Two C^1-Methods to Generate Bézier Surfaces from the Boundary

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Abstract

Two methods to generate tensor-product Bézier surface patches from their boundary curves and with tangent conditions along them are presented. The first one is based on the tetraharmonic equation: we show the existence and uniqueness of the solution of $\Delta^4\overrightarrow{\mathbf{x}} = 0$ with prescribed boundary and adjacent to the boundary control points of a $n \times n$ Bézier surface. The second one is based on the nonhomogeneous biharmonic equation $\Delta^2\overrightarrow{\mathbf{x}} = p$, where p could be understood as a vectorial load adapted to the C^1-boundary conditions.

Key words: Tetraharmonic surfaces, bilaplacian operator, biharmonic surfaces, nonhomogeneous biharmonic equation, fictitious load, thin plate equation, PDE freeform surfaces

1 Introduction

The aim of this work is to develop C^1-boundary based intuitive surface design techniques for Bézier surfaces which are one of the basic types of surfaces widely used in CAGD. The main idea is to find polynomial solutions to some natural PDEs which can only be controlled through the boundary control points and those adjacent to them. The two PDEs used are based on the Laplacian operator, so the solutions can be seen as extremals of the corresponding energy functionals.

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According to previous results in [7], given two opposed boundaries of a Bézier surface there is a unique harmonic surface ($\Delta \overrightarrow{\mathbf{x}} = 0$) with that prescribed partial boundary. And given the four opposed boundaries of a Bézier surface there is a unique biharmonic surface ($\Delta^2 \overrightarrow{\mathbf{x}} = 0$) with that prescribed boundary (see [8] and [4]).

This last result allows us to give a simple method to generate a Bézier surface from its boundary. In this paper we study a similar problem, when not only the boundaries are prescribed but also the derivatives along them.

The idea behind the first method is just, after having seen what happens with harmonic and biharmonic surfaces, to increase the order of the partial differential operator. It is reasonable to think that if the harmonic condition $\Delta \overrightarrow{\mathbf{x}} = 0$ completely determines the Bézier surface $\overrightarrow{\mathbf{x}}$ from just two opposed boundaries, and if the biharmonic condition $\Delta^2 \overrightarrow{\mathbf{x}} = 0$ completely determines the Bézier surface from the whole boundary, then the tetraharmonic condition $\Delta^4 \overrightarrow{\mathbf{x}} = 0$ should completely determine the Bézier surface from the boundaries and the derivatives along the boundaries.

In Section 8 it is proved that given the boundary control points of a Bézier surface and those adjacent to them in the control net, then there exists a unique tetraharmonic polynomial surface satisfying such boundary conditions. Moreover, an explicit algorithm to compute the solution is given. These C^1 conditions on the boundary enable us to control the shape of the surface near these boundary, which can be very useful in a variety of different situations such as engineering or even virtual design.

The second method is based on a modification of the biharmonic condition. If just with the four boundary curves, a unique biharmonic Bézier surface is determined, then, in order to manage with the conditions related to the derivatives along the boundaries, one has to introduce more degrees of freedom. One possibility is to substitute the homogeneous biharmonic equation $\Delta^2 \overrightarrow{\mathbf{x}} = 0$ by the nonhomogeneous one $\Delta^2 \overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{y}}$.

The scalar nonhomogeneous biharmonic equation $\Delta^2 f = p$ describes the deflection of $f(u, v)$ of the middle surface of an elastic isotropic flat plate of uniform thickness and where $p(u, v)$ is the load per unit area, the coordinates u, v being taken in the plane $z = 0$ of the middle surface of the plate before bending. See [5] for a complete study of the biharmonic problem in a rectangle. The homogeneous biharmonic equation can be understood as a thin plate problem without load.

The new degrees of freedom we need will be under the form of an ad hoc vectorial load, which we will choose mainly concentrated along the boundaries. Intuitively, the reasoning could be the following: the prescription of the boundaries completely determines a biharmonic Bézier surface, $\overrightarrow{\mathbf{x}}_0$, which, in general, would not satisfy the conditions related with the derivatives along the boundaries. Therefore, the introduction of a load, mainly concentrated along the boundaries, acting on $\vec{\mathbf{x}}_0$ would bend the surface up to verifying the derivative conditions.

The use of *thin plate* methods in CAGD is well known from the very starting days of the subject. For example, one has to recall the notion of TP splines (see [3]). In this work such guiding principles are particularized to polynomial solutions. The idea of adding a new term to the load in the nonhomogeneous biharmonic equation to obtain polynomial approximations is not new either. It dates back to Biezeno and Koch (see [6]) and it can be said that the new load added by these two authors was also a polynomial load. [1]

Moreover, in the recent paper [2] the authors make use of polynomials solutions of the biharmonic equation as a first approximation to the true solution when the boundaries are not necessarily polynomial.

In section 6 we compare our results with those in [2] which are also compared to Timoshenko's results. The major difference is that Bloor and Wilson's final approximation has a non-polynomial term added while our final solution is polynomial, which is useful for computational purposes.

Let us say too that both methods are related by the following argument: any solution of the tetrahamonic equation $\Delta^4\vec{\mathbf{x}} = 0$ can be seen as a solution of the nonhomogeneous biharmonic equation $\Delta^2\vec{\mathbf{x}} = p$ with p a biharmonic load, this is, with p such that $\Delta^2 p = 0$. Therefore, in the first method we look for solutions of the nonhomogeneous biharmonic equation with a biharmonic load whereas in the second method we look for solutions with a load mainly concentrated along the boundary.

For the sake of clarity all the proofs of the main theorems as well as all the lemmas needed for these proofs have been put at the end of the paper in Annex A and Annex B.

Finally the authors would like to thank the referees for all their useful comments which have helped to make the paper easier to follow and understand.

[1] The new term added to the load has been called in [6] *fictitious load*. In our situation, where there is not a real load from the beginning, we will not use the adjective fictitious.

2 Background on biharmonic Bézier surfaces

The usual statement of the biharmonic problem as can be seen in [5] involves the prescription of the boundaries and of the normal derivatives along the boundaries. The surprising fact is that if we are looking for polynomial solutions of the homogeneous biharmonic equation, then we only need to prescribe a polynomial boundary in order to uniquely determine a solution.

Proposition 1 *Let $\overrightarrow{\mathbf{x}}_n(u,v)=\sum_{k,\ell=0}^{n}B_k^n(u)B_\ell^n(v)P_{k\ell}$ be a biharmonic Bézier chart of degree n with control net $\{P_{k\ell}\}_{k,\ell=0}^{n}$. Then all the inner control points $\{P_{k\ell}\}_{k=1,\ell=1}^{n-1}$ are determined by the boundary control points, $\{P_{0\ell}\}_{\ell=0}^{n}$, $\{P_{n\ell}\}_{\ell=0}^{n}$, $\{P_{k0}\}_{k=0}^{n}$ and $\{P_{kn}\}_{k=0}^{n}$.*

Remark 1 *The boundary control points are*

$$\begin{array}{cccccc}
P_{00} & P_{01} & P_{02} & \dots & P_{0n-1} & P_{0n}\\
P_{10} & * & * & \dots & * & P_{1n}\\
P_{20} & * & * & \dots & * & P_{2n}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
P_{n-1,0} & * & * & \dots & * & P_{n-1,n}\\
P_{n0} & P_{n1} & P_{n2} & \dots & P_{n,n-1} & P_{nn}
\end{array}$$

The proof of Prop. 1 can be seen in [8], moreover, in [9] there is the detailed algorithm which allows to compute, given the boundary, the unique polynomial solution of any 4-th order linear PDE, including the biharmonic equation as a particular case. In [4] additional conditions for the existence in the rectangular case are stated.

3 Tetraharmonic Bézier surfaces

Before stating the main result of the first method, let us do a simple computation of the dimension of the vector space of polynomial solutions of the tetraharmonic equation for a given degree.

Recently some authors have considered polynomial solutions of the biharmonic equation as a part of a method to compute approximate solutions (see [2]). Let us recall their arguments for the biharmonic case.

Putting $z=u+\mathbf{i}v$ and $\bar{z}=u-\mathbf{i}v$, then the usual Laplacian operator can be written as

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} = \frac{1}{2}\frac{\partial^2}{\partial z \partial \bar{z}},$$

In general, for a certain degree n, the complex polynomials

$$z^k, \quad 0 \leq k \leq n,$$

$$\bar{z}^k, \quad 1 \leq k \leq n$$

are a basis of complex polynomials solutions of the harmonic equation. They are in total $(n+1)+n = 2n+1$. But when switching to real polynomials, the real and imaginary parts of $z^k, \quad 0 \leq k \leq n$, provide us with $2n+1$ linear independent harmonic polynomials of degree $\leq n$. When n is even there are no more linear independent harmonic polynomials of the same degree, but when n is odd there is still one more, the imaginary part of z^{n+1}.

This agrees with the well-known result about harmonic Bézier patches, including the different cases depending on the parity of n.

The homogeneous biharmonic equation can be written as

$$\Delta^2 f = \frac{1}{4}\frac{\partial^4 f}{\partial z^2 \partial \bar{z}^2} = 0. \tag{1}$$

In general, for a certain degree n (we are considering not the total degree n but the maximum between the degree of z and the degree of $\bar{z}$) the complex polynomials

$$z^k, \bar{z}z^k, \quad 0 \leq k \leq n,$$

$$\bar{z}^k, z\bar{z}^k \quad 2 \leq k \leq n$$

are a basis of complex polynomial solutions of the biharmonic equation. They are in total $2(n+1)+2(n-1) = 4n$. Therefore the dimension of the vectorial space of degree n polynomial solutions of the biharmonic equation is $4n$. This dimension agrees with the number of boundary control points of a degree n Bézier patch.

For real biharmonic polynomials the situation is the following: The real and imaginary parts of $z^k, \quad 0 \leq k \leq n$ and of $\bar{z}z^k, \quad 1 \leq k \leq n-1$, provide us with $4n-2$ linear independent polynomials of degree $\leq n$. We need two more:

If n is odd, $\mathrm{Im} z^{n+1}$ and $\mathrm{Im} \bar{z} z^n$.

If n is even, $\mathrm{Re}\left((z-\bar{z})z^n\right)$ and $\mathrm{Im}\left((z+\bar{z})z^n\right)$.

For the tetraharmonic equation,

$$\Delta^4 f = \frac{1}{16}\frac{\partial^8 f}{\partial z^4 \partial \bar{z}^4} = 0, \qquad (2)$$

the complex polynomials

$$z^k, \bar{z}z^k, \bar{z}^2 z^k, \bar{z}^3 z^k, \quad 0 \leq k \leq n,$$

$$\bar{z}^k, z\bar{z}^k, z^2\bar{z}^k, z^3\bar{z}^k, \quad 4 \leq k \leq n,$$

are a basis of complex polynomial solutions of the tetraharmonic equation. They are in total $4(n+1)+4(n-3) = 8n-8$. Therefore the dimension of the vectorial space of degree n polynomial solutions of the tetraharmonic equation is $8n-8$. This dimension agrees with the number of boundary and adjacent to the boundary control points of a degree n Bézier patch.

For real tetraharmonic polynomials it is possible to find a polynomial basis with the same number of elements given by the real and imaginary parts of

$$\begin{aligned} z^k, &\quad 0 \leq k \leq n, \\ \bar{z}z^k, &\quad 1 \leq k \leq n-1, \\ \bar{z}^2 z^k, &\quad 2 \leq k \leq n-2, \\ \bar{z}^3 z^k, &\quad 3 \leq k \leq n-3, \end{aligned}$$

and by the linear combinations of the other complex polynomials as for the biharmonic case.

The use of the complex variable z has been useful to compute the dimension of these spaces of polynomial solutions, but it is not well adapted to the problem of finding solutions with a prescribed boundary.

The first method is based on the next result. We will prove that the tetraharmonic condition implies that given the boundary control points of a tetraharmonic Bézier surface, and those adjacent to them, we can express the rest of the points in the control net as linear combinations of them.

Theorem 1 *Given the boundary control points and those adjacent to them of an $n \times n$ net, (i.e. $\{P_{0j}\}_{j=0}^{n}$, $\{P_{1j}\}_{j=1}^{n-1}$, $\{P_{n-1,j}\}_{j=1}^{n-1}$, $\{P_{nj}\}_{j=0}^{n}$, $\{P_{i0}\}_{i=0}^{n}$, $\{P_{i1}\}_{i=1}^{n-1}$, $\{P_{i,n-1}\}_{i=1}^{n-1}$ and $\{P_{in}\}_{i=0}^{n}$), there exists a unique tetraharmonic Bézier surface whose chart expressed in the Bernstein basis is*
$\overrightarrow{\mathbf{x}}(u,v) = \sum_{i,j=0}^{n} B_i^n(u)B_j^n(v)P_{ij}(v)$ (and whose control net has those points as boundary control points and those adjacent to them).

