Proceedings of the Fall Workshop on
Differential Geometry and its Applications
Barcelona '93
pp. 9–14

On the Batchelor trivialization of the tangent supermanifold

Juan Monterde
Departament de Geometria i Topologia,
Universitat de València,
Burjassot

O. A. Sánchez-Valenzuela
Centro de Investigación en Matemáticas,
Guanajuato,
México

1 Introduction

By Batchelor's theorem (also proved by K. Gawedzki [Ga]), any smooth graded manifold $(M, \mathcal{A}_M)$ is isomorphic (although not canonically) to $(M, \Lambda\mathcal{E})$, where $\Lambda\mathcal{E}$ is the sheaf of sections of the exterior algebra bundle $\Lambda E \to M$ of a smooth vector bundle $E \to M$ defined by $\mathcal{A}_M$. $(M, \Lambda\mathcal{E})$ is then called *the Batchelor trivialization of* $(M, \mathcal{A}_M)$.

Our aim in this note is to obtain the Batchelor trivialization of the tangent supermanifold $\mathcal{S}\mathcal{T}(M, \mathcal{A}_M)$ of $(M, \mathcal{A}_M)$ in terms of the initial data M and E, given the fact that its corresponding structure sheaf is $Der\Lambda\mathcal{E}$. We show that the underlying smooth manifold of $\mathcal{S}\mathcal{T}(M, \mathcal{A}_M)$ is not TM but $TM \oplus E^*$. This reflects the intrinsic property that the fermionic part of a graded manifold produces a new (nontrivial and non-expected) bosonic part in its tangent graded manifold. Furthermore, we completely describe the Batchelor bundle as the pullback to $TM \oplus E^*$ of the Whitney sum $T^*M \oplus E \oplus E$ (*cf.* Theorem 2 below). In particular $\dim\mathcal{S}\mathcal{T}(M, \mathcal{A}_M) = (2\dim M + \mathrm{rk}\, E, \dim M + 2\mathrm{rk}\, E)$.

e-mail: `monterde@vm.ci.uv.es`, `saval@redvax1.dgsca.unam.mx`

2 Characterization of the derivations

Let $E \to M$ be a rank-n vector bundle over M, and let $\mathcal{E} = \Gamma(E)$ be its sheaf of sections. We shall also write $\mathcal{E}^*$, and $\Lambda\mathcal{E}$, for the sheaves $\Gamma(E^*)$, and $\Gamma(\Lambda E)$, respectively. Finally, we shall write $\mathcal{X}_M$ for the sheaf of sections of the tangent bundle to M. For this part we shall follow the ideas of [MoMo], [Ro1], and [Ro3] (We shall refer the reader to [Ko] for definitions of graded manifolds and all the related topics except the concept of tangent supermanifold; for the latter we refer to [SV]).

The sheaf of derivations $\mathcal{D}er\Lambda\mathcal{E}$ is a locally free $\Lambda\mathcal{E}$-module (*cf.* [Ko]). Moreover, there is a natural inclusion,

$$0 \to \Lambda\mathcal{E} \otimes \mathcal{E}^* \to \mathcal{D}er\Lambda E, \tag{1}$$

defined by letting the elements of $\mathcal{E}^*$ act on $\Lambda\mathcal{E}$ by contraction. There is also a projection of $\Lambda\mathcal{E}$-modules,

$$\mathcal{D}er\Lambda E \to \Lambda\mathcal{E} \otimes \mathcal{X}_M \to 0, \tag{2}$$

given on homogeneous elements as follows: Let $\mathcal{D}er^k\Lambda\mathcal{E}$ be the sheaf of those sections of $\mathcal{D}er\Lambda\mathcal{E}$ that increase the degree by k. Let $X \in \mathcal{D}er^k\Lambda\mathcal{E}$, and let $f \in \Lambda^0\mathcal{E} \simeq C^\infty(M)$. Then, $X f \in \Lambda^k\mathcal{E}$, and for any k-tuple of sections of $\mathcal{E}^*$, $(\varphi_1, \ldots, \varphi_k)$, the mapping,

$$f \mapsto i(\varphi_k) \circ \cdots \circ i(\varphi_1)(X f), \tag{3}$$

defines a derivation of $C^\infty(M)$. Denote by $\hat{X}(\varphi_1, \ldots, \varphi_k)$ this derivation. It is easy to check that the map $(\varphi_1, \ldots, \varphi_k) \mapsto \hat{X}(\varphi_1, \ldots, \varphi_k)$ is C^∞-linear, and alternating; it therefore defines a section of $\Lambda^k\mathcal{E} \otimes \mathcal{X}_M$. The maps (1), and (2), fit together into an exact sequence,

$$0 \to \Lambda\mathcal{E} \otimes \mathcal{E}^* \to \mathcal{D}er\Lambda\mathcal{E} \to \Lambda\mathcal{E} \otimes \mathcal{X}_M \to 0. \tag{4}$$

