ (see [9]). Consequently, the existence and uniqueness of a solution of the abstract Cauchy problem

$$\begin{cases} u'(t) + \partial\Phi_\varphi(u(t)) \ni 0 & t \in]0, \infty[\\ u(0) = u_0 & u_0 \in L^2(\mathbb{R}^N) \end{cases} \tag{3.14}$$

follows immediately from the nonlinear semigroup theory (see [9]). Now, to get the full strength of the abstract result derived from semigroup theory we need to characterize $\partial\Phi_\varphi$. To get this characterization, we introduce the following operator $\mathcal{B}_\varphi$ in $L^2(\mathbb{R}^N)$.

$$(u, v) \in \mathcal{B}_\varphi \iff u \in BV(\mathbb{R}^N, \varphi\, dx) \cap L^2(\mathbb{R}^N), v \in L^2(\mathbb{R}^N)$$
and $\varphi(x)\mathbf{a}(x, \nabla u) \in X_2(\mathbb{R}^N)$ satisfies:

$$-v = \operatorname{div}(\varphi\mathbf{a}(x, \nabla u)) \quad \text{in } \mathcal{D}'(\mathbb{R}^N) \tag{3.15}$$

$$\varphi\mathbf{a}(x, \nabla u) \cdot D^s u = \varphi f^0(x, D^s u) = \varphi f^0(x, \overrightarrow{D^s u})|D^s u|. \tag{3.16}$$

THEOREM 3.5. *Assume we are under assumptions (H) and φ satisfies* (3.2)*, then the operator $\partial\Phi_\varphi$ has dense domain in $L^2(\mathbb{R}^N)$ and*

$$\partial\Phi_\varphi = \mathcal{B}_\varphi.$$

The proof of Theorem 3.5 follows the same approach used in [4] and we shall not include it here. Let us mention that one of the main tools needed is an approximation lemma similar to the one given by Anzellotti in [7]. The proof uses Lemma 3.1 and is similar to the proof in [7] (see also [4]).

LEMMA 3.6. *Assume that φ satisfies* (3.2). *If $v, u \in BV(\mathbb{R}^N, \varphi\, dx) \cap L^p(\mathbb{R}^N)$, $1 \le p < \infty$, then there exists a sequence $v_j \in C^1(\mathbb{R}^N) \cap W^{1,1}(\mathbb{R}^N, \varphi\, dx) \cap W^{1,p}(\mathbb{R}^N)$ such that*

$$v_j \to v \quad \text{in } L^p(\mathbb{R}^N), \tag{3.17}$$

$$\int_{\mathbb{R}^N} \sqrt{1 + |\nabla v_j(x)|^2}\, \varphi\, dx \to \int_{\mathbb{R}^N} \sqrt{1 + |Dv(x)|^2}\, \varphi\, dx, \tag{3.18}$$

$$\nabla v_j(x) \to \nabla v(x) \quad \mathcal{L}^N\text{-a.e. in } \mathbb{R}^N, \tag{3.19}$$

$$|\nabla v_j(x)| \to \infty \text{ and } \frac{\nabla v_j(x)}{|\nabla v_j(x)|} \to \frac{Dv(x)}{|Dv(x)|} \quad |Dv|^s\text{-a.e. in } \mathbb{R}^N, \tag{3.20}$$

$$|\nabla v_j(x)| \to \infty \text{ and } \frac{\nabla v_j(x)}{|\nabla v_j(x)|} \to \frac{Du(x)}{|Du(x)|} \quad |Du|^{ss}\text{-a.e. in } \mathbb{R}^N, \tag{3.21}$$

where $|Du|^{ss}$ *denotes the part of the singular measure* $|Du|^s$ *which is singular with respect to* $|Dv|^s$.

Standard semigroup theory (see [9]) and the characterization of Φ_φ given in Theorem 3.5 permits us to proof Theorem 3.3.

4. Proof of Theorem 1: Existence

We divide the proof into several steps.

STEP 1. Let $u_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Let $u_{0n} \in L^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be such that $u_{0n} \to u_0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Let $\varphi_n \in \mathcal{S}(\mathbb{R}^N)$ satisfying (3.2), $0 < \varphi_n \leq 1$ and $\varphi_n(x) = 1$ for all $x \in B(0, n)$. By Theorem 3.3, for every $n \in \mathbb{N}$ there exists a solution u_n of (3.1) for $\varphi = \varphi_n$, corresponding to the initial conditions u_{0n}. Therefore, $u_n(t), u_n'(t) \in L^2(\mathbb{R}^N)$, $u_n(t) \in BV(\mathbb{R}^N, \varphi_n\, dx)$, $z_n(t) := \varphi_n \mathbf{a}(x, \nabla u_n(t)) \in X_2(\mathbb{R}^N)$ a.e. $t \in [0, T]$, and for almost all $t \in [0, T]$ $u_n(t)$ satisfies:

$$u_n'(t) = \mathrm{div}(z_n(t)) \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \tag{4.1}$$

$$\begin{cases} z_n(t) \cdot D^s u_n(t) = \varphi_n f^0(x, D^s u_n(t)), \\ z_n(t) \cdot D^s p(u_n(t)) = \varphi_n f^0(x, D^s p(u_n(t))) \quad \forall\, p \in \mathcal{P}. \end{cases} \tag{4.2}$$

From (4.1) and (4.2), it follows that

$$-\int_{\mathbb{R}^N} (w - u_n(t)) u_n'(t)\, dx = \int_{\mathbb{R}^N} (z_n(t), Dw) - \int_{\mathbb{R}^N} \varphi_n h(x, Du_n(t)) \tag{4.3}$$

for every $w \in BV(\mathbb{R}^N, \varphi_n\, dx) \cap L^2(\mathbb{R}^N)$.

Let us prove that $\{u_n\}$ is a Cauchy sequence in $C([0, T]; L^1_{\mathrm{loc}}(\mathbb{R}^N))$. Let $\alpha > N$, $T_k(r) := \max(\min(r, k), -k)$ $(k \geq 0)$ and let j_k^+ be the primitive of $p_k^+(r) = \alpha T_k^+(r)^{\alpha-1}$ vanishing at $r = 0$. We define j_k^- as the primitive of p_k^- which vanishes at $r = 0$, where $p_k^-(r) = -p_k^+(-r)$. If $N = 1$, we take $\alpha \geq 2$, so that $(j_k^\pm)' \in W^{1,\infty}(\mathbb{R})$.