Remark 2 *The boundary control points and those adjacent to them are*

$$\begin{array}{ccccccc}
P_{00} & P_{01} & P_{02} & \cdots & P_{0,n-2} & P_{0,n-1} & P_{0n} \\
P_{10} & P_{11} & P_{12} & \cdots & P_{1,n-2} & P_{1,n-1} & P_{1n} \\
P_{20} & P_{21} & * & \cdots & * & P_{2,n-1} & P_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
P_{n-2,0} & P_{n-2,1} & * & \cdots & * & P_{n-2,n-1} & P_{n-2,n} \\
P_{n-1,0} & P_{n-1,1} & P_{n-1,2} & \cdots & P_{n-1,n-2} & P_{n-1,n-1} & P_{n-1,n} \\
P_{n0} & P_{n1} & P_{n2} & \cdots & P_{n,n-2} & P_{n,n-1} & P_{nn}
\end{array}$$

4 The method based on the nonhomogeneous biharmonic equation

Using the previous method, we have obtained solutions of the tetraharmonic equation $\Delta^4 \overrightarrow{\mathbf{x}} = 0$. Such eighth-order PDE is equivalent to the system of fourth-order PDEs

$$\Delta^2 \overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{y}}, \qquad \Delta^2 \overrightarrow{\mathbf{y}} = 0.$$

According to the terminology of biharmonic problems (see [6]), the system can be understood as a nonhomogeneous biharmonic problem with a vectorial load, $\overrightarrow{\mathbf{y}}$, which is biharmonic.

The basic idea of the second method is to obtain solutions of the nonhomogeneous biharmonic equation with a vectorial load satisfying, not a biharmonic condition as before, but another geometric condition related to Bézier statements.

First of all, since the solutions of the nonhomogeneous biharmonic equation we are looking for are polynomial, then the load must be of the same nature. So the first thing we have to do is to find a way, adapted to our aims, to characterize which polynomial functions can act as a load. This is what we develop along this section.

Let us suppose that

$$\overrightarrow{\mathbf{x}}(u,v) = \sum_{i,j=0}^{n} B_i^n(u) B_j^n(v) P_{ij} = \sum_{i,j=0}^{n} \frac{a_{ij}}{i!j!} u^i v^j, \tag{3}$$

where $a_{ij} \in \mathbb{R}^3$.

The nonhomogeneous biharmonic equation $\Delta^2 \overrightarrow{\mathbf{x}} = p$, where $p(u,v) = \sum_{i,j=0}^{n} \frac{p_{ij}}{i!j!} u^i v^j$, can be translated in terms of the coefficients a_{ij}, as follows

$$a_{k+4,\ell} + 2a_{k+2,\ell+2} + a_{k,\ell+4} = p_{k,\ell}, \tag{4}$$

for $0 \leq k, \ell \leq n$ (where $a_{k,\ell} = 0$ if $k > n$ or $\ell > n$).

We are initially interested in finding polynomial solutions of the nonhomogeneous biharmonic equation with a prescribed boundary (but not with prescribed tangent planes along the boundary).

Let $\mathbb{R}_n[u, v]$ denote the vector space of real polynomials in the variables u, v and of degree less or equal to n.

First, we need to characterize which polynomials $p \in \mathbb{R}_n[u, v]$ are in the image of the biLaplacian operator when it is restricted to the vector space $\mathbb{R}_n[u, v]$.

Definition 1 *A polynomial $p \in \mathbb{R}_n[u, v]$ will be called* admissible *if it belongs to* $\mathrm{Im}\Delta^2|_{\mathbb{R}_n[u,v]}$.

Sometimes, by analogy with the thin plate problem, p will be called an admissible load.

In the rest of this section, and looking at Bézier methodology, we will determine a class of admissible polynomial loads concentrated along the boundary.

The configuration of the control points chosen for the vectorial load is one where all the control points except the exterior ones and those adjacent to them are 0. The reason for this is the following one: the C^1-boundary conditions are responsible for the impossibility of solving the homogeneous biharmonic equation in general. Therefore, if we add a load, it is natural to make the load act on the parts of the homogeneous biharmonic solution where the C^1-boundary conditions are not fulfilled. So it is natural to concentrate the load mainly along the boundaries. The load forces tangent planes to fulfill the C^1-boundary conditions.

Theorem 2 *Given the non-boundary control points of a Bézier load $p \in \mathbb{R}_n[u, v]$, there is a unique configuration of the boundary control points making the polynomial load admissible.*

Previous results, see Prop. 1, imply the impossibility of solving the homogeneous biharmonic equation prescribing the boundary and the normal derivatives along the boundary within the set of polynomial functions. Given just the boundary, a unique polynomial solution of the homogeneous biharmonic equation is fixed. In order to cope with the normal derivative conditions we need to introduce some more degrees of freedom. We shall introduce them as a load mainly concentrated near the boundary.

Theorem 3 *Given a polynomial boundary, or equivalently, the exterior control points of a Bézier surface, and given the normal derivatives along the* *boundary, or equivalently, the adjacent to the exterior control points of the Bézier surface,*

$$\begin{array}{ccccccc}
P_{0,0} & P_{0,1} & P_{0,2} & \dots & P_{0,n-2} & P_{0,n-1} & P_{0,n} \\
P_{1,0} & P_{1,1} & P_{1,2} & \dots & P_{1,n-2} & P_{1,n-1} & P_{1,n} \\
P_{2,0} & P_{2,1} & * & \dots & * & P_{2,n-1} & P_{2,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\
P_{n-2,0} & P_{n-2,1} & * & \dots & * & P_{n-2,n-1} & P_{n-2,n} \\
P_{n-1,0} & P_{n-1,1} & P_{n-1,2} & \dots & P_{n-1,n-2} & P_{n-1,n-1} & P_{n-1,n} \\
P_{n,0} & P_{n,1} & P_{n,2} & \dots & P_{n,n-2} & P_{n,n-1} & P_{n,n}
\end{array}$$

there is a unique admissible polynomial load, $p_a(u,v)$, whose Bézier control net is of the kind

$$\begin{array}{ccccccc}
R_{0,0} & R_{0,1} & R_{0,2} & \dots & R_{0,n-2} & R_{0,n-1} & R_{0,n} \\
R_{1,0} & R_{1,1} & R_{1,2} & \dots & R_{1,n-2} & R_{1,n-1} & R_{1,n} \\
R_{2,0} & R_{2,1} & 0 & \dots & 0 & R_{2,n-1} & R_{2,n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\
R_{n-2,0} & R_{n-2,1} & 0 & \dots & 0 & R_{n-2,n-1} & R_{n-2,n} \\
R_{n-1,0} & R_{n-1,1} & R_{n-1,2} & \dots & R_{n-1,n-2} & R_{n-1,n-1} & R_{n-1,n} \\
R_{n,0} & R_{n,1} & R_{n,2} & \dots & R_{n,n-2} & R_{n,n-1} & R_{nn}
\end{array} \tag{5}$$

and there is a unique polynomial solution of $\Delta^2 f = p_a$ with the given C^1-boundary conditions.

We do not provide the reader with any recursive algorithm of computation of the solution as for the other method (as will be seen in Annex A). The examples have been computed solving first the linear system for a given degree using a symbolic algebraic program (Mathematica) and fixing then the boundary conditions.

4.1 Comparison with an exact solution of the homogeneous biharmonic equation

We will start with the same exact solution of the biharmonic equation than in [2],

$$f(u,v) = u\cos(u)e^v.$$

We will consider it restricted to the domain $[0,1]^2$. Since it is not a polynomial function, we approximate its boundaries by polynomial curves. Applying Prop. 1 we get a unique biharmonic polynomial solution (see Fig. I) with those boundaries (and without any condition on the normal derivatives).

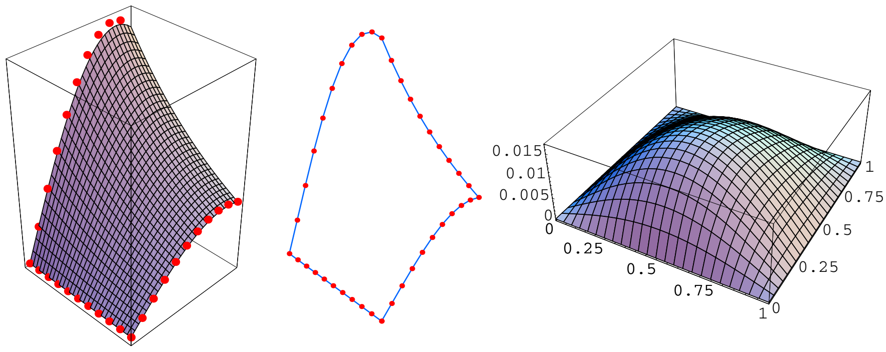

Figure I. Left: Differences between the polynomial and the exact solution. Center: polynomial solution. Right: Degree 10 boundary conditions.

Applying now Th. 3 we can manage the normal derivative conditions (see Fig. II).

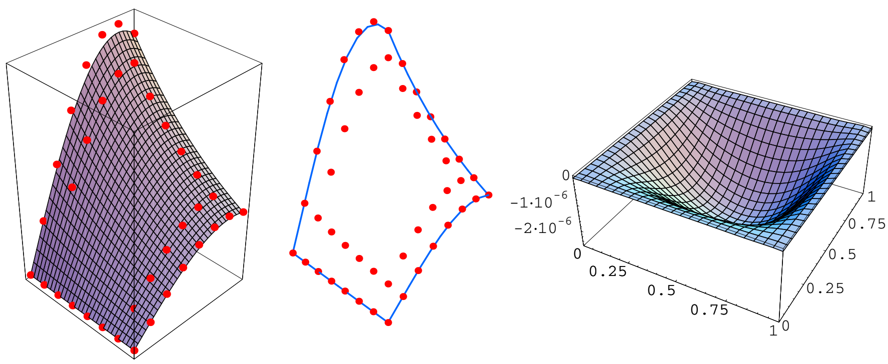

Figure II. Left: Differences between the polynomial and the exact solution. Center: polynomial solution with load. Right: Degree 7 C1-boundary conditions.

It is interesting to see the shape of the load needed. As one can see in Fig. III (right), the load is concentrated mainly along two of the boundaries.

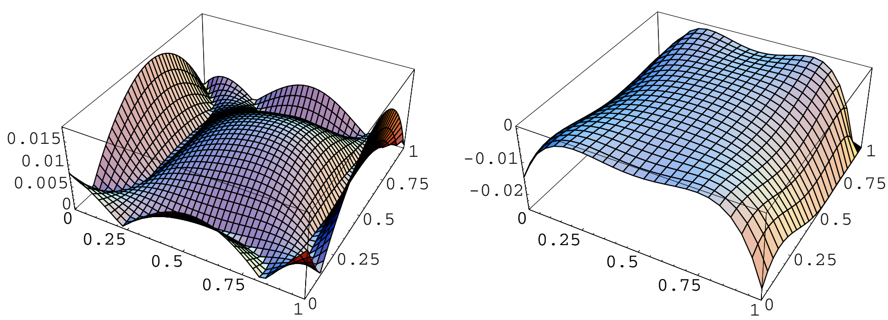

Figure III. The loads for the tetraharmonic solution (left) and the nonhomogenenous biharmonic solution with a load (right).

Finally, let us say that the shape of the tetraharmonic solution is similar to that of the nonhomogeneous biharmonic solution. The maximum difference between the exact and the nonhomogeneous biharmonic solutions is $2.499\ 10^{-6}$ while for the tetraharmonic solution it is $9.108\ 10^{-6}$.

5 Examples

We shall provide some examples of tetraharmonic and nonhomogeneous biharmonic Bézier surfaces generated by the method studied above in the case $n = 7$, which corresponds to an 8×8 control net. In two of the examples the same boundary control points are given, but changing the control points adjacent to them, in order to make clear the usefulness of being able to choose the tangent planes on the boundary. Bear in mind that the tangent plane in a boundary point P (of the surface, not of the control net) is determined by the boundary control points which generate the Bézier curve to which this point belongs, and the control points adjacent to them, not only by these boundary control points and the two neighbouring adjacent control points to P.