When a connection ∇ in the bundle E is given, this sequence splits and therefore,

$$\mathcal{D}er\Lambda\mathcal{E} \simeq \Lambda\mathcal{E} \otimes (\mathcal{X}_M \oplus \mathcal{E}^*). \tag{5}$$

In this description one manifestly reads the fact that $\mathcal{D}er\Lambda\mathcal{E}$ is a $\Lambda\mathcal{E}$-module of rank (m, n). Note that the structure of the supercotangent sheaf can be deduced from (5):

$$(\mathcal{D}er\Lambda\mathcal{E})^* = \mathcal{H}om(\mathcal{D}er\Lambda\mathcal{E}, \Lambda\mathcal{E}) \simeq \Lambda\mathcal{E} \otimes (\Omega^1_M \oplus \mathcal{E}), \tag{6}$$

where, Ω^1_M denotes the sheaf of sections of the cotangent bundle to M.

3 The tangent supermanifold

We shall now use the structures found in (5), and (6) to produce two supermanifolds—the supertangent, and supercotangent manifolds to $(M, \Lambda\mathcal{E})$, respectively—and two submersions—one from each of these supermanifolds onto $(M, \Lambda\mathcal{E})$—in such a way that the sheaf-theoretic sections of $\mathcal{D}er\Lambda\mathcal{E}$, and $(\mathcal{D}er\Lambda\mathcal{E})^*$ correspond in a one-to-one fashion with the geometric sections of these submersions. Thus,

$$\begin{aligned} \mathcal{D}er\Lambda\mathcal{E} &\leftrightarrow \Gamma\big((M, \Lambda\mathcal{E}), (STM, \Lambda\mathcal{A})\big) \\ (\mathcal{D}er\Lambda\mathcal{E})^* &\leftrightarrow \Gamma\big((M, \Lambda\mathcal{E}), (ST^*M, \Lambda\mathcal{B})\big). \end{aligned} \tag{7}$$

In order to determine these supervector bundles we shall take into account the following (*cf.* [SV]):

1. Supervector bundles over $(M, \Lambda\mathcal{E})$ correspond functorially to locally free sheaves of $\Lambda\mathcal{E}$-modules over M, and this functor commutes with $\mathcal{H}om$, $\otimes$, and $\times$.
2. There is a universal object in the category of supermanifolds, $\mathcal{R}^{1|1}$, such that
$$\Lambda\mathcal{E} \leftrightarrow \mathcal{M}aps\big((M, \Lambda\mathcal{E}), \mathcal{R}^{1|1}\big),$$
3. Supervector bundles are locally products of the base with a fiber; the latter being isomorphic to a fixed supermanifold.

Now, the determination of the underlying smooth manifolds STM, and ST^*M follows from general principles: each supermanifold $(M, \Lambda\mathcal{E})$ comes equipped with a sheaf epimorphism, $\Lambda\mathcal{E} \to \mathcal{C}^\infty_M$ and hence, with an exact sequence,

$$0 \to \mathcal{N} \to \Lambda\mathcal{E} \to \mathcal{C}^\infty_M \to 0, \tag{8}$$

where $\mathcal{N}$ denotes the nilpotent ideal of $\Lambda\mathcal{E}$. The sheaf $\mathcal{E}$ of sections of the Batchelor bundle E can be recovered from this sequence by looking at $\mathcal{C}^\infty \supset \mathcal{N} \supset \mathcal{N}^2 \cdots$, and observing that $\mathcal{E} \simeq \mathcal{N}/\mathcal{N}^2$. This has the structure of the odd part of the supercotangent sheaf since $\mathcal{N}$ is contained in the maximal ideal of vanishing superfunctions (*cf.* [Ro2]).