Let $\phi \in C_0^\infty(\mathbb{R}^N)$, $\phi \geq 0$, and suppose that $m \geq n$. Then, by (4.1) and Green's formula, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} p_k^+(u_n(t) - u_m(t))\phi(u_n'(t) - u_m'(t)) \\
&\quad = -\int_{\mathbb{R}^N} (z_n(t) - z_m(t), D(p_k^+(u_n(t) - u_m(t))\phi)) \\
&\quad = -\int_{\mathbb{R}^N} \phi(z_n(t) - z_m(t), D(p_k^+(u_n(t) - u_m(t))) \\
&\qquad - \int_{\mathbb{R}^N} \nabla\phi \cdot (z_n(t) - z_m(t)) p_k^+(u_n(t) - u_m(t)).
\end{aligned}$$

Now, since $\phi\varphi_n = \phi$ for all $n \geq n(\phi)$, having in mind (4.2) and (2.5), it is easy to see that

$$\int_{\mathbb{R}^N} \phi(z_n(t) - z_m(t), D(p_k^+(u_n(t) - u_m(t))) \geq 0.$$

Consequently, for every $m \geq n \geq n(\phi)$, we get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N} j_k^+(u_n(t) - u_m(t))\phi \leq -\int_{\mathbb{R}^N} \nabla\phi \cdot (z_n(t) - z_m(t)) p_k^+(u_n(t) - u_m(t)) \\
&\quad \leq 2M \int_{\mathbb{R}^N} |\nabla\phi|\, |p_k^+(u_n(t) - u_m(t))|.
\end{aligned}$$

Then, choosing $\phi = \varphi^\alpha$, with $0 \leq \varphi \in C_0^\infty(\mathbb{R}^N)$,

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N} j_k^+(u_n(t) - u_m(t))\varphi^\alpha \leq 2\alpha M \int_{\mathbb{R}^N} |p_k^+(u_n(t) - u_m(t))|\, \varphi^{\alpha-1}|\nabla\varphi| \\
&\quad \leq 2\alpha M \left(\int_{\mathbb{R}^N} (|p_k^+(u_n(t) - u_m(t))|\, \varphi^{\alpha-1})^{\frac{\alpha}{\alpha-1}}\right)^{\frac{\alpha-1}{\alpha}} \left(\int_{\mathbb{R}^N} |\nabla\varphi|^\alpha\right)^{\frac{1}{\alpha}} \\
&\quad \leq 2\alpha^2 M \left(\int_{\mathbb{R}^N} |T_k^+(u_n(t) - u_m(t))|^\alpha\, \varphi^\alpha\right)^{\frac{\alpha-1}{\alpha}} \left(\int_{\mathbb{R}^N} |\nabla\varphi|^\alpha\right)^{\frac{1}{\alpha}}.
\end{aligned}$$

Now, we observe that $T_k^+(r)^\alpha \leq j_k^+(r)$ for all $r \in \mathbb{R}$. Hence

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N} j_k^+(u_n(t) - u_m(t))\varphi^\alpha \\
&\quad \leq 2\alpha^2 M \left(\int_{\mathbb{R}^N} j_k^+(u_n(t) - u_m(t))\, \varphi^\alpha\right)^{\frac{\alpha-1}{\alpha}} \left(\int_{\mathbb{R}^N} |\nabla\varphi|^\alpha\right)^{\frac{1}{\alpha}}
\end{aligned}$$

and therefore,

$$\frac{d}{dt}\left(\int_{\mathbb{R}^N} j_k^+(u_n(t) - u_m(t))\varphi^\alpha\right)^{\frac{1}{\alpha}} \leq 2\alpha M \left(\int_{\mathbb{R}^N} |\nabla\varphi|^\alpha\right)^{\frac{1}{\alpha}}.$$

Setting $\psi_r(x) = \varphi(\frac{x}{r})$ instead of $\varphi(x)$ we get there exists $n_r \in \mathbb{N}$ such that for all $m \geq n \geq n_r$, we have

$$\frac{d}{dt}\left(\int_{\mathbb{R}^N} j_k^+(u_n(t) - u_m(t))\psi_r^\alpha\right)^{\frac{1}{\alpha}} \leq 2\alpha M \left(\int_{\mathbb{R}^N} |\nabla \psi_r|^\alpha\right)^{\frac{1}{\alpha}}$$
$$= 2M\alpha r^{\frac{N-\alpha}{\alpha}} \left(\int_{\mathbb{R}^N} |\nabla \varphi|^\alpha\right)^{\frac{1}{\alpha}}.$$

Integrating from 0 to s, with $0 \leq s \leq T$, we obtain

$$\left(\int_{\mathbb{R}^N} j_k^+(u_n(s) - u_m(s))\psi_r^\alpha\right)^{\frac{1}{\alpha}}$$
$$\leq \left(\int_{\mathbb{R}^N} j_k^+(u_{0n} - u_{0m})\psi_r^\alpha\right)^{\frac{1}{\alpha}} + 2TM\alpha r^{\frac{N-\alpha}{\alpha}} \left(\int_{\mathbb{R}^N} |\nabla \varphi|^\alpha\right)^{\frac{1}{\alpha}},$$

for all $m \geq n \geq n_r$ and $s \in [0, T]$. Given $\epsilon > 0$, since $\alpha > N$, we can find $r_\epsilon \in \mathbb{N}$ such that

$$2TM\alpha r^{\frac{N-\alpha}{\alpha}} \left(\int_{\mathbb{R}^N} |\nabla \varphi|^\alpha\right)^{\frac{1}{\alpha}} \leq \frac{\epsilon}{2} \qquad \forall r \geq r_\epsilon.$$

Now, let $n_\epsilon \in \mathbb{N}$ be such that $n_\epsilon \geq n_{r_\epsilon}$ and

$$\left(\int_{\mathbb{R}^N} j_k^+(u_{0n} - u_{0m})\psi_{r_\epsilon}^\alpha\right)^{\frac{1}{\alpha}} \leq \frac{\epsilon}{2} \qquad \forall m \geq n \geq n_\epsilon.$$

Then, we obtain that

$$\left(\int_{\mathbb{R}^N} j_k^+(u_n(s) - u_m(s))\psi_{r_\epsilon}^\alpha\right)^{\frac{1}{\alpha}} \leq \epsilon \quad \forall m \geq n \geq n_\epsilon \text{ and } s \in [0, T],$$

from where it follows that $\{u_n\}$ is a Cauchy sequence in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$. Thus we may assume that $u_n \to u$ in $C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$ for some function $u \in C([0, T]; L^1_{\text{loc}}(\mathbb{R}^N))$. In particular, we have that $u(t) \to u_0$ in $L^1_{\text{loc}}(\mathbb{R}^N)$ as $t \to 0+$.

