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A characterization of graded symplectic structures

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Abstract: We give a characterization of graded symplectic forms by studying the module of derivations of a graded sheaf. When the graded sheaf is the sheaf of differentiable forms on the underlying manifold M, we find canonical liftings from metrics on TM to odd symplectic forms, and from symplectic forms on M and metrics on TM to even symplectic forms. These graded symplectic forms give rise to canonical Poisson brackets on the graded manifold.

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Introduction

Recently some papers studying extensions of Poisson brackets on the algebra of differentiable functions on a manifold M, to Poisson brackets on different algebras have appeared. The algebras involved are, mainly, the Cartan algebra, i.e., the algebra of differential forms on the manifold, $A(M)$, or the algebra of multivectors, $V(M)$. See for example [11, 2, 8].

In classical mechanics, the first examples of Poisson brackets are those defined by means of a symplectic form. Similarly, it would be convenient to find out if it is possible to define a kind of objects related with the new algebra in such a way that they play the role of symplectic structures and then to associate a Poisson bracket to them.

The pairs $(M, A(M))$ and $(M, V(N))$ are typical examples of graded manifolds in the sense of Kostant [7]. Then, it seems natural to look for Poisson brackets on bigger algebras using the definitions and methods of the theory of graded manifolds.

The purpose of the present work is twofold. First, to give a characterization of graded symplectic structures using an intrinsic representation of graded vector fields,

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and second, to apply this characterization in order to build new Poisson brackets.

Graded symplectic forms are already defined in [7], and the author shows there that such graded structures define a unique symplectic structure on the underlying manifold. This is the first step towards a full characterization of graded symplectic forms.

A second step can be found in [5]. The authors study the problem of constructing graded symplectic forms from a given symplectic form on the underlying manifold. They find a non-explicit method to build exact graded symplectic structures on a graded manifold from a given exact symplectic structure on the underlying manifold. The general case has been solved in [16]. There the author finds an explicit description of even graded symplectic forms which can be seen as complementary of the present work.

In [13] we give a characterization of the derivations of the algebra of local sections of an exterior bundle. This result can be applied to the theory of graded manifolds via Batchelor's theorem. This theorem asserts that the graded sheaf of any graded manifold is isomorphic to the sheaf of local sections of an exterior bundle, $\Gamma(\cdot, \Lambda E)$, for a suitable vector bundle over the underlying manifold, E. We have thus a manageable representation of graded vector fields that will allow us to characterize graded forms.

Given a linear connection on E, we associate, in a unique way, to any graded symplectic form a set of fields defined on some bundles over M. Among them we find, for the case of an even symplectic form, the underlying symplectic form and a metric on the dual vector bundle E^*; and for the case of an odd symplectic form, a linear isomorphism between the tangent bundle of the underlying manifold and the bundle E. These three fields are independent of the linear connection. Moreover, we show that any closed graded 2-form induces a homomorphism, independent of the linear connection, between $TM \oplus E^*$ and $T^*M \oplus E$, and that the closed graded 2-form is symplectic if and only if it induces an isomorphism.

As an application of this characterization we construct three kinds of graded symplectic forms, all of them independent of the linear connection, and study their Poisson bracket.

First, for any Riemannian manifold there is an associated odd symplectic form of the graded manifold $(M, \Gamma(\Lambda T^*M))$.

Second, if in addition we have a symplectic form on M, we obtain a canonical lifting to an even symplectic form, in such a way that the graded Poisson bracket associated to the even symplectic form is an extension of the initial Poisson bracket. When the underlying manifold is of dimension 2, we show a relation between the graded symplectic form and the Gauss curvature. All these constructions are natural.

And third, following these methods, we prove a known fact: the Schouten-Nijenhuis bracket is the Poisson bracket of an odd symplectic form.