In all the figures in this section, we will give the boundary control points (dark ones) and those adjacent to them (lighter ones), and the two resulting surfaces, sometimes plotted using gray tones according to the mean curvature.

5.1 Cylindrical boundary conditions

In the first example we are going to study, we will place the boundary control points lying equally spaced on a cylinder and vary the adjacent control points. In Fig. IV the adjacent control points are placed in the same cylinder, and the obtained figure is as one would expect.

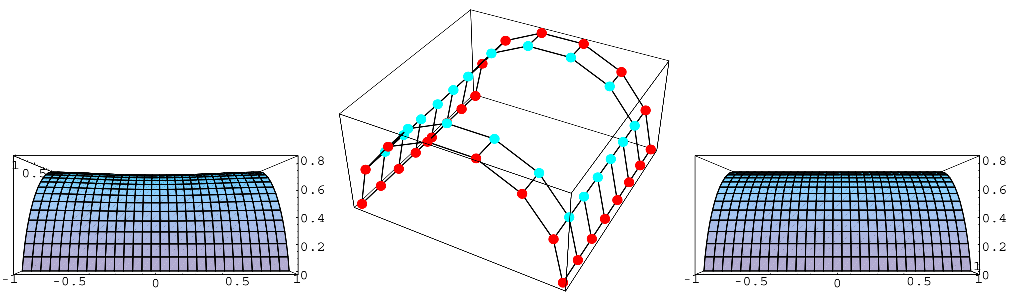

Figure IV. For cylindrical boundary data, the two methods lead to a similar shape, more cylindrical for the tetraharmonic one (right) than for the nonhomogeneous biharmonic one (left). Meanwhile, in Fig. V, by lowering the adjacent control points, the central part of the resultant surfaces sinks, but more noticeably in the tetraharmonic case.

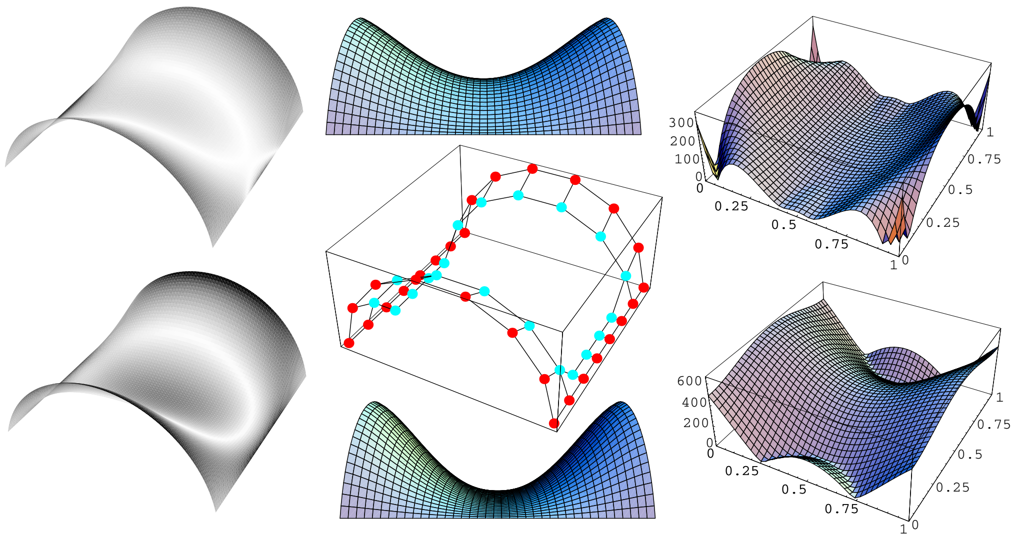

Figure V. Center: boundary control data. Top: nonhomogeneous biharmonic surface. Bottom: tetraharmonic surface. In both cases: Left: with gray tones using mean curvature. Middle: From a point of view showing their shapes in the central part. Right: plot of the corresponding load. Gray levels in the plots of the solutions correspond to the values of the mean curvature on a scale common to both surfaces.

5.2 Spherical boundary conditions

In this example (see Fig. VI) the boundary conditions are taken from a degree 7 polynomial approximation to a piece of a unit sphere.

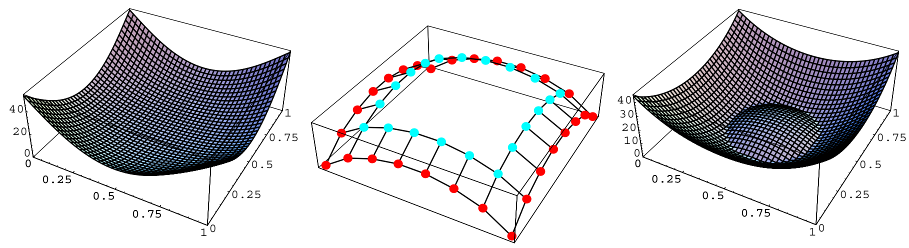

Figure VI. Boundary conditions from an approximation to a piece of a sphere (center). Loads of the non homogeneous biharmonic solution (left) and of the tetraharmonic solution (right). The two corresponding Bézier solutions are not plotted because they are similar to the piece of the sphere. In fact, the maximum differences between the two solution and the sphere are 0.00822637 for the nonhomogeneous biharmonic solution and 0.0141973 for the tetraharmonic solution.

5.3 Arched boundary conditions

In this case, the boundary control points are equally distributed in four vertical semi-circumferences. Fig. VII shows these boundary control points with a collection of adjacent control points following a natural pattern, and the surface obtained is certainly not surprising.

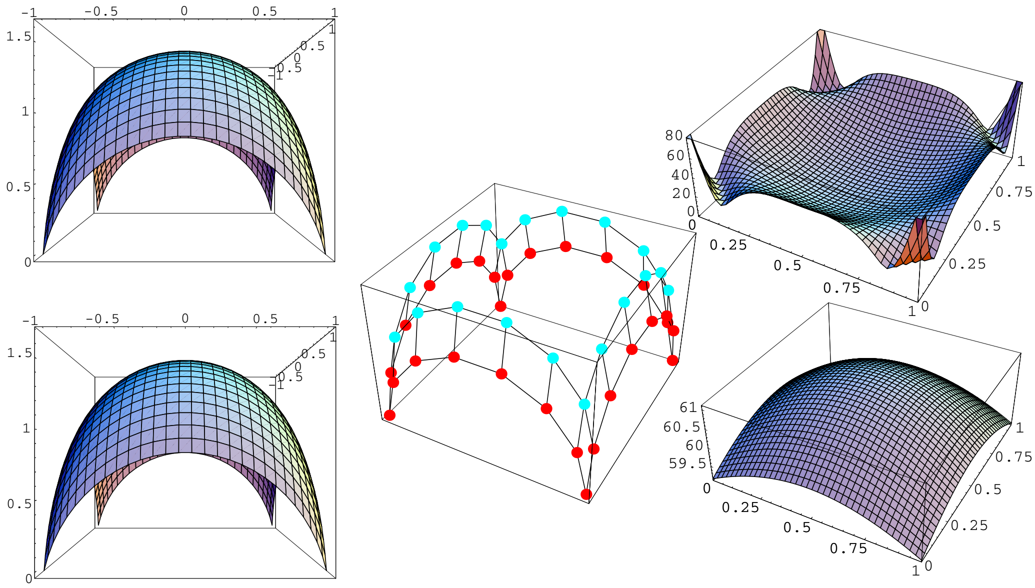

Figure VII. Middle, the C^1-boundary configuration. Top, the nonhomogeneous biharmonic solution and its corresponding load. Bottom, the tetraharmonic solution and its corresponding load.

On the other hand, in Fig. VIII, the adjacent points are in a horizontal plane over the boundary ones. The tetraharmonic surface is rather unexpected whereas the shape of the nonhomogeneous biharmonic surface seems to be more natural.

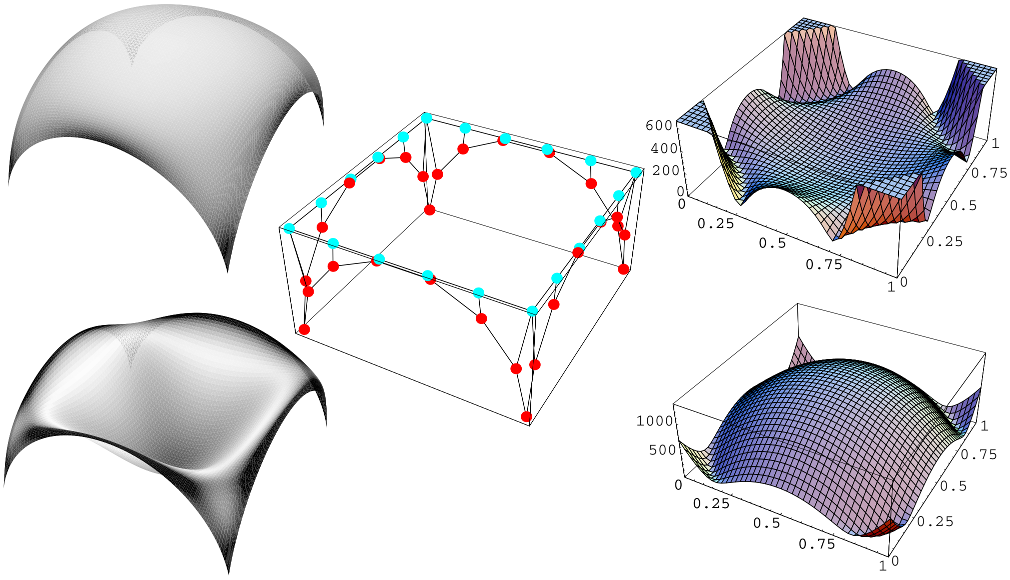

Figure VIII. Similar scheme of plotting than that of Fig. VII. Gray levels in the plots of the solutions correspond to the values of the mean curvature on a scale common to both surfaces.

5.4 Other type of arched boundary conditions

In this final example, the boundary control points are also equally spaced along four semi-circumferences, but unlike the previous example, two of the semi-circumferences corresponding to opposite sides are lying in the horizontal plane $z = 0$. In the first two rows in Fig. IX, the adjacent control points are distributed in a similar way to the boundary ones, but shifted towards the interior, and the resulting surface shows an interesting upwards slope in the tetraharmonic case, but not in the nonhomogeneous biharmonic case. In the last row, on the other hand, the adjacent control points are shifted towards the exterior, and the tetraharmonic figure obtained is blown up like a balloon, whereas the nonhomogeneous one is quite different and even has some self-intersections.

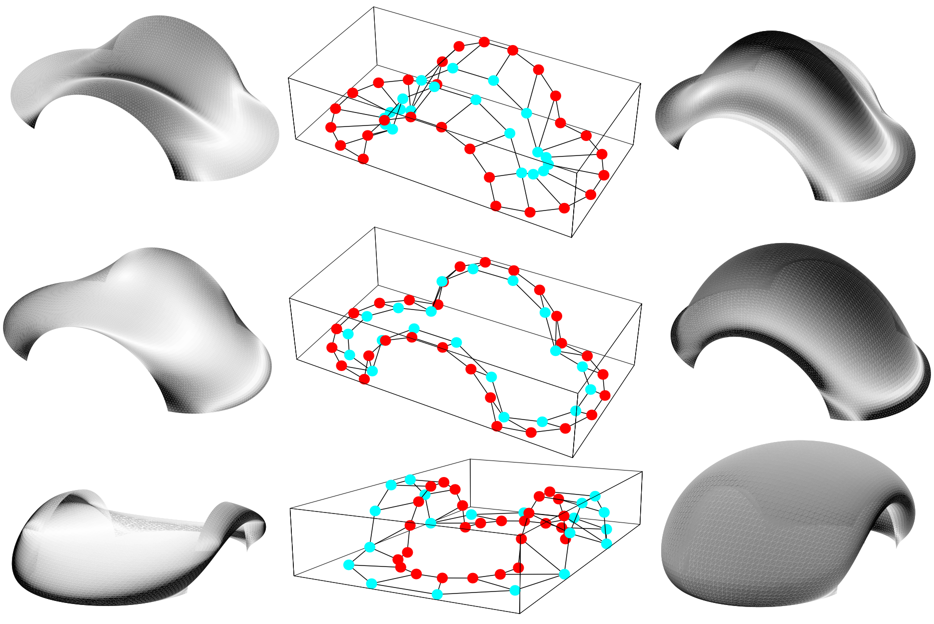

Figure IX. In the middle column the same C^0-boundary but with three different C^1-boundary configurations. In the left column, the corresponding nonhomogeneous biharmonic solutions. In the right column, the tetraharmonic solutions.