STEP 2. Convergence of the derivatives and identification of the limit. sSince the map $t \mapsto u_n'(t)$ is strongly measurable from $[0, T]$ into $L^2(\mathbb{R}^N)$ and, by (2.10),

$$\|u_n'(t)\|_{BV(\mathbb{R}^N)_2^*} \leq \|u_n'(t)\|_{L^2(\mathbb{R}^N)},$$

it follows that this map is strongly measurable from $[0, T]$ into $BV(\mathbb{R}^N)_2^*$. Moreover, for every $w \in BV(\mathbb{R}^N)_2$, if we take $u_n(t) - w$ as test function in (4.3), since

$$\int_{\mathbb{R}^N} \varphi_n h(x, Du_n(t)) = \int_{\mathbb{R}^N} (z_n(t), Du_n(t)),$$

we get

$$\int_{\mathbb{R}^N} u_n'(t) w \, dx = -\int_{\mathbb{R}^N} (z_n(t), Dw).$$

Hence

$$\left| \int_{\mathbb{R}^N} u_n'(t) w \, dx \right| \leq M \int_{\mathbb{R}^N} |Dw| \leq M_1 \|w\|_{BV(\mathbb{R}^N)_2} \quad \forall\, n \in \mathbb{N}.$$

Thus,

$$\|u_n'(t)\|_{BV(\mathbb{R}^N)_2^*} \leq M_1 \quad \forall\, n \in \mathbb{N} \text{ and } t \in [0, T].$$

Consequently, $\{u_n'\}_{n\in\mathbb{N}}$ is a bounded sequence in $L^\infty(0, T; BV(\mathbb{R}^N)_2^*)$. Now, since the space $L^\infty(0, T; BV(\mathbb{R}^N)_2^*)$ is a vector subspace of the dual space $(L^1(0, T; BV(\mathbb{R}^N)_2))^*$, we can find a subnet $\{u_\alpha'\}$ such that

$$u_\alpha' \to \xi \in (L^1(0, T; BV(\mathbb{R}^N)_2))^* \quad \text{weakly}^*. \tag{4.4}$$

Since $\|z_n(t)\|_\infty \leq M$ for all $n \in \mathbb{N}$ and a.e. $t \in [0, T]$, we may also assume that

$$z_n \to z \in L^\infty(Q_T, \mathbb{R}^N) \qquad \text{weakly}^*. \tag{4.5}$$

Obviously, we have

$$\xi = \operatorname{div}_x(z) \qquad \text{in } \mathcal{D}'(Q_T) \tag{4.6}$$

and

$$\xi(t) = \operatorname{div}_x(z(t)) \qquad \text{in } \mathcal{D}'(\mathbb{R}^N) \text{ a.e. } t \in [0, T]. \tag{4.7}$$

Consequently, $(z(t), \xi(t)) \in Z(\mathbb{R}^N)$ for almost all $t \in [0, T]$.

With a similar proof to the one given for Lemma 4.1 of [3], we get the following result.

LEMMA 4.1. *ξ is the time derivative of u in the sense of the Definition* 2.3.

STEP 3. Next, we prove that $\xi = \operatorname{div}(z)$ in $(L^1(0, T, BV(\mathbb{R}^N)_2))^*$ in the sense of the Definition 2.4. To do that, let us first observe that (z, Dw), defined by (2.11), is a Radon measure in Q_T for all $w \in L^1_w(0, T, BV(\mathbb{R}^N)_2) \cap L^\infty(Q_T)$. Let $\phi \in \mathcal{D}(Q_T)$, then

$$\langle (z, Dw), \phi \rangle = -\int_0^T \langle \xi(t) - u_\alpha'(t), w(t)\phi(t) \rangle \, dt$$
$$- \int_{Q_T} w(z - z_\alpha) \cdot \nabla_x \phi \, dx dt + \int_0^T \langle (z_\alpha(t), Dw(t)), \phi(t) \rangle \, dt.$$

Taking limits in α, and using (4.4), we get

$$\langle (z, Dw), \phi \rangle = \lim_\alpha \int_0^T \langle (z_\alpha(t), Dw(t)), \phi(t) \rangle \, dt. \tag{4.8}$$

Therefore

$$|\langle (z, Dw), \phi\rangle| \le M\|\phi\|_\infty \int_0^T \int_{\mathbb{R}^N} |Dw(t)|\, dt.$$

Hence, (z, Dw) is a Radon measure in Q_T. Moreover, from (4.8), applying Green's formula we obtain that

$$\begin{aligned}\int_{Q_T} (z, Dw) &= \lim_\alpha \int_0^T (z_\alpha(t), Dw(t))\, dt \\ &= -\lim_\alpha \int_0^T \int_{\mathbb{R}^N} \operatorname{div}(z_\alpha(t)) w(t)\, dxdt = -\int_0^T \langle \xi(t), w(t)\rangle\, dt,\end{aligned}$$

that is,

$$\int_{Q_T} (z, Dw) + \int_0^T \langle \xi(t), w(t)\rangle\, dt = 0. \tag{4.9}$$

As a consequence of the boundedness of $\{u_n'\}$, (4.4) and the above statement, we have

$$u_n' \to \xi \in (L^1(0, T; BV(\mathbb{R}^N)_2))^* \quad \text{weakly}^*. \tag{4.10}$$

STEP 4. Convergence of the energy.

Let $0 \le \varphi \in C_0^\infty(\mathbb{R}^N)$. If n is large enough, we have $\varphi\varphi_n = \varphi$. From now on, we assume that this is the case. Multiplying (4.1) by $(w - p(u_n(t)))\varphi$, integrating in $\mathbb{R}^N$ and using (4.2), we have that

$$\begin{aligned}&-\int_{\mathbb{R}^N} (w - p(u_n(t)))\varphi u_n'(t)\, dx \\ &\quad = \int_{\mathbb{R}^N} (z_n(t), Dw)\varphi - \int_{\mathbb{R}^N} h(x, Dp(u_n(t)))\varphi + \int_{\mathbb{R}^N} z_n(w - p(u_n)) D\varphi\end{aligned} \tag{4.11}$$

for every $w \in BV_{loc}(\mathbb{R}^N) \cap L^2_{loc}(\mathbb{R}^N)$ and all $p \in \mathcal{P}$.

First, we observe that setting $w = 0$ in (4.11) and integrating in $(0, T)$, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N} J_p(u_n(T))\varphi dx + \int_0^T \int_{\mathbb{R}^N} h(x, Dp(u_n(t)))\varphi\, dxdt \\ &\quad = \int_{\mathbb{R}^N} J_p(u_{0,n})\varphi - \int_0^T \int_{\mathbb{R}^N} z_n p(u_n(t)) D\varphi\, dxdt,\end{aligned}$$

where $J_p'(r) = p(r)$. In particular, we have

$$\int_0^T \int_{\mathbb{R}^N} J_p(u_n(t))\varphi \, dxdt \leq C, \tag{4.12}$$

$$\int_0^T \int_{\mathbb{R}^N} h(x, Dp(u_n(t)))\varphi \, dxdt \leq C. \tag{4.13}$$