The canonical epimorphism $\Lambda\mathcal{E} \to \mathcal{C}^\infty_M$ can be used to also define a functor from the category of locally free $\Lambda\mathcal{E}$-modules, into the category of locally free $\mathcal{C}^\infty_M$-modules over M; namely, any locally free $\Lambda\mathcal{E}$-module, $\mathcal{M}$, gives rise to the locally free $\mathcal{C}^\infty_M$-module, $\mathcal{M}/(\mathcal{N}\,\mathcal{M})$. For the supertangent, and the supercotangent sheaves, this functor produces,

$$\begin{aligned} \mathcal{D}er\Lambda\mathcal{E} &\mapsto \mathcal{D}er\Lambda\mathcal{E}/(\mathcal{N}\,\mathcal{D}er\Lambda\mathcal{E}) = \mathcal{X}_M \oplus \mathcal{E}^* \\ (\mathcal{D}er\Lambda\mathcal{E})^* &\mapsto (\mathcal{D}er\Lambda\mathcal{E})^*/(\mathcal{N}\,(\mathcal{D}er\Lambda\mathcal{E})^*) = \Omega^1_M \oplus \mathcal{E} \end{aligned} \tag{9}$$

Lemma *These isomorphisms are independent of the connection used to split the sequence* (4)

In particular, the underlying manifolds to the supertangent and to the supercotangent spaces to $(M, \Lambda\mathcal{E})$, are respectively given by,

$$STM = TM \oplus E^*, \qquad \text{and,} \qquad ST^*M = T^*M \oplus E, \tag{10}$$

which are the ordinary Whitney sums of the given smooth vector bundles over M.

To understand the structure of the Batchelor bundles A, and B, of $(STM, \Lambda\mathcal{A})$, and $(ST^*M, \Lambda\mathcal{B})$ we refer ourselves to Proposition 2.9 of [SV]. If we apply the general results there obtained to the supertangent and supercotangent sheaves, we obtain:

Theorem *Let $(M, \Lambda\mathcal{E})$ be a graded manifold of graded dimension (m, n). The Batchelor trivializations of the tangent and cotangent supermanifolds are*

$$\begin{aligned}(STM, \Lambda\mathcal{A}) &= \left(TM \oplus E^*, \Lambda\widetilde{\pi}^*(\Omega^1_M \oplus \mathcal{E} \oplus \mathcal{E})\right)\\ (ST^*M, \Lambda\mathcal{B}) &= \left(T^*M \oplus E, \Lambda\widetilde{\pi}^*(\mathcal{X}_M \oplus \mathcal{E}^* \oplus \mathcal{E})\right).\end{aligned} \tag{12}$$

Note: The graded dimension of the tangent supermanifold is $(2m + n, 2n + m)$.

Brief review of the argument Let $(M, \Lambda\mathcal{E})$ be a supermanifold. Let $F_0 \to M$, and $F_1 \to M$ be two smooth vector bundles of finite rank over M (say, p, and q, respectively), and let $\mathcal{F}_0$, and $\mathcal{F}_1$ be their corresponding sheaves of smooth sections. Let $\mathcal{M}$ be a locally free sheaf of $\Lambda\mathcal{E}$-modules over M, and assume it has the following structure:

$$\mathcal{M} \simeq \Lambda\mathcal{E} \otimes (\mathcal{F}_0 \oplus \mathcal{F}_1), \tag{13}$$

so that $\mathcal{M}$ has $\mathbb{Z}_2$-rank (p, q). Then, there is a canonical isomorphism,

$$\mathcal{M} \simeq \mathcal{H}om\left((\mathcal{F}_0 \oplus \mathcal{F}_1)^*, \Lambda\mathcal{E}\right). \tag{14}$$

In particular, each section of the sheaf $\mathcal{H}om\left((\mathcal{F}_0 \oplus \mathcal{F}_1)^*, \Lambda\mathcal{E}\right)$ extends uniquely to a section,

$$\mathbb{Z}_2\text{-}\mathcal{A}lg\left(\Lambda(\mathcal{F}_0 \oplus \mathcal{F}_1)^*, \Lambda\mathcal{E}\right), \tag{15}$$

of $\mathbb{Z}_2$-graded algebra homomorphisms between sheaves of $\mathbb{Z}_2$-graded algebras. The sections of the latter, in turn, are in one-to-one correspondence with maps from the base supermanifold $(M, \Lambda\mathcal{E})$, into a supermanifold whose structure sheaf is $\Lambda(\mathcal{F}_0 \oplus \mathcal{F}_1)^*$. The claim is that these are precisely the local geometric sections of the supervector bundle: maps from the base into the *superfiber*.