6 Uniformly loaded plate

We are now interested in the problem of a uniform rectangular plate of unit length in the x direction, clamped at its edges and bent by uniform pressure p applied to one face. The transverse displacement w satisfies the equation

$$D\Delta^2 w = p$$

subject to the boundary conditions that w and its normal derivatives are zero at the edges, and where D is the flexural rigidity of the plate. The aspect ratio of the plate is taken to be a.

In order to compare the results with that of [2] let us compute the deflection at the middle point of a squared plate, supposing w is a bivariate polynomial of degree n and $p/D = 1$. Our results are the following

degree n	calculated deflection
7	0.00096377
9	0.00122758
11	0.00126654
15	0.00126628

Whereas in [2] the result is 0.00126532.

The load for each degree is plotted in Fig. X. It may be seen that as the degree increases, the load decreases, except for at small regions around the corners.

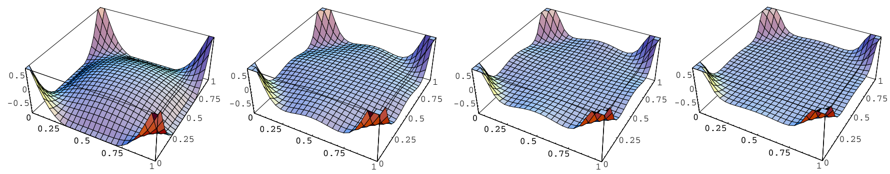

Figure X. Load for the uniformly loaded plate with $p/D = 1$ for degrees $n = 7, 9, 11$ and 15. The four figures are plotted with the same scale factors and the same plot range.

Finally, let us check the behavior of our method with respect to the aspect ratio of the rectangular plate.

aspect ratio	calculated deflection $n = 11$	calculated deflection $n = 15$	results of Bloor&Wilson	results of Timoshenko
1.0	0.00126654	0.00126628	0.00126532	0.00126
1.3	0.00191234	0.00191027	0.00191168	0.00191
1.5	0.00218658	0.00219831	0.00219655	0.00220
1.7	0.00234867	0.00239123	0.00238207	0.00238
2.0	0.00246161	0.00254655	0.00253297	0.00254

The results are quite good in comparison with those existing in the literature even for degrees as $n = 11$ or 15 which are not too high. Nevertheless, the computation of normal reactions along the boundaries needs higher degrees if results similar to the ones in [2] or [6] are wanted.

7 Conclusions

As a continuation of previous work we have presented here two methods to generate Bézier surface from the boundary and adjacent to the boundary control points.

Both methods enable us to control the shape of the surface near its boundary. As for the central part of the generated surface, one may obtain rather unexpected results. The problem of determining which method is better suited in a particular case depends mainly on the boundary information.

If the boundary conditions are more or less natural, for example, they come from a regular surface (Fig. IV, V, VI or VII) both methods give surfaces which have a similar shape. The difference is only quantitative.

For wilder boundary conditions, see Fig. VIII and IX, the solutions clearly show different shapes. In Fig. VII or in the middle row of Fig. IX, the nonhomogeneous biharmonic solution seems to be nicer than the tetraharmonic one. On the contrary, in the last row of Figure IX, the nohomogeneous biharmonic solution even shows unpleasant self-intersections.

Finally, let us say that we have dealt here with the whole set of C^1-boundary conditions. Nevertheless, both methods can be adapted to manage with a partial set of C^1-boundary conditions. For example, if the whole C^0-boundary is prescribed but only the tangent planes along two opposed boundary curves are prescribed, it is reasonable to think that instead of looking for solutions of the tetraharmonic equation, $\Delta^4\overrightarrow{\mathbf{x}} = 0$, one has to look for solutions of the triharmonic equation $\Delta^3\overrightarrow{\mathbf{x}} = 0$.

Finally, for the nonhomogeneous biharmonic equation, the possibility to cope with partial C^1-conditions is to work with a load vanishing along the boundary curves with no normal derivative prescribed.

8 Annex A. Proof of Th. 1

Given a Bézier surface $\overrightarrow{\mathbf{x}} : [0,1] \times [0,1] \to \mathbb{R}^3$ let us compute $\Delta^4\overrightarrow{\mathbf{x}}$ but considering $\overrightarrow{\mathbf{x}}$ in the usual basis instead of the Bernstein basis, i.e. $\overrightarrow{\mathbf{x}}(u,v) = \sum_{i,j=0}^{n} \frac{a_{ij}}{i!j!} u^i v^j$, in order to obtain conditions over the coefficients for a Bézier surface to be tetraharmonic in a simpler way.

$$\begin{aligned}
\Delta^4\ \overrightarrow{\mathbf{x}}(u,v) &= \left(\tfrac{\partial^8}{\partial u^8}+4\tfrac{\partial^8}{\partial u^6\partial v^2}+6\tfrac{\partial^8}{\partial u^4\partial v^4}+4\tfrac{\partial^8}{\partial u^2\partial v^6}+\tfrac{\partial^8}{\partial v^8}\right)\overrightarrow{\mathbf{x}}(u,v)\\
&= \textstyle\sum_{i=8,j=0}^{n}\frac{a_{ij}}{(i-8)!j!}u^{i-8}v^{j}+4\sum_{i=6,j=2}^{n}\frac{a_{ij}}{(i-6)!(j-2)!}u^{i-6}v^{j-2}\\
&\quad \textstyle +6\sum_{i=4,j=4}^{n}\frac{a_{ij}}{(i-4)!(j-4)!}u^{i-4}v^{j-4}+4\sum_{i=2,j=6}^{n}\frac{a_{ij}}{(i-2)!(j-6)!}u^{i-2}v^{j-6}\\
&\quad \textstyle +\sum_{i=0,j=8}^{n}\frac{a_{ij}}{i!(j-8)!}u^{i}v^{j-8}\\
&= \textstyle\sum_{i,j=0}^{n}\frac{1}{i!j!}\left(a_{i+8,j}+4a_{i+6,j+2}+6a_{i+4,j+4}+4a_{i+2,j+6}+a_{i,j+8}\right)u^{i}v^{j}
\end{aligned} \tag{6}$$

using the convention $a_{i,j}=0$ for $i>n$ or $j>n$.

Therefore $\Delta^4\overrightarrow{\mathbf{x}}=0$ if and only if

$$a_{i+8,j}+4a_{i+6,j+2}+6a_{i+4,j+4}+4a_{i+2,j+6}+a_{i,j+8}=0, \qquad \forall i,j\in\mathbb{N} \tag{7}$$

Lemma 1 *If the coefficients in the first eight rows, $\{a_{0,j},a_{1,j},...,a_{7,j}\}_{j=0}^{n}$, are known, then the linear system defined by the equations of the kind given in equation (7) has a unique solution, which is, for $k\geq 4$:*

$$\begin{cases}
a_{2k,j} = \frac{(-1)^{(k+1)}}{6}(k-3)(k-2)(k-1)k \\
\qquad\qquad \left(\frac{a_{6,2k+j-6}}{k-3}+3\frac{a_{4,2k+j-4}}{k-2}+3\frac{a_{2,2k+j-2}}{k-1}+\frac{a_{0,2k+j}}{k}\right) \\
a_{2k+1,j} = \frac{(-1)^{(k+1)}}{6}(k-3)(k-2)(k-1)k \\
\qquad\qquad \left(\frac{a_{7,2k+j-6}}{k-3}+3\frac{a_{5,2k+j-4}}{k-2}+3\frac{a_{3,2k+j-2}}{k-1}+\frac{a_{1,2k+j}}{k}\right)
\end{cases} \tag{8}$$

Proof: Let us write for $k\geq 4$ the solution as follows:

$$a_{2k,j}=(-1)^{(k+1)}\left(A_k a_{6,2k+j-6}+B_k a_{4,2k+j-4}+C_k a_{2,2k+j-2}+D_k a_{0,2k+j}\right). \tag{9}$$

It is easy to check that all of the four sequences $\{A_k,B_k,C_k,D_k\}_{k\in\mathbb{N}}$ verify the recurrence equation given by equation (7). The usual techniques for computing the general term of a sequence defined by a recurrence equation give the next results:

$$\begin{aligned}
A_k &= \tfrac{1}{6}(k-2)(k-1)k, &\quad B_k &= \tfrac{1}{2}(k-3)(k-1)k,\\
C_k &= \tfrac{1}{2}(k-3)(k-2)k, &\quad D_k &= \tfrac{1}{6}(k-3)(k-2)(k-1).
\end{aligned} \tag{10}$$

Simplifying now formula 9 we get the first of the answers.

For $a_{2k+1,j}$ the solution is a linear combination of $a_{7,i+j-6}$, $a_{5,i+j-4}$, $a_{3,i+j-2}$ and $a_{1,i+j}$ with the same coefficients as in the previous case. $\square$

Lemma 2 *Let $r \geq 8$. If the coefficients a_{r0}, $a_{r-1,1}$, $a_{r-2,2}$, $a_{r-3,3}$, $a_{3,r-3}$, $a_{2,r-2}$, $a_{1,r-1}$ and a_{0r} are known, then the linear system defined by the equations of the kind given in equation (7) for $i+j+8=r$ ($r-7$ equations) and with unknowns $\{a_{ij}\}$, $i+j=r$, $i,j \geq 4$, has a unique solution.*

Remark 3 *These are the coefficients shown as a matrix. Notice that the systems which will be solved in the following proof correspond to the positive slope diagonals.*

$$
\begin{pmatrix}
a_{0,0} & a_{0,1} & a_{0,2} & \cdots & & & \mathbf{a_{0,r}} & \cdots & a_{0,n-1} & a_{0,n} \\
a_{1,0} & a_{1,1} & a_{1,2} & \cdots & & \mathbf{a_{1,r-1}} & & \cdots & a_{1,n-1} & a_{1,n} \\
a_{2,0} & a_{2,1} & a_{2,2} & \cdots & \mathbf{a_{2,r-2}} & & & \cdots & a_{2,n-1} & a_{2,n} \\
 & \vdots & & & & & & & \vdots & \\
 & & \mathbf{a_{r-2,2}} & \cdots & & & & \cdots & & \\
 & \mathbf{a_{r-1,1}} & & \cdots & & & & \cdots & & \\
\mathbf{a_{r,0}} & & & \cdots & & & & \cdots & & \\
 & \vdots & & & & & & & \vdots & \\
a_{n-1,0} & a_{n-1,1} & a_{n-1,2} & \cdots & & & & \cdots & a_{n-1,n-1} & a_{n-1,n} \\
a_{n,0} & a_{n,1} & a_{n,2} & \cdots & & & & \cdots & a_{n,n-1} & a_{n,n}
\end{pmatrix}
$$

Proof: Let us write the first equations starting with a_{r0}. The known values are written in bold type. We shall consider first the case where r is even:

$$
\begin{array}{llllllll}
\mathbf{a_{r,0}} & +4\mathbf{a_{r-2,2}} & +6a_{r-4,4} & +4a_{r-6,6} & +a_{r-8,8} & & & = 0 \\
 & \mathbf{a_{r-2,2}} & +4a_{r-4,4} & +6a_{r-6,6} & +4a_{r-8,8} & +a_{r-10,10} & & = 0 \\
 & & \vdots & & & & & \\
 & & & & \dots & +4a_{4,r-4} & +\mathbf{a_{2,r-2}} & = 0 \\
 & & & a_{8,r-8} & +4a_{6,r-6} & +6a_{4,r-4} & +4\mathbf{a_{2,r-2}} +\mathbf{a_{0,r}} & = 0
\end{array}
$$

Here we have $\frac{r}{2}-3$ equations and the same number of unknowns. In the case where r is odd, the system has $\left[\frac{r}{2}\right]-3$ equations and unknowns:

$$
\begin{array}{llllllll}
\mathbf{a_{r,0}} & +4\mathbf{a_{r-2,2}} & +6a_{r-4,4} & +4a_{r-6,6} & +a_{r-8,8} & & & = 0 \\
 & \mathbf{a_{r-2,2}} & +4a_{r-4,4} & +6a_{r-6,6} & +4a_{r-8,8} & +a_{r-10,10} & & = 0 \\
 & & \vdots & & & & & \\
 & & & & \dots & +4a_{5,r-5} & +\mathbf{a_{3,r-3}} & = 0 \\
 & & & a_{9,r-9} & +4a_{7,r-7} & +6a_{5,r-5} & +4\mathbf{a_{3,r-3}} +\mathbf{a_{1,r-1}} & = 0
\end{array}
$$