Hence, by (2.7),

$$\int_0^T \int_{\mathbb{R}^N} \varphi |Dp(u_n(t))| \, dt \leq C \tag{4.14}$$

where C is a constant depending on u_0, φ, $\|p\|_\infty$ and the constants in (2.1). Since the functional $\Phi_\varphi : L^1_{loc}(\mathbb{R}^N) \to]-\infty, +\infty]$, defined by

$$\Phi_\varphi(w) = \begin{cases} \displaystyle\int_{\mathbb{R}^N} \varphi \, d|Dw| & \text{if } \ w \in BV_{loc}(\mathbb{R}^N) \\ +\infty & \text{if } \ w \in L^1_{loc}(\mathbb{R}^N) \setminus BV_{loc}(\mathbb{R}^N), \end{cases} \tag{4.15}$$

is lower semicontinuous in $L^1_{loc}(\mathbb{R}^N)$, we have that

$$\Phi_\varphi(p(u(t))) \leq \liminf_{n\to\infty} \Phi_\varphi(p(u_n(t)) = \liminf_{n\to\infty} \int_{\mathbb{R}^N} \varphi |Dp(u_n(t))|.$$

On the other hand, by Lemma 5 in [3], the map $t \mapsto \int_{\mathbb{R}^N} \varphi |Dp(u_n(t))|$ is measurable, then by the Fatou's Lemma and (4.14), it follows that

$$\begin{aligned} \int_0^T \Phi_\varphi(p(u(t))) &\leq \int_0^T \liminf_{n\to\infty} \left(\int_{\mathbb{R}^N} \varphi |Dp(u_n(t))| \right) \, dt \\ &\leq \liminf_{n\to\infty} \int_0^T \left(\int_{\mathbb{R}^N} \varphi |Dp(u_n(t))| \right) \, dt \leq C. \end{aligned} \tag{4.16}$$

As a consequence of (4.16), we obtain that $p(u(t)) \in BV_{loc}(\mathbb{R}^N)$ for almost all $t \in [0, T]$.

From Lemma 4.2 in [3], if $0 \leq \eta = \psi(t)\varphi(x)$, $\psi(t) \in \mathcal{D}(]0, T[)$, $\varphi(x) \in C_0^\infty(\mathbb{R}^N)$, the map $t \mapsto p(u(t))\eta(t)$, from $[0, T]$ into $BV(\mathbb{R}^N)$, is weakly measurable.

Using the same technique than in the proofs of Lemmas 4.3 and Lemma 4.4 of [3], we obtain the following two results.

LEMMA 4.2. *For any $\tau > 0$, we define the function ψ^τ, as the Dunford integral* (see [12])

$$\psi^\tau(t) := \frac{1}{\tau} \int_{t-\tau}^t \eta(s) p(u(s)) \, ds \in BV(\mathbb{R}^N)^{**},$$

that is,

$$\langle \psi^\tau(t), w\rangle = \frac{1}{\tau}\int_{t-\tau}^{t} \langle \eta(s)p(u(s)), w\rangle \, ds$$

for any $w \in BV(\mathbb{R}^N)^*$. *Then* $\psi^\tau \in C([0,T]; BV(\mathbb{R}^N))$. *Moreover,* $\psi^\tau(t) \in L^2(\mathbb{R}^N)$*, thus,* $\psi^\tau(t) \in BV(\mathbb{R}^N)_2$ *and* ψ^τ *admits a weak derivative in* $L^1_w(0, T, BV(\mathbb{R}^N)) \cap L^\infty(Q_T)$.

LEMMA 4.3. *For* $\tau > 0$ *small enough, we have*

$$\int_0^T \langle \psi^\tau(t), \xi(t)\rangle \, dt \le -\int_0^T \int_{\mathbb{R}^N} \frac{\eta(t-\tau)-\eta(t)}{-\tau} J_p(u(t)) \, dxdt. \tag{4.17}$$

We need the following result.

LEMMA 4.4. *Let*

$$A_n := \int_0^T \int_{\mathbb{R}^N} h(x, Dp(u_n))\varphi \, dt,$$

$\eta = \psi(t)\phi(x)$, $\psi \in \mathcal{D}(0,T)$, $\phi \in C_0^\infty(\mathbb{R}^N)$*, and*

$$(\eta p(u))^\tau(t) = \frac{1}{\tau}\int_{t-\tau}^{t} \eta(s)p(u(s))ds.$$

Then

$$\limsup_{n\to\infty} A_n \le \int_0^T \int_{\mathbb{R}^N} z(t)\cdot \nabla p(u(t))\, \varphi \, dxdt$$
$$+ \liminf_{\eta\uparrow 1} \liminf_{\tau\to 0} \int_0^T (z(t), D^s(\eta p(u))^\tau(t))\varphi \, dt.$$

Proof. Let $w \in W^{1,1}((0,T)\times\mathbb{R}^N)$. We use as test function $(\eta p(w(t)))^\tau$ in (4.11) and integrate in $(0,T)$ to obtain

$$-\int_0^T \int_{\mathbb{R}^N} (\eta p(w(t)))^\tau \varphi u_n'(t) \, dxdt + \int_0^T \int_{\mathbb{R}^N} p(u_n(t))\varphi u_n'(t)\, dxdt$$
$$+\int_0^T \int_{\mathbb{R}^N} h(x, Dp(u_n(t)))\varphi \, dt$$
$$= \int_0^T \int_{\mathbb{R}^N} (z_n(t), D(\eta p(w(t)))^\tau)\, \varphi dt + \int_0^T \int_{\mathbb{R}^N} z_n(t) D\varphi(\eta p(w(t)))^\tau dt$$
$$-\int_0^T \int_{\mathbb{R}^N} z_n(t) D\varphi p(u_n(t)),$$

where $\eta^\tau(t) = \frac{1}{\tau}\int_{t-\tau}^{t} \eta(s)ds$. Our purpose is to take limits in the above expression as $n \to \infty$, $w \to u$ in $L^1_{loc}((0,T) \times \mathbb{R}^N)$, $\tau \to 0$ and $\eta \uparrow 1$. We take $\tau > 0$ small enough. Let us analyze the first term

$$-\int_0^T \int_{\mathbb{R}^N} (\eta p(w(t)))^\tau \varphi u_n'(t)\, dxdt = \int_0^T \int_{\mathbb{R}^N} (\eta p(w(t)))_t^\tau \varphi u_n(t)\, dxdt$$
$$\to \int_0^T \int_{\mathbb{R}^N} (\eta p(w(t)))_t^\tau \varphi u(t)\, dxdt \quad \text{as } n \to \infty.$$

Now, using Lemma 4.7 and Lemma 4.9,

$$\int_0^T \int_{\mathbb{R}^N} (\eta p(w(t)))_t^\tau \varphi u = \int_0^T \int_{\mathbb{R}^N} \frac{\eta(t)p(w(t)) - \eta(t-\tau)p(w(t-\tau))}{\tau} \varphi u(t)\, dxdt$$
$$\to \int_0^T \int_{\mathbb{R}^N} \frac{\eta(t)p(u(t)) - \eta(t-\tau)p(u(t-\tau))}{\tau} \varphi u(t)\, dxdt, \quad \text{as } w \to u \text{ in } L^1_{loc},$$
$$= -\int_0^T \langle \xi(t), (\varphi\eta p(u(t)))^\tau \rangle\, dt \geq \int_0^T \int_{\mathbb{R}^N} \frac{\eta(t-\tau) - \eta(t)}{-\tau} \varphi J_p(u(t))\, dxdt$$
$$\to \int_0^T \int_{\mathbb{R}^N} \eta_t \varphi J_p(u(t))\, dxdt, \quad \text{as } \tau \to 0$$
$$\to \int_{\mathbb{R}^N} (J_p(u(0)) - J_p(u(T)))\varphi\, dx \quad \text{as } \eta \uparrow 1.$$