Finally let me say a word about the role that graded symplectic forms play in graded (or super) mechanics. It is well-known the relation that exists between Lagrangian and Hamiltonian mechanics, and that symplectic structures are a basic geometric tool for the Hamiltonian formulation of mechanics. A similar relation must be true for super- or graded mechanics, and an unavoidable step in the way from Lagrangian problems to Hamiltonian ones, in the setting of graded manifolds, is the concept of graded symplectic form.

A deduction of the variational equations for graded and Berezinian Lagrangian densities has been recently presented in [15]. In [14] some significant examples are fully developed. On the other hand, in [17] and [14] there are particular examples of graded Hamilton equations. We hope that the characterization of graded symplectic forms will make easier to translate Lagrangian problems to Hamiltonian ones.

1. Definitions

For the definition of graded manifolds we will follow [7].

Let $\pi : E \to M^m$ be a real vector bundle where the dimension of M is m and the fibre dimension is n, and let $\pi : \Lambda E \to M$ be the exterior bundle of E.

Let $\Gamma(\cdot, \Lambda E)$ be the sheaf of exterior $\mathbb{R}$-algebras of local smooth sections of ΛE (all objects are C^∞). The pair $(M, \Gamma(\cdot, \Lambda E))$ is a graded manifold of graded dimension (m, n). Batchelor's theorem [1] asserts that any graded manifold is isomorphic, but not canonically, to a graded manifold $(M, \Gamma(\cdot, \Lambda E))$ for a suitable E. Thus we will focus our attention on such kind of graded manifolds.

If $s \in \Gamma(\Lambda E)$ is homogeneous, say of degree p, then we will write $|s| = p$. If $s \in \Gamma(\Lambda E)$, we denote by $s_{(p)}$ the component of s of degree p; thus $s = \sum_{p=0}^{n} s_{(p)}$.

The map $\kappa : \Gamma(\Lambda E) \to C^\infty(M)$ defined by $\kappa(s) = s_0$ is called the natural morphism of the graded manifold $(M, \Gamma(\cdot, \Lambda E))$.

A linear endomorphism $D : \Gamma(\Lambda E) \to \Gamma(\Lambda E)$ is said to be homogeneous of degree $|D|$ if $|D(s_{(p)})| = p + |D|$.

A homogeneous linear homomorphism $D : \Gamma(\Lambda E) \to \Gamma(\Lambda E)$ is called a derivation of degree $|D|$ if, for homogeneous $s_1, s_2 \in \Gamma(\Lambda E)$,

$$D(s_1 s_2) = (D s_1) s_2 + (-1)^{|D||s_1|} s_1 (D s_2).$$

We fix the following terminology: when we refer to the $\mathbb{Z}_2$-grading, we shall talk of *even* or *odd* objects if they are of degree 0 or 1, respectively. We shall reserve the expression *object of degree* ... for the $\mathbb{Z}$-grading.

Every homogeneous derivation is determined by its action on the elements of degree 0 and 1. Thus all derivations of degree less than -1 are zero. A linear endomorphism of $\Gamma(\Lambda E)$ is called a derivation if its homogeneous parts are derivations.

Let $\mathcal{M}, \mathcal{N}$ be graded $\Gamma(\Lambda E)$-modules. A morphism $\mathcal{S} : \mathcal{M} \to \mathcal{N}$ is called of degree $|\mathcal{S}|$ if $\mathcal{S}(x_{(p)})$ is of degree $|\mathcal{S}| + p$, and is called a $\Gamma(\Lambda E)$-endomorphism if $\mathcal{S}(sx) = (-1)^{|\mathcal{S}||s|} s\mathcal{S}(x)$, $s \in \Gamma(\Lambda E), x \in \mathcal{M}$.