The coefficient matrix for these systems is the following $k \times k$ square matrix:

$$M_k = \begin{pmatrix} 6 & 4 & 1 & 0 & & & \ldots & \\ 4 & 6 & 4 & 1 & & & \ldots & \\ 1 & 4 & 6 & 4 & & & \ldots & \\ 0 & 1 & 4 & 6 & & & \ldots & \\ & & & & \ddots & & & \\ & \ldots & & & & 6 & 4 & 1 \\ & \ldots & & & & 4 & 6 & 4 \\ & \ldots & & & & 1 & 4 & 6 \end{pmatrix}$$

This is a scalar pentadiagonal matrix and using the recurrence formula for its determinant due to [11] or [1], which is:

$$\begin{aligned} H_k = CH_{k-1} + (AE - BD)H_{k-2} + (B^2E + AD^2 - 2AEC)H_{k-3} \\ +AE(AE - BD)H_{k-4} + A^2B^2CH_{k-5} - A^3E^3H_{k-6} \end{aligned} \tag{11}$$

where $H_k = det(M_k)$, and in our case, $A = 1 = E$, $B = 4 = D$ and $C = 6$, we obtain:

$$H_k = 6H_{k-1} - 15H_{k-2} + 20H_{k-3} - 15H_{k-4} + 6H_{k-5} - H_{k-6} \tag{12}$$

The associated characteristic polynomial is therefore $\lambda^6 - 6\lambda^5 + 15\lambda^4 - 20\lambda^3 + 15\lambda^2 - 6\lambda + 1 = (\lambda - 1)^6$ and this gives us that the general term is of the following type: $(-1)^k(c_0 + c_1k + c_2k^2 + c_3k^3 + c_4k^4 + c_5k^5)$. Calculating H_k for $k = 1, \ldots, 5$ we have the initial conditions to obtain the coefficients c_i in the general term, and so the final solution (i.e. the determinant of the matrix as a function of the number of equations or unknowns) is : $\frac{1}{12}(12 + 28k + 23k^2 + 8k^3 + k^4)$ which is greater than 0 for all k. This means that the system has a unique solution.

Note that this system involves coefficients with the second index even. To obtain the rest of the a_{ij} in the same diagonal we start the equations with $a_{r-1,1}$ instead of a_{r0}. $\square$

Remark 4 *For $i + j \geq n + 4$, the linear system that appears in the proof is homogeneous, taking into account the convention $a_{i,j} = 0$ for i or $j > n$, and therefore $a_{i,j} = 0$ for $i + j \geq n + 4$.*

8.1 Proof of Th. 1

Proof: Let us write the Bézier chart in the usual basis of polynomials

$$\overrightarrow{\mathbf{x}}(u,v) = \sum_{i,j=0}^{n} \frac{a_{ij}}{i!j!} u^i v^j,$$

with $a_{ij} \in \mathbb{R}^3$. Note that $\{P_{0j}\}_{j=0}^n$ determine

$$\sum_{j=0}^{n} B_j^n(v) P_{0j} = \overrightarrow{\mathbf{x}}(0,v) = \sum_{j=0}^{n} \frac{a_{0j}}{j!} v^j \tag{13}$$

and so determine $\{a_{0j}\}_{j=0}^n$. We also have that $\{P_{1j}\}_{j=0}^n$ together with the points given before determine

$$\begin{aligned} m\textstyle\sum_{j=0}^n B_j^n(v)(P_{1j}-P_{0j}) &= m\textstyle\sum_{j=0}^n B_j^n(v)\Delta^{1,0}P_{0j} = \frac{\partial}{\partial u}\overrightarrow{\mathbf{x}}(0,v) \\ &= \textstyle\sum_{j=0}^n \frac{a_{1j}}{j!} v^j \end{aligned} \tag{14}$$

and so determine $\{a_{1j}\}_{j=0}^n$. Analogously, from $\{P_{i0}\}_{i=0}^n$ and $\{P_{i1}\}_{i=0}^n$ we obtain $\{a_{i0}\}_{i=0}^n$ and $\{a_{i1}\}_{i=0}^n$.

Now we have

$$\sum_{j=0}^{n} B_j^n(v) P_{nj} = \overrightarrow{\mathbf{x}}(1,v) = \sum_{j=0}^{n} \frac{1}{j!} \left(\sum_{i=0}^{n} \frac{a_{ij}}{i!} \right) v^j \tag{15}$$

and

$$\begin{aligned} m\textstyle\sum_{j=0}^n B_j^n(v)(P_{nj}-P_{n-1,j}) &= m\textstyle\sum_{j=0}^n B_j^n(v)\Delta^{1,0}P_{n-1,j} = \frac{\partial}{\partial u}\overrightarrow{\mathbf{x}}(1,v) \\ &= \textstyle\sum_{j=0}^n \frac{1}{j!}\left(\sum_{i=1}^n \frac{a_{ij}}{(i-1)!}\right) v^j \end{aligned} \tag{16}$$

which means that knowing $\{P_{nj}\}_{j=0}^n$ and $\{P_{n-1,j}\}_{j=0}^n$ gives the system shown in the following paragraph by comparing the coefficients of v^j in each side of the equalities:

From $\frac{\partial}{\partial u}\overrightarrow{\mathbf{x}}(1,v)$ we get that $a_{1j} + \mathbf{a_{2j}} + \frac{\mathbf{a_{3j}}}{2} + \frac{a_{4j}}{6} + \ldots$ is a known value, and from $\overrightarrow{\mathbf{x}}(1,v)$ we get that $a_{0j} + a_{1j} + \frac{\mathbf{a_{2j}}}{2} + \frac{\mathbf{a_{3j}}}{6} + \ldots$ is a known value too, so for each j we have a system of two linear equations, which we will name S^j, whose unknowns are a_{2j} and a_{3j}. This system has a unique solution because its coefficient matrix is

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} \end{pmatrix}$$

whose determinant is $-\frac{1}{12} \neq 0$.

Analogously, from $\vec{\mathbf{x}}(u,1)$ and $\frac{\partial}{\partial v}\vec{\mathbf{x}}(u,1)$, we can obtain the values of $a_{i1} + \mathbf{a_{i2}} + \frac{\mathbf{a_{i3}}}{2} + \frac{a_{i4}}{6} + \ldots$ and $a_{i0} + a_{i1} + \frac{\mathbf{a_{i2}}}{2} + \frac{\mathbf{a_{i3}}}{6} + \ldots$, therefore, for each i we have a system of two linear equations with a unique solution, which we will call S_i, whose unknowns are a_{i2} and a_{i3}.

We must take into account that we know that $a_{ij} = 0$ for $i + j \geq n + 4$ by Remark (4).

Starting from $k = n$ and working recursively backwards until $k = 2$, we can solve the systems S^k and S_k, bearing in mind that we already know a_{0k}, a_{1k}, a_{k0} and a_{k1}, and we also know a_{ij} for $i + j \geq k + 4 (= n + 4)$. This way, we can get the values of a_{k2}, a_{k3}, a_{2k} and a_{3k}. Then, as we know the values of $a_{0,k+3}$, $a_{1,k+2}$, $a_{2,k+1}$, a_{3k}, a_{k3}, $a_{k+1,2}$, $a_{k+2,1}$ and $a_{k+3,0}$, we can use Lemma (2) to compute a_{ij} for $i + j = k + 3$, so we know a_{ij} for $i + j = (k-1) + 4$. This way we can start the next step of the downwards recursion.

Remark 5 *Having reached this point, it may be convenient for the neatness of the proof to introduce a series of formulas which will be used in the final part of the proof. From now on, we shall use the notation* $B_i^n(v) = \sum_{j=0}^{n-i} c_{ij}^n v^{i+j}$, *where* $c_{ij}^n = c_{ij} = (-1)^j \binom{n}{i}\binom{n-i}{j}$.

From Equation (13) we obtain:

$$a_{0j} = j! \sum_{\ell=0}^{j} c_{\ell,j-\ell} P_{0\ell} \qquad j = 0, \ldots, n. \tag{17}$$

From equation (14) we get:

$$a_{1j} = j! n \sum_{\ell=0}^{j} c_{\ell,j-\ell}(P_{1\ell} - P_{0\ell}) \qquad j = 0, \ldots, n. \tag{18}$$

From equation (15) we get:

$$\sum_{i=0}^{n} \frac{a_{ij}}{i!} = j! \sum_{\ell=0}^{j} c_{\ell,j-\ell} P_{n\ell} \qquad j = 0, \ldots, n. \tag{19}$$

From equation (16) we get:

$$\sum_{i=1}^{n} \frac{a_{ij}}{(i-1)!} = j!n \sum_{\ell=0}^{j} c_{\ell,j-\ell}(P_{n\ell} - P_{n-1,\ell}) \qquad j = 0, \ldots, n. \tag{20}$$

Analogously, from $\overrightarrow{\mathbf{x}}(u,0)$*,* $\overrightarrow{\mathbf{x}}(u,1)$ *and their partial derivatives in the second variable we obtain similar equations:*

$$a_{i0} = i! \sum_{\ell=0}^{i} c_{\ell,i-\ell} P_{\ell 0} \qquad i = 0, \ldots, n, \tag{21}$$

$$a_{i1} = i!n \sum_{\ell=0}^{i} c_{\ell,i-\ell}(P_{\ell 1} - P_{\ell 0}) \qquad i = 0, \ldots, n, \tag{22}$$

$$\sum_{j=0}^{n} \frac{a_{ij}}{j!} = i! \sum_{\ell=0}^{i} c_{\ell,i-\ell} P_{\ell n} \qquad i = 0, \ldots, n, \tag{23}$$

$$\sum_{j=1}^{n} \frac{a_{ij}}{(j-1)!} = i!n \sum_{\ell=0}^{i} c_{\ell,i-\ell}(P_{\ell n} - P_{\ell,n-1}) \qquad i = 0, \ldots, n. \tag{24}$$

And now we shall continue with the rest of the proof.

We reach a problem when we get to solving a_{33}, because it can be obtained as solution to two different systems, S^3 and S_3, so we must check for compatibility. We will first calculate a_{33} in S_3. Using formulas (23) and (24) for $i = 3$ we get the following system:

$$\begin{aligned} \textstyle\sum_{j=1}^{n} \frac{a_{3j}}{(j-1)!} &= 6n \textstyle\sum_{\ell=0}^{3} c_{\ell,3-\ell}(P_{\ell n} - P_{\ell,n-1}), \\ \textstyle\sum_{j=0}^{n} \frac{a_{3j}}{j!} &= 6 \textstyle\sum_{\ell=0}^{3} c_{\ell,3-\ell} P_{\ell n}, \end{aligned} \tag{25}$$

with unknowns a_{32} and a_{33}. From here it follows that:

$$\begin{aligned} -\tfrac{a_{33}}{12} = 6 \textstyle\sum_{\ell=0}^{3} c_{\ell,3-\ell} P_{\ell n} - 3n \textstyle\sum_{\ell=0}^{3} c_{\ell,3-\ell}(P_{\ell n} - P_{\ell,n-1}) \\ -a_{30} - \tfrac{1}{2} a_{31} - \tfrac{1}{2} \textstyle\sum_{j=4}^{n} \tfrac{2-j}{j!} a_{3j} \end{aligned} \tag{26}$$