The analysis of the second term is easy. Letting $n \to \infty$ we have

$$\int_0^T \int_{\mathbb{R}^N} p(u_n(t))\varphi u_n'(t)\, dxdt = \int_0^T \frac{d}{dt} \int_{\mathbb{R}^N} J_p(u_n(t))\varphi\, dx$$
$$= \int_{\mathbb{R}^N} (J_p(u_n)(T) - J_p(u_n(0)))\varphi\, dx \to \int_{\mathbb{R}^N} (J_p(u(T)) - J_p(u(0)))\varphi\, dx.$$

Let us deal together the first two terms of the right hand side of the equality we are analyzing. Having in mind Steps 3, 4 and (4.10), taking limits as $n \to \infty$, $w \to u$ in L^1_{loc} and $\tau \to 0$, we get:

$$\int_0^T \int_{\mathbb{R}^N} (z_n(t), D[(\eta p(w))^\tau \varphi])\, dt \to$$
$$-\int_0^T \langle \xi(t), (\eta p(w))^\tau \varphi \rangle\, dt = \int_0^T \int_{\mathbb{R}^N} u(t)(\eta p(w))_t^\tau \varphi\, dxdt$$
$$= \int_0^T \int_{\mathbb{R}^N} u(t) \frac{\eta(t)p(w)(t) - \eta(t-\tau)p(w)(t-\tau)}{\tau} \varphi\, dxdt$$
$$\to \int_0^T \int_{\mathbb{R}^N} u(t) \frac{\eta(t)p(u)(t) - \eta(t-\tau)p(u)(t-\tau)}{\tau} \varphi\, dxdt$$

$$
\begin{aligned}
&= \int_0^T \int_{\mathbb{R}^N} u(t)(\eta p(u))_t^\tau \varphi \, dxdt = -\int_0^T \langle \xi(t), (\eta p(u))^\tau \varphi \rangle \, dt \\
&= \int_0^T \int_{\mathbb{R}^N} z(t) \cdot \nabla[(\eta p(u))^\tau \varphi] \, dxdt + \int_0^T \int_{\mathbb{R}^N} z(t) \cdot D^s[(\eta p(u))^\tau] \varphi \, dt \\
&\quad \to \int_0^T \int_{\mathbb{R}^N} z(t) \cdot \nabla(\eta p(u) \varphi) \, dxdt + \liminf_{\tau \to 0} \int_0^T \int_{\mathbb{R}^N} z(t) \cdot D^s(\eta p(u))^\tau \varphi \, dt \\
&\quad \to \int_0^T \int_{\mathbb{R}^N} z(t) \cdot \nabla[p(u(t))\varphi] \, dxdt + \liminf_{\eta \uparrow 1} \liminf_{\tau \to 0} \\
&\qquad \int_0^T \int_{\mathbb{R}^N} z(t) \cdot D^s(\eta p(u))^\tau \varphi \, dt \\
&\qquad\qquad \text{as } \eta \uparrow 1.
\end{aligned}
$$

The last term $\int_0^T \int_{\mathbb{R}^N} z_n(t) D\varphi p(u_n(t))$ easily converges to $\int_0^T \int_{\mathbb{R}^N} z(t) D\varphi p(u(t))$. □

The lemma follows by collecting all these facts.

The proof of next Lemma is similar to the proof of Lemma 9 in [3].

LEMMA 4.5. *Let* $\varphi \in C_0^\infty(\mathbb{R}^N)$. *Let*

$$\Psi_{p,\varphi}(p(u(t))) = \int_{\mathbb{R}^N} f(x, Dp(u(t)))\varphi.$$

Then

$$\int_0^T \Psi_{p,\varphi}(p(u(t))) \, dt = \lim_{n \to \infty} \int_0^T \Psi_{p,\varphi}(p(u_n(t))) \, dt. \tag{4.18}$$

As a consequence, we also have that

$$\Psi_{p,\varphi}(p(u(t))) = \lim_{n \to \infty} \Psi_{p,\varphi}(p(u_n(t))) \quad \text{a.e. in } t. \tag{4.19}$$

From Lemma 4.5 it follows that

$$\int_{\mathbb{R}^N} h(x, Dp(u(t)))\varphi = \lim_{n \to \infty} \int_{\mathbb{R}^N} h(x, Dp(u_n(t)))\varphi, \tag{4.20}$$

a.e. in $t \in (0, T)$. Indeed, if we consider the $\mathbb{R}^N$-valued measures μ_n, μ on $\mathbb{R}^N$ which are defined by

$$\mu_n(B) := \int_B \varphi Dp(u_n), \qquad \mu(B) := \int_B \varphi Dp(u)$$

for all Borel sets $B \subseteq \mathbb{R}^N$, we have

$$\mu_n \to \mu \qquad \text{weakly as measures in } \mathbb{R}^N.$$

Moreover,

$$\Psi_{p,\varphi}(p(u)) = \int_{\mathbb{R}^N} \tilde{f}(x,\lambda) \quad \text{and} \quad \Psi_{p,\varphi}(p(u_n)) = \int_{\mathbb{R}^N} \tilde{f}(x,\lambda_n),$$

with $\lambda = (\mu, \varphi\mathcal{L}^N)$ and $\lambda_n = (\mu_n, \varphi\mathcal{L}^N)$. Hence, (4.18) yields

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \tilde{f}(x,\lambda_n) = \int_{\mathbb{R}^N} \tilde{f}(x,\lambda).$$

Then, applying Theorem 3 of [19] (see also [14], Theorem 1, page 90), it follows that

$$\int_{\mathbb{R}^N} \tilde{h}(x,\lambda) = \lim_{n\to\infty} \int_{\mathbb{R}^N} \tilde{h}(x,\lambda_n) = \lim_{n\to\infty} \int_{\mathbb{R}^N} h(x, Dp(u_n))\varphi.$$

Now, it is easy to see that

$$\int_{\mathbb{R}^N} \tilde{h}(x,\lambda) = \int_{\mathbb{R}^N} h(x, Dp(u))\varphi,$$

consequently, we obtain (4.20).