Two classes of derivations on $\Gamma(\Lambda E)$. Let $TM \to M$ be the tangent bundle over M and $\Gamma(TM \otimes \Lambda E)$ the $\Gamma(\Lambda E)$-module of smooth sections of $TM \otimes \Lambda E$. We can define in $\Gamma(TM \otimes \Lambda E)$ a $\mathbb{Z}$- and a $\mathbb{Z}_2$-grading. Every $\Psi \in \Gamma(TM \otimes \Lambda E)$ can be expressed as a finite sum of decomposable homogeneous sections $X \otimes s_{(p)}$ where $s_{(p)}$ is a homogeneous section of $\Gamma(\Lambda E)$ of degree p, and $X \in \Gamma(TM)$.

Let ∇ be a linear connection in E. If $\Psi = X \otimes s_{(p)}$, we define the endomorphism $\nabla_\Psi : \Gamma(\Lambda E) \to \Gamma(\Lambda E)$ by $\nabla_\Psi u = s_{(p)} \nabla_X u$, where $u \in \Gamma(\Lambda E)$, and if $\Psi \in \Gamma(TM \otimes \Lambda E)$, we define ∇_Ψ by linear extension.

∇_Ψ is a derivation and we call it the *proper derivation associated to* Ψ *through* ∇. If Ψ is a homogeneous element, in whatever grading, then ∇_Ψ is a derivation of degree $|\Psi|$.

Now, we shall define another type of derivations, the algebraic ones.

Let $\pi : E^* \to M$ be the dual bundle of E, and let $\Gamma(E^* \otimes \Lambda E)$ be the $\Gamma(\Lambda E)$-module of smooth sections of $E^* \otimes \Lambda E$. We define in $\Gamma(E^* \otimes \Lambda E)$ a $\mathbb{Z}$- and a $\mathbb{Z}_2$-grading.

Every $\Phi \in \Gamma(E^* \otimes \Lambda E)$ can be expressed as a finite sum of decomposable homogeneous sections $\alpha \otimes s_{(p)}$ where $s_{(p)}$ is a homogeneous section of $\Gamma(\Lambda E)$ of degree p, and $\alpha \in \Gamma(E^*)$.

If $\Phi = \alpha \otimes s_{(p)}$, we define the endomorphism $i_\Phi : \Gamma(\Lambda E) \to \Gamma(\Lambda E)$ by $i_\Phi u = s_{(p)} i_\alpha u$, where $u \in \Gamma(\Lambda E)$, and where i_α is the interior multiplication; and if $\Phi \in \Gamma(E^* \otimes \Lambda E)$, we define i_Φ by linear extension.

i_Φ is a derivation that acts trivially on the smooth functions on M, and we call it the *algebraic derivation associated to* Φ. If Φ is a homogeneous element, in either grading, then i_Φ is a derivation of degree $|\Phi| - 1$ (modulo 2 if we are dealing with the $\mathbb{Z}_2$-grading).

Characterization of the derivations on $\Gamma(\Lambda E)$. The graded vector fields of a graded manifold are defined as the derivation of the sheaf of algebras, like vector fields of a differentiable manifold are the derivations of the sheaf of algebras of differentiable functions. Therefore, the graded vector fields of the graded manifold $(M, \Gamma(\cdot, \Lambda E))$ are the derivations on $\Gamma(\Lambda E)$. With the help of a linear connection on E, we can characterize the graded vector fields on $(M, \Gamma(\cdot, \Lambda E))$.

The following characterization can be found in [13] and is analogous to that of Frölicher-Nijenhuis [4].

Proposition 1.1. *Let D be a derivation on $\Gamma(\Lambda E)$, and let ∇ be a connection in E. Then, there are unique fields $\Psi \in \Gamma(TM \otimes \Lambda E)$ and $\Phi \in \Gamma(E^* \otimes \Lambda E)$ such that*

$$D = i_\Phi + \nabla_\Psi .$$

If ∇' is another connection in E, then we can express the derivation ∇_X in terms of ∇'. For $\nabla_X - \nabla'_X$ is an algebraic derivation, so there exists $A(X) \in \Gamma(E^* \otimes E)$ such that $\nabla_X = \nabla'_X + i_{A(X)}$.

Let us recall the following concept.