In fact, if $\mathcal{M}$ is to give rise to a supermanifold $(F, \Lambda\mathcal{F})$, equiped with a supermanifold epimorphism $\pi: (F, \Lambda\mathcal{F}) \to (M, \Lambda\mathcal{E})$, in such a way that the geometric sections (*i.e.*, maps $\sigma: (M, \Lambda\mathcal{E}) \to (F, \Lambda\mathcal{F})$ such that $\pi \circ \sigma = id$) correspond to the sheaf theoretic sections of $\mathcal{M}$, then, there must be a canonical embedding $\Lambda\mathcal{E} \to \Lambda\mathcal{F}$ that defines π. This must be so, since each section σ gives rise to a superalgebra epimorphism, $\sigma^*: \Lambda\mathcal{F} \to \Lambda\mathcal{E}$, such that, $\sigma^* \circ \pi^* = id^*$. In other words, $\Lambda\mathcal{E}$ must be a canonical summand —and in fact, a subalgebra— of $\Lambda\mathcal{F}$. Therefore,

$$\Lambda\mathcal{F} \simeq \Lambda(\cdots \oplus \mathcal{E}). \tag{16}$$

This yields the global result of the assertion that the supermanifold $(F, \Lambda\mathcal{F})$ must be locally trivial; *i.e.*, locally the product, $(M, \Lambda\mathcal{E}) \times (V, \Lambda\mathcal{V})$, of the base with the superfiber $(V, \Lambda\mathcal{V})$. In this situation,

$$\Gamma\big((M, \Lambda\mathcal{E}), (F, \Lambda\mathcal{F})\big) \leftrightarrow \mathcal{M}aps\big((M, \Lambda\mathcal{E}), (V, \Lambda\mathcal{V})\big) \leftrightarrow \mathbb{Z}_2\text{-}\mathcal{A}lg\,(\Lambda\mathcal{V}, \Lambda\mathcal{E}). \tag{17}$$

This result, together with (15), completes the picture given by (16); namely,

$$\Lambda\mathcal{F} \simeq \Lambda\big((\mathcal{F}_0 \oplus \mathcal{F}_1)^* \oplus \mathcal{E}\big). \tag{18}$$

The only technical point is that the Whitney sum of the bundles F_0^*, F_1^*, and E now occurs over the underlying total space of the supervector bundle; *i.e.*, over $F = F_0 \oplus F_1$. This is done by taking the pullback of such bundles along $\widetilde{\pi}: F_0 \oplus F_1 \to M$.

Corollary *Let $\Omega(M) = \Gamma(\Lambda T^* M)$ be the Cartan algebra of differentiable forms on a smooth manifold M. The tangent supermanifold of the graded manifold $(M, \Omega(M))$ is*

$$\big(TM \oplus TM, \Lambda\widetilde{\pi}^*(\Omega^1_M \oplus \Omega^1_M \oplus \Omega^1_M)\big).$$

References

[Ba] BATCHELOR, M., "The Structure of Supermanifolds", *Trans. Amer. Math. Soc.* **253** (1979) 329–338.

[Ga] GAWEDZKI, K., "Supersymmetries—Mathematics of supergeometry", *Ann. Inst. H. Poincaré Sect. A* **27** (1977) 335–366.

[Ko] KOSTANT, B., "Graded Manifolds, Graded Lie Theory and Prequantization", *Lecture Notes in Math.* vol. 570, (Bleuler, K. and

Reetz, A., eds.), Proc. Conf. on Diff. Geom. Methods in Math. Phys., Bonn 1975, Springer Verlag, Berlin and New York, 1977, pp. 177–306.

[MoMo] MONTERDE, J. AND MONTESINOS, A., "Integral Curves of Derivations", *Annals of Global Analysis and Geometry* **6 no. 2** (1988) 177–189.

[Ro1] ROTHSTEIN, M., "Deformations of complex supermanifolds", *Proc. Amer. Math. Soc.* **95 no. 2** (1985) 255–260.

[Ro2] ROTHSTEIN, M., "The axioms of supermanifolds and a new structure arising from them", *Trans. Amer. Math. Soc.* **299** (1987) 387–396.

[Ro3] ROTHSTEIN, M., "The Structure of Supersymplectic Supermanifolds", *Lecture Notes in Physics* vol. 375, (Bartozzi, Bruzzo and Cianci, eds.), Proc. XIX Int. Conf. on Diff. Geom. Methods in Math. Phys., Rapallo Italy 1990, Springer Verlag, Berlin and New York, 1991, pp. 331–343.

[SV] SÁNCHEZ-VALENZUELA, O. A., "On Supergeometric Structures", *Ph.D. thesis, Harvard University–Cambridge* (1986);
"On Supervector Bundles", *Comunicaciones Técnicas IIMAS-UNAM (Serie Naranja)* **457** (1986).