Let us now prove that

$$\int_0^T \int_{\mathbb{R}^N} z(t)\cdot\nabla p(u(t))\varphi\, dxdt \leq \int_0^T \int_{\mathbb{R}^N} \mathbf{a}(x,\nabla u(t))\cdot\nabla p(u(t))\varphi\, dxdt. \tag{4.21}$$

In fact, from the convexity of f in ξ, we have

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^N} \mathbf{a}(x,\nabla p(u_n))\cdot\nabla p(u)\varphi\, dxdt &\leq \int_0^T \int_{\mathbb{R}^N} \mathbf{a}(x,\nabla p(u_n))\cdot\nabla p(u_n)\varphi\, dxdt \\
&\quad + \int_0^T \int_{\mathbb{R}^N} f(x,\nabla p(u))\varphi\, dxdt - \int_0^T \int_{\mathbb{R}^N} f(x,\nabla p(u_n))\varphi\, dxdt \\
&= \int_0^T \int_{\mathbb{R}^N} h(x,\nabla p(u_n))\varphi\, dxdt + \int_0^T \int_{\mathbb{R}^N} f^0(x, D^s p(u_n))\varphi\, dt \\
&\quad - \int_0^T \int_{\mathbb{R}^N} f^0(x, D^s p(u_n))\varphi\, dt + \int_0^T \int_{\mathbb{R}^N} f(x,\nabla p(u))\varphi\, dxdt \\
&\quad - \int_0^T \int_{\mathbb{R}^N} f(x,\nabla p(u_n))\varphi\, dxdt = \int_0^T \int_{\mathbb{R}^N} h(x, Dp(u_n))\varphi\, dt \\
&\quad - \int_0^T \int_{\mathbb{R}^N} f(x, Dp(u_n))\varphi\, dt + \int_0^T \int_{\mathbb{R}^N} f(x,\nabla p(u))\varphi\, dxdt.
\end{aligned}$$

Now, since

$$\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^N} [\mathbf{a}(x,\nabla u_n(t)) - \mathbf{a}(x,\nabla p(u_n(t)))]\cdot\nabla p(u(t))\varphi\, dxdt = 0,$$

we have

$$\lim_{n\to\infty}\int_0^T\int_{\mathbb{R}^N}\mathbf{a}(x,\nabla p(u_n(t)))\cdot\nabla p(u(t))\varphi\,dxdt$$
$$=\int_0^T\int_{\mathbb{R}^N} z(t)\cdot\nabla p(u(t))\varphi\,dxdt.$$

Then, letting $n\to\infty$, using Lemma 4.11 and (4.20), we deduce that

$$\int_0^T\int_{\mathbb{R}^N} z(t)\cdot\nabla p(u(t))\varphi\,dxdt \le \int_0^T\int_{\mathbb{R}^N} h(x,Dp(u))\varphi\,dt$$
$$-\int_0^T\int_{\mathbb{R}^N} f(x,Dp(u))\varphi\,dt+\int_0^T\int_{\mathbb{R}^N} f(x,\nabla p(u))\varphi\,dt$$
$$=\int_0^T\int_{\mathbb{R}^N}\mathbf{a}(x,\nabla u(t))\cdot\nabla p(u(t))\,dxdt,$$

and (4.21) holds.

STEP 5. Identification of the limit. Let us now prove that

$$z(t,x)=\mathbf{a}(x,\nabla u(t,x))\qquad \text{a.e. } (t,x)\in(0,T)\times\mathbb{R}^N. \tag{4.22}$$

Let $0\le\phi\in C_0^1((0,T)\times\mathbb{R}^N)$ and $g\in C^1([0,T]\times\mathbb{R}^N)$. We observe that

$$\int_0^T\int_{\mathbb{R}^N}\phi[(\mathbf{a}(x,\nabla u_n),Dp(u_n-g))-\mathbf{a}(x,\nabla g)Dp(u_n-g)]$$
$$=\int_0^T\int_{\mathbb{R}^N}\phi[\mathbf{a}(x,\nabla u_n)-\mathbf{a}(x,\nabla g)]\cdot\nabla p(u_n-g)\,dxdt$$
$$+\int_{\mathbb{R}^N}\phi[\mathbf{a}(x,\nabla u_n)-\mathbf{a}(x,\nabla g)]\cdot D^s p(u_n-g).$$

Since both terms at the right hand side of the above expression are positive, we have

$$\int_0^T\int_{\mathbb{R}^N}\phi[(\mathbf{a}(x,\nabla u_n),Dp(u_n-g))-\mathbf{a}(x,\nabla g)Dp(u_n-g)]\ge 0. \tag{4.23}$$

Our purpose is to take limits as $n\to\infty$ in the above inequality. We assume that $\phi(t,x)=\eta(t)\psi(x)$, where $\eta\in\mathcal{D}(0,T)$, $\psi\in\mathcal{D}(\mathbb{R}^N)$, $\eta\ge 0$, $\psi\ge 0$. First, integrating by parts in the first term, we have

$$\int_0^T\int_{\mathbb{R}^N}\phi(\mathbf{a}(x,\nabla u_n),Dp(u_n-g))\,dt=-\int_0^T\int_{\mathbb{R}^N}p(u_n-g)\nabla_x\phi\cdot\mathbf{a}(x,\nabla u_n)\,dxdt$$
$$-\int_0^T\int_{\mathbb{R}^N}\phi\,\mathrm{div}(\mathbf{a}(x,\nabla u_n))p(u_n-g)\,dxdt$$

$$= -\int_0^T \int_{\mathbb{R}^N} p(u_n - g)\nabla_x\phi \cdot \mathbf{a}(x, \nabla u_n)\, dxdt - \int_0^T \int_{\mathbb{R}^N} \phi u_n'(t) p(u_n - g)\, dxdt$$
$$= -\int_0^T \int_{\mathbb{R}^N} p(u_n - g)\nabla_x\phi \cdot \mathbf{a}(x, \nabla u_n)\, dxdt - \int_0^T \int_{\mathbb{R}^N} \phi \frac{d}{dt} J_p(u_n - g)\, dxdt$$
$$- \int_0^T \int_{\mathbb{R}^N} \phi g_t p(u_n - g)\, dxdt = -\int_0^T \int_{\mathbb{R}^N} p(u_n - g)\nabla_x\phi \cdot \mathbf{a}(x, \nabla u_n)\, dxdt$$
$$+ \int_0^T \int_{\mathbb{R}^N} \phi_t J_p(u_n - g)\, dxdt - \int_0^T \int_{\mathbb{R}^N} \phi g_t p(u_n - g)\, dxdt.$$

Letting $n \to \infty$ in (4.23), taking into account the above equalities, we obtain

$$- \int_0^T \int_{\mathbb{R}^N} p(u - g)\nabla_x\phi \cdot z\, dxdt + \int_0^T \int_{\mathbb{R}^N} \phi_t J_p(u - g)\, dxdt$$
$$- \int_0^T \int_{\mathbb{R}^N} \phi g_t p(u - g)\, dxdt - \int_0^T \int_{\mathbb{R}^N} \phi(a(x, \nabla g), Dp(u - g))\, dt \geq 0. \tag{4.24}$$

Now,

$$\int_0^T \int_{\mathbb{R}^N} \phi_t J_p(u - g)\, dxdt$$
$$= \lim_{\tau \to 0} \int_0^T \int_{\mathbb{R}^N} \frac{\phi(t - \tau) - \phi(t)}{-\tau} J_p(u - g)\, dxdt$$
$$= \lim_{\tau \to 0} \int_0^T \int_{\mathbb{R}^N} \psi(x) \frac{\eta(t - \tau) - \eta(t)}{-\tau} J_p(u - g)\, dxdt. \tag{4.25}$$