Our current objective is to give this expression as a linear combination of the control points, except for one term, for reasons which will become clearer later on. Now, by formulas (21) and (22) we have:

$$a_{30} = 6 \sum_{\ell=0}^{3} c_{\ell,3-\ell} P_{\ell 0}, \qquad a_{31} = 6n \sum_{\ell=0}^{3} c_{\ell,3-\ell}(P_{\ell 1} - P_{\ell 0}) \tag{27}$$

and from solving the system (19), (20) for a_{2j} and a_{3j} we obtain the general expression for $a_{3j}, j \geq 4$

$$a_{3j} = 12\left(a_{0j} + \frac{a_{1j}}{2} - \frac{j!}{2}\sum_{\ell=0}^{j} c_{\ell,j-\ell}\left((2-n)P_{n\ell} - nP_{n-1,\ell}\right) - \sum_{i=4}^{n}\frac{2-i}{2i!}a_{ij}\right) \quad (28)$$

and so $\frac{1}{2}\sum_{j=4}^{n}\frac{2-j}{j!}a_{3j} =$

$$\begin{aligned}
&= \tfrac{1}{2}\textstyle\sum_{j=4}^{n}\frac{2-j}{j!}12\left(a_{0j} + \frac{a_{1j}}{2} - \frac{j!}{2}\sum_{\ell=0}^{j} c_{\ell,j-\ell}((2-n)P_{n\ell} + nP_{n-1,\ell}) - \sum_{i=4}^{n}\frac{2-i}{2i!}a_{ij}\right)\\
&= 6\textstyle\sum_{j=4}^{n}(2-j)\left(\sum_{\ell=0}^{j} c_{\ell,j-\ell}P_{0\ell} + \frac{1}{2}n\sum_{\ell=0}^{j} c_{\ell,j-\ell}(P_{1\ell} - P_{0\ell})\right.\\
&\qquad\qquad \left. -\tfrac{1}{2}\textstyle\sum_{\ell=0}^{j} c_{\ell,j-\ell}((2-n)P_{n\ell} + nP_{n-1,\ell}) - \frac{1}{j!}\sum_{i=4}^{n}\frac{2-i}{2i!}a_{ij}\right)\\
&= 3\textstyle\sum_{j=4}^{n}(2-j)\left(\sum_{\ell=0}^{j} c_{\ell,j-\ell}((2-n)P_{0\ell} + nP_{1\ell}\right.\\
&\qquad\qquad \left. -(2-n)P_{n\ell} - nP_{n-1,\ell}) - 3\textstyle\sum_{j=4}^{n}\sum_{i=4}^{n}\frac{(2-j)(2-i)}{j!i!}a_{ij}\right) \qquad (29)\\
&= 3\textstyle\sum_{\ell=0}^{n}(\sum_{j=\ell}^{n}(2-j)c_{\ell,j-\ell})((2-n)P_{0\ell} + nP_{1\ell} - (2-n)P_{n\ell} - nP_{n-1,\ell})\\
&\qquad\qquad -3\textstyle\sum_{\ell=0}^{3}(\sum_{j=\ell}^{3}(2-j)c_{\ell,j-\ell})((2-n)P_{0\ell} + nP_{1\ell} - (2-n)P_{n\ell} - nP_{n-1,\ell})\\
&\qquad\qquad -3\textstyle\sum_{j=4}^{n}\sum_{i=4}^{n}\frac{(2-j)(2-i)}{j!i!}a_{ij}
\end{aligned}$$

Notice that in the last equality we have changed the order of the finite double series. Now, using that $B_i^n(v) = \sum_{j=0}^{n-i} c_{ij}^n v^{i+j}$, we have the following expression:

$$\begin{aligned}
\textstyle\sum_{j=\ell}^{n}(2-j)c_{\ell,j-\ell} &= \textstyle\sum_{j=0}^{n-\ell}(2-(j+\ell))c_{\ell,j} = 2B_\ell^n(1) - \frac{\partial}{\partial v}B_\ell^n(1)\\
&= 2B_\ell^n(1) - n(B_\ell^n(1) - B_\ell^{n-1}(1)) \qquad (30)\\
&= 2\delta_\ell^n - n(\delta_\ell^n - \delta_\ell^{n-1}) = (2-n)\delta_\ell^n + n\delta_\ell^{n-1}.
\end{aligned}$$

Using this expression, we can simplify the calculation above:

$$\begin{aligned}
&\tfrac{1}{2}\textstyle\sum_{j=4}^{n}\frac{2-j}{j!}a_{3j} =\\
&\qquad = 3\textstyle\sum_{\ell=0}^{n}((2-n)\delta_\ell^n + n\delta_\ell^{n-1})((2-n)P_{0\ell} + nP_{1\ell} - (2-n)P_{n\ell} - nP_{n-1,\ell})\\
&\qquad\; -3\textstyle\sum_{\ell=0}^{3}(\sum_{j=\ell}^{3}(2-j)c_{\ell,j-\ell})((2-n)P_{0\ell} + nP_{1\ell} - (2-n)P_{n\ell} - nP_{n-1,\ell}) \qquad (31)\\
&\qquad\; -3\textstyle\sum_{j=4}^{n}\sum_{i=4}^{n}\frac{(2-j)(2-i)}{j!i!}a_{ij}
\end{aligned}$$

Finally, using equations (27) and (31), we get this final expression for equation (26):

$$
\begin{aligned}
-\tfrac{a_{33}}{12} = {} & 6\sum_{\ell=0}^{3} c_{\ell,3-\ell}P_{\ell n} - 3n\sum_{\ell=0}^{3} c_{\ell,3-\ell}(P_{\ell n} - P_{\ell,n-1}) \\
& -6\sum_{\ell=0}^{3} c_{\ell,3-\ell}P_{\ell 0} - 3n\sum_{\ell=0}^{3} c_{\ell,3-\ell}(P_{\ell 1} - P_{\ell 0}) \\
& -3\sum_{\ell=0}^{n}((2-n)\delta_\ell^n + n\delta_\ell^{n-1})((2-n)P_{0\ell} + nP_{1\ell} - (2-n)P_{n\ell} - nP_{n-1,\ell}) \\
& +3\sum_{\ell=0}^{3}(\sum_{j=\ell}^{3}(2-j)c_{\ell,j-\ell})((2-n)P_{0\ell} + nP_{1\ell} - (2-n)P_{n\ell} - nP_{n-1,\ell}) \\
& +3\sum_{j=4}^{n}\sum_{i=4}^{n}\tfrac{(2-j)(2-i)}{j!i!}a_{ij}
\end{aligned}
\tag{32}
$$

Analogously, from system S^3 we get that:

$$
\begin{aligned}
-\tfrac{a_{33}}{12} = {} & 6\sum_{\ell=0}^{3} c_{\ell,3-\ell}P_{n\ell} - 3n\sum_{\ell=0}^{3} c_{\ell,3-\ell}(P_{n\ell} - P_{n-1,\ell}) \\
& -6\sum_{\ell=0}^{3} c_{\ell,3-\ell}P_{0\ell} - 3n\sum_{\ell=0}^{3} c_{\ell,3-\ell}(P_{1\ell} - P_{0\ell}) \\
& -3\sum_{\ell=0}^{n}((2-n)\delta_\ell^n + n\delta_\ell^{n-1})((2-n)P_{\ell 0} + nP_{\ell 1} - (2-n)P_{\ell n} - nP_{\ell,n-1}) \\
& +3\sum_{\ell=0}^{3}(\sum_{i=\ell}^{3}(2-i)c_{\ell,i-\ell})((2-n)P_{\ell 0} + nP_{\ell 1} - (2-n)P_{\ell n} - nP_{\ell,n-1}) \\
& +3\sum_{i=4}^{n}\sum_{j=4}^{n}\tfrac{(2-i)(2-j)}{i!j!}a_{ij}
\end{aligned}
\tag{33}
$$

and it can easily be checked that a_{33} is the same in the two previous equations, comparing the coefficients of each P_{ij} and taking into account that i and j only appear in the coefficients and not as indexes of the control points.

In the final step of the recursion (in which we solve systems S^2 and S_2), we already know a_{23} and a_{32} from S^3 and S_3, so the only variable left is a_{22}. We can prove in a similar fashion that the solutions obtained for a_{22} from each of the four equations that form systems S^2 and S_2 are the same, so we also have compatibility in this final step of the process.

We have obtained all the coefficients $\{a_{ij}\}_{i,j=0}^n$ of the tetraharmonic Bézier chart $\vec{\mathbf{x}}(u,v)$ in the usual basis given the boundary control points and those adjacent to them. Comparing it to the Bézier chart expressed in the Bernstein basis we obtain the rest of the control points. □

9 Annex B. Proofs of Th. 2 and 3

We will need some results characterizing admissible loads.

Lemma 3 *If a polynomial p is admissible then its part of degree less or equal to $(n-2)$ uniquely determines the parts of degree $n-1$ and n.*

Proof: Let us suppose that there exists $f \in \mathbb{R}_n[u,v]$ such that $\Delta^2 f = p$. Let us write the following scheme for the coefficients of f

$$\begin{array}{cccccccc}
a_{0,0} & a_{0,1} & \cdots & a_{0,k} & a_{0,k+1} & \cdots & a_{0,n-1} & a_{0,n} \\
a_{1,0} & a_{1,1} & \cdots & a_{1,k} & a_{1,k+1} & \cdots & a_{1,n-1} & a_{1,n} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{k,0} & a_{k,1} & \cdots & a_{k,k} & a_{k,k+1} & \cdots & a_{k,n-1} & \mathbf{a_{k,n}} \\
a_{k+1,0} & a_{k+1,1} & \cdots & a_{k+1,k} & a_{k+1,k+1} & \cdots & \mathbf{a_{k+1,n-1}} & a_{k+1,n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \boldsymbol{\cdot^{\cdot^{\cdot}}} & \vdots & \vdots \\
a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,k} & \mathbf{a_{n-1,k+1}} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\
a_{n,0} & a_{n,1} & \cdots & \mathbf{a_{n,k}} & a_{n,k+1} & \cdots & a_{n,n-1} & a_{n,n}
\end{array}$$

For each $m \geq n+3$, let us consider the linear system LS_m made of the equations (4) relating the coefficients $a_{i,j}$ in a positive slope diagonal with $i+j=m$. Each one of these systems can be split into two independent linear subsystems of the same kind. Just as an example let us see the case $m = n+4$. The initial linear system is

$$\left\{\begin{array}{rl}
p_{n,0} & = a_{n+4,0} + 2a_{n+2,2} + a_{n,4}, \\
p_{n-1,1} & = a_{n+3,1} + 2a_{n+1,3} + a_{n-1,5}, \\
p_{n-2,2} & = a_{n+2,2} + 2a_{n,4} + a_{n-2,6}, \\
\cdots & \\
p_{2,n-2} & = a_{6,n-2} + 2a_{4,n} + a_{2,n+2}, \\
p_{1,n-1} & = a_{5,n-1} + 2a_{3,n+1} + a_{1,n+3}, \\
p_{0,n} & = a_{4,n} + 2a_{2,n+2} + a_{0,n+4}.
\end{array}\right. \tag{34}$$

Recall that $a_{i,j} = 0$ if $i > n$ or $j > n$.

This linear system is split into two linear subsystems depending on the parity of the second index: (if we suppose n even)

$$\left\{\begin{array}{rl}
p_{n,0} & = a_{n,4}, \\
p_{n-2,2} & = 2a_{n,4} + a_{n-2,6}, \\
p_{n-4,4} & = a_{n,4} + 2a_{n-2,6} + a_{n-4,8}, \\
\cdots & \\
p_{4,n-4} & = a_{8,n-4} + 2a_{6,n-2} + a_{4,n}, \\
p_{2,n-2} & = a_{6,n-2} + 2a_{4,n}, \\
p_{0,n} & = a_{4,n}.
\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{rl}
p_{n-1,1} & = a_{n-1,5}, \\
p_{n-3,3} & = 2a_{n-1,5} + a_{n-3,7}, \\
p_{n-5,5} & = a_{n-1,5} + 2a_{n-3,7} + a_{n-5,9}, \\
\cdots & \\
p_{3,n-3} & = a_{7,n-3} + 2a_{5,n-1}, \\
p_{1,n-1} & = a_{5,n-1}.
\end{array}\right. \tag{35}$$

Let us concentrate in the first subsystem because the same arguments are valid for the second one. The first subsystem involves $\frac{n}{2}+1$ equations and $\frac{n}{2}-1$ unknowns. If we remove the first and the last equations what we get is a linear system whose matrix of coefficients is a scalar tridiagonal matrix, which we will denote by A, with 2 in the diagonal and 1 in the two lines parallel to the diagonal. This matrix is invertible, then it is easy to compute the unique solution $\{a_{n,4}, a_{n-2,6}, \ldots, a_{6,n-2}, a_{4,n}\}$ of

$$\begin{cases} p_{n-2,2} &= 2a_{n,4} + a_{n-2,6}, \\ p_{n-4,4} &= a_{n,4} + 2a_{n-2,6} + a_{n-4,8}, \\ \quad \cdots & \\ p_{4,n-4} &= a_{4,n} + 2a_{6,n-2} + a_{8,n-4}, \\ p_{2,n-2} &= 2a_{4,n} + a_{6,n-2}. \end{cases} \tag{36}$$

Therefore, we get $p_{n,0} = a_{n,4}$ and $p_{0,n} = a_{4,n}$ in terms of the coefficients $\{p_{n-2,2}, p_{n-4,4}, \ldots, p_{4,n-4}, p_{2,n-2}\}$.