For simplicity, let us write $v = u - g$. Since

$$J_p(v(t)) - J_p(v(t + \tau)) \leq (v(t) - v(t + \tau)) p(v(t)),$$

for τ small enough, we have

$$\int_0^T \int_{\mathbb{R}^N} \frac{v(t + \tau) - v(t)}{\tau} \eta(t)\psi(x) p(v(t))\, dxdt$$
$$\leq \int_0^T \int_{\mathbb{R}^N} \frac{J_p(v(t + \tau)) - J_p(v(t))}{\tau} \eta(t)\psi(x)\, dxdt$$
$$= \int_0^T \int_{\mathbb{R}^N} \frac{\eta(t - \tau) - \eta(t)}{\tau} \psi(x) J_p(v)\, dxdt. \tag{4.26}$$

By Lemma 4.1, we have

$$\int_0^T \int_{\mathbb{R}^N} \frac{v(t + \tau) - v(t)}{\tau} \eta(t)\psi(x) p(v(t))\, dxdt$$
$$= -\int_0^T \int_{\mathbb{R}^N} v(t) \frac{d}{dt} (\eta p(v))^\tau(t) \psi(x)\, dxdt$$
$$= \int_0^T \int_{\mathbb{R}^N} (\xi - g_t)(t) (\eta p(v))^\tau(t) \psi(x)\, dxdt. \tag{4.27}$$

Collecting these inequalities, we obtain

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^N} \frac{\eta(t-\tau)-\eta(t)}{-\tau} \psi J_p(v)\, dxdt \\
&\quad \le -\int_0^T \int_{\mathbb{R}^N} (\xi - g_t)(t)(\eta p(v))^\tau(t)\psi(x)\, dxdt \\
&\quad = -\lim_n \int_0^T \left\langle u_n'(t) - g_t, (\eta p(v))^\tau(t)\psi \right\rangle dt \\
&\quad = -\lim_n \int_0^T \frac{1}{\tau} \int_{t-\tau}^t \eta(s) \left\langle \operatorname{div}(z_n(t)) - g_t(t), p(v(s))\psi \right\rangle \, dsdt \\
&\quad = \lim_n \int_0^T \frac{1}{\tau} \int_{t-\tau}^t \eta(s) \left[\int_{\mathbb{R}^N} (z_n(t), D(p(v(s))\psi)) + \langle g_t, p(v(s))\psi \rangle \right] dsdt \\
&\quad = \lim_n \int_0^T \frac{1}{\tau} \int_{t-\tau}^t \eta(s) \int_{\mathbb{R}^N} (z_n(t), Dp(v(s)))\psi \, dsdt \\
&\qquad + \lim_n \int_0^T \frac{1}{\tau} \int_{t-\tau}^t \eta(s) \int_{\mathbb{R}^N} p(v(s)) z_n(t) \cdot \nabla\psi \, dsdt \\
&\qquad + \int_0^T \frac{1}{\tau} \int_{t-\tau}^t \eta(s) \langle g_t, p(v(s))\psi \rangle \, dsdt.
\end{aligned}
$$

Since

$$
Dp(v(s)) = \nabla p(u(s) - g(s)) + D^s p(u(s) - g(s))
$$

and

$$
\begin{aligned}
z_n(t) \cdot D^s p(u(s) - g(s)) &= \mathbf{a}(x, \nabla u_n(t, x)) \cdot D^s p(u(s) - g(s)) \\
&\le h^0(x, D^s p(u(s) - g(s))),
\end{aligned}
$$

from the above inequality, it follows that

$$
\begin{aligned}
&\int_0^T \int_{\mathbb{R}^N} \frac{\eta(t-\tau)-\eta(t)}{-\tau} \psi J_p(u-g)\, dxdt \\
&\quad \le \int_0^T \frac{1}{\tau} \int_{t-\tau}^t \eta(s) \int_{\mathbb{R}^N} z(t) \cdot \nabla p(u(s) - g(s))\psi \, dxdsdt \\
&\qquad + \int_0^T \frac{1}{\tau} \int_{t-\tau}^t \eta(s) \int_{\mathbb{R}^N} \psi h^0(x, D^s p(u(s) - g(s)))\, dsdt \\
&\qquad + \int_0^T \frac{1}{\tau} \int_{t-\tau}^t \eta(s) \int_{\mathbb{R}^N} p(u(s) - g(s)) z(t) \cdot \nabla\psi \, dxdsdt \\
&\qquad + \int_0^T \frac{1}{\tau} \int_{t-\tau}^t \eta(s) \int_{\mathbb{R}^N} g_t(t) p(u(s) - g(s))\psi(x)\, dxdsdt. \qquad (4.28)
\end{aligned}
$$

Hence, letting $\tau \to 0$ in (4.28), we obtain

$$\int_0^T \int_{\mathbb{R}^N} \phi_t J_p(u-g) \leq \int_0^T \int_{\mathbb{R}^N} \eta(t) z(t) \cdot \nabla p(u(t)-g(t)) \psi \, dxdt$$
$$+ \int_0^T \eta(t) \int_{\mathbb{R}^N} \psi h^0(x, D^s p(u(t)-g(t))) \, dt$$
$$+ \int_0^T \int_{\mathbb{R}^N} \eta(t) p(u(t)-g(t)) z(t) \cdot \nabla \psi \, dxdt$$
$$+ \int_0^T \int_{\mathbb{R}^N} \eta(t) g_t(t) p(u(t)-g(t)) \psi(x) \, dxdsdt. \tag{4.29}$$

Taking into account (4.24) and (4.29), we get

$$\int_0^T \int_{\mathbb{R}^N} \phi([z-\mathbf{a}(x,\nabla g)] \cdot \nabla p(u-g)$$
$$+ h^0(x, D^s p(u-g)) - \mathbf{a}(x,\nabla g) \cdot D^s p(u-g)) \geq 0$$

for all $\phi(t,x) = \eta(t)\psi(x)$, $\eta \in \mathcal{D}(0,T)$, $\psi \in \mathcal{D}(\mathbb{R}^N)$, $\eta, \psi \geq 0$. Thus, the measure

$$([z-\mathbf{a}(x,\nabla g)] \cdot \nabla p(u-g) + h^0(x, D^s p(u-g)) - \mathbf{a}(x,\nabla g) \cdot D^s p(u-g)) \geq 0.$$

Then its absolutely continuous part

$$[z-\mathbf{a}(x,\nabla g)] \cdot \nabla p(u-g) \geq 0 \quad \text{a.e. in } \mathbb{R}^N.$$

Hence

$$[z-\mathbf{a}(x,\nabla g)] \cdot \nabla (u-g) \geq 0 \quad \text{a.e. in } \mathbb{R}^N.$$