Analogous arguments applied now to the second linear subsystem allow to express $p_{n-1,1}$ and $p_{1,n-1}$ in terms of the coefficients $\{p_{n-3,3}, p_{n-5,5}, \ldots, p_{5,n-5}, p_{3,n-3}\}$. □

Proposition 2 *A polynomial $p(u,v) = \sum_{i,j=0}^{n} \frac{p_{i,j}}{i!j!} u^i v^j \in \mathbb{R}_n[u,v]$ is admissible if and only if for $m = n-1, n$ and $i = 0, 1, \ldots, n$*

$$\begin{cases} p_{i,m} = \frac{1}{[\frac{n-i}{2}]} \sum_{\ell=1}^{[\frac{n-i}{2}]-1} (-1)^{\ell-1} ([\frac{n-i}{2}] - \ell)\ p_{i+2\ell, m-2\ell}, \\ p_{m,i} = \frac{1}{[\frac{n-i}{2}]} \sum_{\ell=1}^{[\frac{n-i}{2}]-1} (-1)^{\ell-1} ([\frac{n-i}{2}] - \ell)\ p_{m-2\ell, i+2\ell}. \end{cases} \tag{37}$$

Remark 6 *Notice that when $i \geq n-3$ in expressions* (37)*, the upper limit in the sum is* 0 *and thus the corresponding equations reduce to $p_{i,m} = 0$ and $p_{m,i} = 0$, so the coefficients of p are*

$$\begin{array}{cccccccc} p_{0,0} & p_{0,1} & \cdots & p_{0,n-4} & p_{0,n-3} & p_{0,n-2} & \mathbf{p_{0,n-1}} & \mathbf{p_{0,n}} \\ p_{1,0} & p_{1,1} & \cdots & p_{1,n-4} & p_{1,n-3} & p_{1,n-2} & \mathbf{p_{1,n-1}} & \mathbf{p_{1,n}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{n-4,0} & p_{n-4,1} & \cdots & p_{n-4,n-4} & p_{n-4,n-3} & p_{n-4,n-2} & \mathbf{p_{n-4,n-1}} & \mathbf{p_{n-4,n}} \\ p_{n-3,0} & p_{n-3,1} & \cdots & p_{n-3,n-4} & p_{n-3,n-3} & p_{n-3,n-2} & 0 & 0 \\ p_{n-2,0} & p_{n-2,1} & \cdots & p_{n-2,n-4} & p_{n-2,n-3} & p_{n-2,n-2} & 0 & 0 \\ \mathbf{p_{n-1,0}} & \mathbf{p_{n-1,1}} & \cdots & \mathbf{p_{n-1,n-4}} & 0 & 0 & 0 & 0 \\ \mathbf{p_{n,0}} & \mathbf{p_{n,1}} & \cdots & \mathbf{p_{n,n-4}} & 0 & 0 & 0 & 0 \end{array} \tag{38}$$

Proof: As we have said before, the matrix of coefficients is invertible and its inverse is $A^{-1} = (b^r_{i,j})_{i,j=1,\ldots,r}$, where $r = \mathrm{rank} A$, defined by

$$b^r_{i,j} = \frac{(-1)^{i+j}}{r+1}(i+1)(r-j), \qquad i \leq j,$$

and if $i > j$, then $b^r_{i,j} = b^r_{j,i}$. Therefore the solution of the linear systems (36) and the corresponding one for the other subsystems can be explicitly computed. Thus, for $m = n-1$ and $m = n$

$$p_{i,m} = \sum_{\ell=1}^{[\frac{n-i}{2}]-1} b_{\ell-1,0}^{[\frac{n-i}{2}]-1} \; p_{i+2\ell,m-2\ell} = \frac{1}{[\frac{n-i}{2}]} \sum_{\ell=1}^{[\frac{n-i}{2}]-1} (-1)^{\ell-1}([\frac{n-i}{2}] - \ell) \; p_{i+2\ell,m-2\ell},$$

where we have applied that $b_{\ell-1,0}^{[\frac{n-i}{2}]-1} = b_{0,\ell-1}^{[\frac{n-i}{2}]-1}$. Analogously for $p_{m,i}$.

Reciprocally, if conditions (37) are satisfied we have to show that there exists $f \in \mathbb{R}_n[u,v]$ such that $\Delta^2 f = p$, or equivalently, we have to show that all the linear systems LS_m for any m have a solution.

In Lemma 3, we have shown that conditions (37) were necessary, but also sufficient, to assure the existence of solution of the linear systems LS_m with $m \geq n+3$. Let us see what happens for $m < n+3$.

Let us recall that the homogeneous linear systems defined by Eq. 4 with $p_{k,\ell} = 0$ are related with the homogeneous biharmonic equation. In [8] and [4] it is shown that such linear systems have a unique solution once the first two rows and the first two columns of coefficients are given. Therefore the same happens for the linear systems LS_m because they have the same nonsingular matrix of coefficients. In fact, the matrix of coefficients is the same scalar tridiagonal matrix A appearing in the proof of Lemma 3. $\square$

The last part of the proof and results given in [8] allow to state the next result:

Corollary 1 *Given an admissible polynomial p_a there is a unique polynomial solution of $\Delta^2 f = p_a$ with a prescribed boundary.*

9.1 Proof of Th. 2

Let M_n denote the triangular matrix of the change of basis from the Bernstein basis $\{B_i^n(t)\}_{i=0}^n$ to the usual power basis $\{t^i\}_{i=0}^n$.

Let $R = \{r_{ij}\}_{i,j=0}^n$ be the control net of a polynomial function $p = p(u,v)$. The matrix of coefficients of p in the power basis is $M_n^T.R.M_n$.

Lemma 4 *Let $R = \{r_{ij}\}_{i,j=0}^{n}$, with $r_{ij} = 0$ for all $i, j \in \{1, 2, \ldots, n-1\}$,*

$$
\begin{array}{cccccc}
r_{00} & r_{01} & r_{02} & \ldots & r_{0n-1} & r_{0n} \\
r_{10} & 0 & 0 & \ldots & 0 & r_{1n} \\
r_{20} & 0 & 0 & \ldots & 0 & r_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
r_{n-1,0} & 0 & 0 & \ldots & 0 & r_{n-1,n} \\
r_{n0} & r_{n1} & r_{n2} & \ldots & r_{n,n-1} & r_{nn}
\end{array}
$$

be the control net of a polynomial function $p = p(u, v)$. Then, the matrix of coefficients of p in the power basis, $A = \{a_{ij}\}_{i,j=0}^{n} = M_n^T.R.M_n$, verifies the following condition:

For any $i, j \in \{0, \ldots, n-2\}$, the entry $a_{i,j}$ is a linear combination of $a_{i,n-1}$, $a_{n-1,j}$ and $a_{n-1,n-1}$, in the following way: there are scalars $\lambda_{i,j}^{k}$, $k = 1, 2, 3$, which don't depend on R, such that

$$a_{i,j} = \lambda_{i,j}^{1} a_{i,n-1} + \lambda_{i,j}^{2} a_{n-1,i} + \lambda_{i,j}^{3} a_{n-1,n-1}.$$

The next scheme tries to represent the three entries related to $a_{i,j}$.

$$
\begin{array}{ccccccc}
a_{0,0} & a_{0,1} & \ldots & a_{0,j} & \ldots & a_{0,n-1} & a_{0,n} \\
a_{1,0} & a_{1,1} & \ldots & a_{1,j} & \ldots & a_{1,n-1} & a_{1,n} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
a_{i,0} & a_{i,1} & \ldots & \mathbf{a_{i,j}} & \leftarrow & \mathbf{a_{i,n-1}} & a_{i,n} \\
\vdots & \vdots & \vdots & \uparrow & \nwarrow & \vdots & \vdots \\
a_{n-1,0} & a_{n-1,1} & \ldots & \mathbf{a_{n-1,j}} & \ldots & \mathbf{a_{n-1,n-1}} & a_{n-1,n} \\
a_{n,0} & a_{n,1} & \ldots & a_{n,j} & \ldots & a_{n,n-1} & a_{n,n}
\end{array}
$$

Proof: For any $i, j \in \{1, \ldots, n-2\}$ let $C_{i,j}(\lambda)$ (resp. $D_{i,j}(\mu)$) be the $(n+1) \times (n+1)$ matrix whose entries are

$$
\begin{array}{ll}
1, & \text{entry } i, j, \\
\lambda \text{ (resp. } \mu), & \text{entry } i, n-1 \text{ (resp. } n-1, j), \\
0, & \text{elsewhere.}
\end{array}
$$

Note that $C_{i,j}(\lambda)^T = D_{j,i}(\lambda)$.

Note that $C_{i,j}(\lambda)AD_{i,j}(\mu)$ is a matrix with null entries except the entry i, j whose value is

$$a_{i,j} + \lambda a_{n-1,j} + \mu a_{i,n-1} + \lambda\mu a_{n-1,n-1}.$$

We will prove that for some values of λ and μ the product $C_{i,j}(\lambda)AD_{i,j}(\mu)$ vanishes. This means that

$$a_{i,j} = -\lambda a_{n-1,j} - \mu a_{i,n-1} - \lambda\mu a_{n-1,n-1},$$

and the lemma is proved.

Let us define the polynomials $H_i^n(t) = \binom{n}{i}(t-1)^i$ for $n \in \mathbb{N}$ and $i \in \{0, 1, \ldots, n\}$. It is easy to check that the family $\{H_i^n(t)\}_{i=0}^n$ is a basis of polynomials of degree $\leq n$ and that the matrix of the change of basis from the basis $\{H_i^n(t)\}_{i=0}^n$ to the usual power basis $\{t^i\}_{i=0}^n$ is M_n^T.

We will understand the matrix $C_{i,j}(\lambda)$ as the matrix of control points with respect to the basis $\{H_i^n(t)\}_{i=0}^n$ of the polynomial function

$$\begin{aligned} f_{i,j}(u,v) &= H_i^n(u)H_j^n(v) + \lambda H_i^n(u)H_{n-1}^n(v) = H_i^n(u)\left(H_j^n(v) + \lambda H_{n-1}^n(v)\right) \\ &= \tbinom{n}{i}(u-1)^i\left(\tbinom{n}{j}(v-1)^j + \lambda n(v-1)^{n-1}\right). \end{aligned}$$

Note that the expression of $f_{i,j}(u,v)$ in the usual polynomial basis has no terms in $u^n v^k$, nor in $u^k v^n$, for any $k \in \{0, 1, \ldots, n\}$. Moreover,

$$f_{i,j}(u,0) = \binom{n}{i}(u-1)^i\left(\binom{n}{j}(-1)^j + \lambda n(-1)^{n-1}\right).$$

If we choose $\lambda_j = \frac{1}{n}\binom{n}{j}(-1)^{n-j}$ then $f_{i,j}(u,0) = 0$. Therefore, $f_{i,j}(u,v)$ has no terms in $u^k v^0$, for any $k \in \{0, 1, \ldots, n\}$.

That means that the product $M_n C_{i,j}(\lambda_j) M_n^T$, recall that it represents the matrix of coefficients in the usual power basis of $f_{i,j}(u,v)$, is a matrix with three of the border lines vanishing, i.e., a matrix of the kind shown in the next scheme (Scheme 39, left):

$$\begin{pmatrix} 0 & * & * & \ldots & * & 0 \\ 0 & * & * & \ldots & * & 0 \\ 0 & * & * & \ldots & * & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & * & * & \ldots & * & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ * & * & * & \ldots & * & 0 \\ * & * & * & \ldots & * & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & \ldots & * & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \tag{39}$$

Analogously for the matrix $D_{i,j}(\mu)$. There is a μ_i such that the product $M_n D_{i,j}(\mu_i) M_n^T$, is a matrix of the same kind, but with a different non vanishing border line (Scheme 39, right).

Finally, note that, since the matrix M_n is invertible, the product $C_{i,j}(\lambda_j) A D_{i,j}(\mu_i)$ vanishes if and only if the same happens for the product

$$M_n C_{i,j}(\lambda_j) A D_{i,j}(\mu_i) M_n^T = (M_n C_{i,j}(\lambda_j) M_n^T) R (M_n D_{i,j}(\mu_i) M_n^T).$$

Now, it is easy to check that

$$\begin{pmatrix} 0 * * \ldots * 0 \\ 0 * * \ldots * 0 \\ 0 * * \ldots * 0 \\ \vdots \vdots \vdots \ddots \vdots \vdots \\ 0 * * \ldots * 0 \\ 0\,0\,0 \ldots 0\,0 \end{pmatrix} \begin{pmatrix} r_{00} & r_{01} & r_{02} & \ldots & r_{0n-1} & r_{0n} \\ r_{10} & 0 & 0 & \ldots & 0 & r_{1n} \\ r_{20} & 0 & 0 & \ldots & 0 & r_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{n-1,0} & 0 & 0 & \ldots & 0 & r_{n-1,n} \\ r_{n0} & r_{n1} & r_{n2} & \ldots & r_{n,n-1} & r_{nn} \end{pmatrix} \begin{pmatrix} 0\,0\,0 \ldots 0\,0 \\ * * * \ldots * 0 \\ * * * \ldots * 0 \\ \vdots \vdots \vdots \ddots \vdots \vdots \\ * * * \ldots * 0 \\ 0\,0\,0 \ldots 0\,0 \end{pmatrix} = \begin{pmatrix} 0\,0\,0 \ldots 0\,0 \\ 0\,0\,0 \ldots 0\,0 \\ 0\,0\,0 \ldots 0\,0 \\ \vdots \vdots \vdots \ddots \vdots \vdots \\ 0\,0\,0 \ldots 0\,0 \\ 0\,0\,0 \ldots 0\,0 \end{pmatrix}$$

$\square$

Finally, the proof of Th. 2:

Proof (Th. 2): Let us consider the net of control points of a polynomial load $\{r_{i,j}\}_{i,j=0}^n$ and let us rewrite the polynomial load in terms of the usual basis

$$\sum_{i,j=0}^{n} B_i^n(u) B_j^n(v) r_{i,j} = p(u,v) = \sum_{i,j=0}^{n} \frac{p_{i,j}}{i!j!} u^i v^j.$$

Recall that coefficients $p_{i,j}$ can be computed from the control points $r_{i,j}$ using linear expressions.

According to Corollary 1, the determination of the exterior control points is equivalent to solving the linear system given by Eqs. (37) after substituting the coefficients $p_{i,j}$ in terms of the $r_{i,j}$. This linear system has $4n$ equations and the same number of unknowns, the $4n$ boundary control points. In order to prove that the linear system has a solution and it is unique, it is enough to check that when the interior control points are all zero, then the trivial solution is the only possible solution.

Comparing the matrix in Lemma 4 with the matrix (38) we get that coefficients in the right bottom corner vanish:

$p_{i,j}$	0	1	...	$n-4$	$n-3$	$n-2$	$n-1$	n
0	*	*	...	*	*	*	*	*
1	*	*	...	*	*	*	*	*
⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮	⋮
$n-4$	*	*	...	*	*	*	*	*
$n-3$	*	*	...	*	*	*	0	0
$n-2$	*	*	...	*	*	*	0	0
$n-1$	*	*	...	*	0	0	0	0
n	*	*	...	*	0	0	0	0

(40)

Now, Lemma 4 implies that

$$p_{n-3,n-3} = p_{n-2,n-3} = p_{n-3,n-2} = p_{n-2,n-2} = 0.$$

$p_{i,j}$	0	1	...	$n-4$	$n-3$	$n-2$	$n-1$	n
0	*	*	...	*	*	*	*	*
1	*	*	...	*	*	*	*	*
⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮	⋮
$n-4$	*	*	...	*	*	*	*	*
$n-3$	*	*	...	*	0	0	0	0
$n-2$	*	*	...	*	0	0	0	0
$n-1$	*	*	...	*	0	0	0	0
n	*	*	...	*	0	0	0	0

(41)

Proposition 2 implies that

$$p_{n-5,n} = p_{n-4,n} = p_{n-5,n-1} = p_{n-4,n-1} = 0$$
$$p_{n,n-5} = p_{n,n-4} = p_{n-1,n-5} = p_{n-1,n-4} = 0.$$

$p_{i,j}$	0	1	...	$n-4$	$n-3$	$n-2$	$n-1$	n
0	*	*	...	*	*	*	*	*
1	*	*	...	*	*	*	*	*
⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋮	⋮
$n-4$	*	*	...	*	*	*	0	0
$n-3$	*	*	...	*	0	0	0	0
$n-2$	*	*	...	*	0	0	0	0
$n-1$	*	*	...	0	0	0	0	0
n	*	*	...	0	0	0	0	0

(42)

After a finite process, we get that all $p_{i,j} = 0$, and so $p(u, v) = 0$. The only possible solution is the trivial one. This shows that the linear system we have to solve is compatible and determined; therefore, there is a solution and it is unique. □

Let us see what happens for an analogous problem in terms of Bézier curves. We will need two lemmas:

Lemma 5 *If a Bézier curve of degree $n-2$, $\alpha = \mathcal{B}[P_0, P_1, \ldots, P_{n-2}]$, can be written as $\alpha = \mathcal{B}[Q_0, 0, \ldots, 0, Q_n]$ when it is considered as a Bézier curve of degree n, then $\alpha \equiv 0$.*

Proof: The well-known formulas of raising the degree of a Bézier curve say that the relation between the control points $\{Q_i\}_{i=0}^n$ and $\{P_i\}_{i=0}^{n-2}$ is

$$Q_i = \frac{1}{n(n-1)} \left((n-i)(n-i-1)P_i + 2i(n-i-1)P_{i-1} + i(i+1)P_{i-2}\right),$$

for $i = 0, 1, \ldots, n$, where $P_{-1} = P_{-2} = P_{n-1} = P_n = 0$.

Conditions $Q_i = 0$ for $i = 1, 2, \ldots, n-1$ can be written as a homogeneous linear system of $n-1$ equations with $n-1$ unknowns $\{P_i\}_{i=0}^{n-2}$. The matrix of coefficients, $M_{n-1} = (m_{ij})_{i,j=1}^{n-1}$ is given by

$$m_{ii} = 2i(n-i-1), m_{i+1,i} = (n-i)(n-i-1), m_{i,i+1} = i(i+1),$$

and the rest of entries null. Now it is just a matter of computation to check that $\det M_{n-1} \neq 0$. In fact $\det M_n = \frac{n(n-1)}{2} \det M_{n-1}$ and $\det M_1 = 1$. $\square$

Note that the matrix of coefficients only has the three main diagonal lines with no null entries.

Lemma 6 *If a Bézier curve of degree n, $\alpha = \mathcal{B}[0, P_1, \ldots, P_{n-1}, 0]$, is such that $\alpha'' \equiv 0$, then $\alpha \equiv 0$.*

Proof: Since the control points of α'' are $n(n-1)(P_i - 2P_{i+1} + P_{i+2})$, with $P_0 = P_n = 0$, then condition $\alpha'' \equiv 0$ can be written as a homogeneous linear system of $n-1$ equations with $n-1$ unknowns whose matrix is the regular Toeplitz matrix $T^{n-1}_{1,-2,1}$. $\square$

Lemma 7 *Let $\alpha = \mathcal{B}[0, P_1, \ldots, P_{n-1}, 0]$ be a Bézier curve of degree n, such that $\alpha'' = \mathcal{B}[Q_0, 0, \ldots, 0, Q_n]$, then $\alpha \equiv 0$.*

Proof: The second derivative α'' is a Bézier curve of degree $n-2$. If their control points, when α'' is considered as a Bézier curve of degree n, are of the kind, $Q_0, 0, \ldots, 0, Q_n$, then by Lemma 5, we get $\alpha'' \equiv 0$. Applying now Lemma 6, we get $\alpha \equiv 0$. $\square$

In this case it is also possible to reduce the problem to show that the unique solution of some homogeneous system of linear equations is the trivial one. The associated matrix of coefficients is the product of the matrices M_{n-1} and $T^{n-1}_{1,-2,1}$. Now, it is a matrix that has the five main diagonal lines with no null entries.

It is not difficult to check the next statement for Bézier curves again.

Lemma 8 *Let* $\alpha = \mathcal{B}[0,0,P_2,\ldots,P_{n-2},0,0]$ *be a Bézier curve of degree* n*, such that* $\alpha^{(iv)} = \mathcal{B}[Q_0,Q_1,0,\ldots,0,Q_{n-1},Q_n]$*, then* $\alpha \equiv 0$.

Its proof is again a consequence of factorizing the matrix of coefficients associated to the problem as a product of two regular matrices. The first associated to the raising of degree of Bézier curves. The second, a Toeplitz matrix associated to the incremental operator applied to control points. The difference, with respect to the previous lemmas, is that now the non null main diagonals in the two factor matrices is five, and in the matrix of coefficients it is nine.

The analogous result for Bézier surfaces and for the biharmonic operator can be stated as follows:

Lemma 9 *Let* f *and* p *be polynomial functions of degree* n *in* u *and* v *with* $\{P_{ij}\}_{i,j=0}^n$ *and* $\{R_{ij}\}_{i,j=0}^n$*, respectively, the associated control points verifying*

(1) $P_{ij} = 0$ *if* $i,j \notin \{2,\ldots,n-2\}$,
(2) $R_{ij} = 0$ *if* $i,j \in \{2,\ldots,n-2\}$*, and*
(3) $\Delta^2 f = p$

then $f = p \equiv 0$.

We have checked using a symbolic program (Mathematica) that the result is true for degrees $n \leq 20$. Nevertheless, we have been unable to find a general and simple proof for Lemma 9.

Finally, the proof of Th. 3

Proof (Th. 3): First of all, note that uniqueness is a consequence of Lemma 9. We will now prove the existence.

Let f_0 be the unique polynomial solution of $\Delta^2 f = 0$ with the same boundary, and let $\{Q_{i,j}\}_{i,j,=0}^n$ be its control net. So the problem can be reduced to the following situation:

Finding an admissible load and a polynomial solution of $\Delta^2 f = p_a$ whose boundary and adjacent to the boundary control points are

$$\begin{array}{ccccccc}
0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & P_{1,1}-Q_{1,1} & P_{1,2}-Q_{1,2} & \cdots & P_{1,n-2}-Q_{1,n-2} & P_{1,n-1}-Q_{1,n-1} & 0 \\
0 & P_{2,1}-Q_{2,1} & * & \cdots & * & P_{2,n-1}-Q_{2,n-1} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\
0 & P_{n-2,1}-Q_{n-2,1} & * & \cdots & * & P_{n-2,n-1}-Q_{n-2,n-1} & 0 \\
0 & P_{n-1,1}-Q_{n-1,1} & P_{n-1,2}-Q_{n-1,2} & \cdots & P_{n-1,n-2}-Q_{n-1,n-2} & P_{n-1,n-1}-Q_{n-1,n-1} & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{array} \tag{43}$$

We only have to consider a load of the kind (5) with unknown boundary control points

$$\begin{array}{ccccccc}
* & * & * & \cdots & * & * & * \\
* & R_{1,1} & R_{1,2} & \cdots & R_{1,n-2} & R_{1,n-1} & * \\
* & R_{2,1} & 0 & \cdots & 0 & R_{2,n-1} & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\
* & R_{n-2,1} & 0 & \cdots & 0 & R_{n-2,n-1} & * \\
* & R_{n-1,1} & R_{n-1,2} & \cdots & R_{n-1,n-2} & R_{n-1,n-1} & * \\
* & * & * & \cdots & * & * & *
\end{array} \tag{44}$$

Theorem 2 implies that, in order for p_a to be admissible, the boundary control points are totally determined as linear combinations of the adjacent to the exterior ones.

For a given admissible load, p_a, and taking zero as the boundary, there is a unique solution of $\Delta^2 f = p_a$. Let us denote it by f_{p_a}. The map, $M : (\mathbb{R}^3)^{4n-8} \to (\mathbb{R}^3)^{4n-8}$, that assigns to any configuration of the kind (44) of p_a the adjacent to the boundary control points of the unique solution f_{p_a} is a linear map. Indeed, the adjacent to the boundary control points of f_{p_a} are a linear combination of the adjacent to the boundary control points of p_a (bear in mind that the boundary control points of p_a are also a linear combination of the adjacent to the boundary control points of p_a).

Finally, uniqueness implies that the linear map M is a bijective map. Therefore, for any boundary configuration there always exists an admissible load of the kind (44). $\square$

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