Since we may take a countable set dense in $C^1_{loc}([0,T] \times \mathbb{R}^N)$, we have that the above inequality holds for all $(t,x) \in S$, where $S \subseteq (0,T) \times \mathbb{R}^N$ is such that $\mathcal{L}^N(((0,T) \times \mathbb{R}^N) \setminus S) = 0$, and all $g \in C^1_{loc}([0,T] \times \mathbb{R}^N)$. Now, fixe $(t,x) \in S$, and given $y \in \mathbb{R}^N$ there is $g \in C^1_{loc}([0,T] \times \mathbb{R}^N)$ such that $\nabla g(t,x) = y$. Then

$$(z(t,x) - \mathbf{a}(x,y)) \cdot (\nabla u(t,x) - y) \geq 0 \quad \forall y \in \mathbb{R}^N,$$

and we get that

$$z(t,x) = \mathbf{a}(x, \nabla u(t,x)) \qquad \text{a.e.} \quad (t,x) \in Q_T. \tag{4.30}$$

Then, we have

$$\operatorname{div}(z(t)) = \operatorname{div}(\mathbf{a}(x, \nabla u(t))) \quad \text{in } \mathcal{D}'(\mathbb{R}^N), \quad \text{a.e. } t \in [0,T].$$

STEP 6. Conclusion. Finally, we are going to prove that u verifies:

$$-\int_0^T \int_{\mathbb{R}^N} j(u(t)-l)\eta_t \, dxdt + \int_0^T \int_{\mathbb{R}^N} \eta(t)h(x, Dp(u(t)-l))\, dt \\ + \int_0^T \int_{\mathbb{R}^N} z(t)\cdot\nabla\eta(t)p(u(t)-l)\, dxdt \le 0, \tag{4.31}$$

for all $\eta \in C^\infty(\overline{Q_T})$, with $\eta \ge 0$, $\eta(t,x) = \phi(t)\psi(x)$, being $\phi \in \mathcal{D}(]0,T[)$, $\psi \in C_0^\infty(\mathbb{R}^N)$, and $p \in \mathcal{T}$, where $j(r) = \int_0^r p(s)\, ds$.

Let $\eta \in C_0^\infty(Q_T)$, with $\eta \ge 0$, $\eta(t,x) = \phi(t)\psi(x)$, $\phi \in \mathcal{D}(]0,T[)$, $\psi \in C_0^\infty(\mathbb{R}^N)$, $p \in \mathcal{P}$ and $a \in \mathbb{R}$. Let $G_p(r) = \int_a^r p(s)\, ds$. Since $u_n'(t) = \operatorname{div}(z_n(t))$, multiplying by $p(u_n(t))\eta(t)$ and integrating, we obtain that

$$\int_0^T \int_{\mathbb{R}^N} \frac{d}{dt} G_p(u_n(t))\eta(t)\, dxdt = \int_0^T \int_{\mathbb{R}^N} p(u_n(t))u_n'(t)\eta(t)\, dxdt \\ = \int_0^T \int_{\mathbb{R}^N} \operatorname{div}(z_n(t))p(u_n(t))\eta(t)\, dxdt \\ = -\int_0^T \int_{\mathbb{R}^N} (z_n(t), D(p(u_n(t))\eta(t)))\ dt \\ = -\int_0^T \int_{\mathbb{R}^N} \eta(t)h(x, Dp(u_n(t)))\, dt - \int_0^T \int_{\mathbb{R}^N} z_n(t)\cdot\nabla\eta(t)\ p(u_n(t))\, dxdt.$$

Hence, having in mind that $\eta(0) = \eta(T) = 0$, we get

$$\int_0^T \int_{\mathbb{R}^N} \eta(t)h(x, Dp(u_n(t)))\, dt = -\int_0^T \int_{\mathbb{R}^N} z_n(t)\cdot\nabla\eta(t)\ p(u_n(t))\, dxdt \\ -\int_0^T \int_{\mathbb{R}^N} \frac{d}{dt} G_p(u_n(t))\eta(t)\, dxdt = -\int_0^T \int_{\mathbb{R}^N} z_n(t)\cdot\nabla\eta(t)\ p(u_n(t))\, dxdt \\ + \int_0^T \int_{\mathbb{R}^N} G_p(u_n(t))\ \eta_t\, dxdt.$$

Now, observe that, by the Lemma 4.5, we have that

$$\int_{\mathbb{R}^N} \eta(t,x)f(x, Dp(u_n)) \to \int_{\mathbb{R}^N} \eta(t,x)f(x, Dp(u))$$

a.e. in $t \in (0,T)$, and, therefore,

$$\int_{\mathbb{R}^N} \eta(t,x)h(x, Dp(u_n)) \to \int_{\mathbb{R}^N} \eta(t,x)h(x, Dp(u)),$$

a.e. in $t \in (0, T)$. Hence, integrating in $(0, T)$ and using Fatou's Lemma, it follows that

$$\int_0^T \int_{\mathbb{R}^N} \eta(t) h(x, Dp(u(t)))\, dt \leq \lim_{n \to \infty} \int_0^T \int_{\mathbb{R}^N} \eta(t) h(x, Dp(u_n(t)))\, dt$$

$$= \lim_{n\to\infty} \left[- \int_0^T \int_{\mathbb{R}^N} z_n(t) \cdot \nabla \eta(t)\ p(u_n(t))\, dxdt + \int_0^T \int_{\mathbb{R}^N} G_p(u_n(t))\ \eta_t\, dxdt \right]$$

$$= - \int_0^T \int_{\mathbb{R}^N} z(t) \cdot \nabla\eta(t)\ p(u(t))\, dxdt + \int_0^T \int_{\mathbb{R}^N} G_p(u(t))\ \eta_t\, dxdt.$$

We have

$$- \int_0^T \int_{\mathbb{R}^N} G_p(u(t))\eta_t\, dxdt + \int_0^T \int_{\mathbb{R}^N} \eta(t) h(x, Dp(u(t)))\, dt + \int_0^T \int_{\mathbb{R}^N} z(t) \cdot \nabla\eta(t)\ p(u(t))\, dxdt \leq 0. \tag{4.32}$$

Finally, given $l \in \mathbb{R}$ and $p \in \mathcal{T}$, since $q(r) := p(r - l)$ is an element of $\mathcal{P}$, and taking $a = l$, we obtain (4.31) as a consequence of (4.32). The proof of the existence is finished.

5. Proof of Theorem 1: Uniqueness

Uniqueness of entropy solutions can be proved using the same technique used to prove uniqueness in the case of a bounded domain (see [2], [3]), a technique inspired by the doubling variables method introduced by Kruzhkov [17] (see also [10]) to prove the L^1-contraction estimate for entropy solutions for scalar conservation laws. This technique has been also applied in [8] to prove uniqueness of entropy solutions of the Total Variation flow in $\mathbb{R}^N$. Since the methods are similar to the above mentioned works, we shall not give the details here.

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