Exterior covariant derivative. Let F be a real vector bundle on M and let ∇ be a connection on F and denote by d^∇ the exterior covariant derivative

$$d^\nabla : \Gamma(\Lambda^k(T^*M) \otimes F) \to \Gamma(\Lambda^{k+1}(T^*M) \otimes F),$$

given by

$$(d^\nabla\Phi)(X_0,\ldots,X_k)=\sum_{i=0}^{p}(-1)^i\nabla_{X_i}(\Phi(X_0,\ldots,\widehat{X_i},\ldots,X_k))$$
$$+\sum_{i<j}(-1)^{i+j}\Phi([X_i,X_j],X_0,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_k),$$

where $\Phi\in\Gamma(\Lambda^k(T^*M)\otimes F)$.

As an example of application of proposition 1.1 let us apply it to a well known derivation. When $E=T^*M$, the sheaf is that of differentiable forms on M and we shall denote it by $A(M)$. When $E=TM$, the sheaf is that of polivectors on M and we shall denote it by $V(M)$. If X is a vector field, the Lie derivative with respect to X, $\mathcal{L}_X$ is a derivation both on $A(M)$ and on $V(M)$.

Lemma 1.2. *Let ∇ be a linear connection on TM, then,*

(1) *as derivations of $A(M)$, $\mathcal{L}_X=\nabla_X+i_{d^\nabla X+T^\nabla(X,\cdot)}$,*
(2) *as derivations of $V(M)$, $\mathcal{L}_X=\nabla_X-i_{d^\nabla X+T^\nabla(X,\cdot)}$,*

where $T^\nabla\in\Gamma(\Lambda^2T^*M\otimes TM)$ is the torsion of ∇.

Proof. (1) If $f\in C^\infty(M)$, then $\mathcal{L}_X(f)=\nabla_X(f)$. Let α be a 1-form. $(\mathcal{L}_X-\nabla_X)(\alpha)$ is a 1-form. Let Z be a vector field on M, then

$$\begin{aligned}(\mathcal{L}_X-\nabla_X)(\alpha)Z&=X(\alpha(Z))-\alpha([X,Z])-X(\alpha(Z))+\alpha(\nabla_XZ)\\&=\alpha(\nabla_ZX+T^\nabla(X,Z))\\&=i_{d^\nabla X+T^\nabla(X,\cdot)}(\alpha)Z.\end{aligned}$$

(2) Simply note that $(\mathcal{L}_X-\nabla_X)(Y)=[X,Y]-\nabla_XY=-\nabla_YX-T^\nabla(X,Y)$. $\square$

2. Graded forms on $(M,\Gamma(\Lambda E))$

Graded forms are $\Gamma(\Lambda E)$-multilinear alternating homomorphisms from the module of graded vector fields into $\Gamma(\Lambda E)$. A graded p-form, $p\in\mathbb{N}$, is then a p-multilinear alternating homomorphism.

To study graded p-forms it is convenient to define a $\mathbb{Z}$-grading for them.

Definition. A graded p-form λ is said to be of degree $k\in\mathbb{N}$ if for all $D_1,\ldots,D_p\in\mathrm{Der}(\Gamma(\Lambda E)))$

$$|\lambda(D_1,\ldots,D_p)|=\sum_{i=1}^{p}|D_i|+k.$$

Remark. We will denote by $\lambda(D)$ the action of the graded form on the derivation, but having in mind that the action is on the right, i.e., in the notation of [7], it is $\langle D,\lambda\rangle$.

The natural morphism $\kappa : \Gamma(\Lambda E) \to C^\infty(M)$ can be extended to an algebra homomorphism, that will also be called κ, that sends graded p-forms to differentiable p-forms on M. (See [7, page 257]). If λ is a graded p-form, then $\kappa\lambda$ is the differentiable p-form defined by

$$(\kappa\lambda)(X_1,\ldots,X_p) = \left(\lambda(\nabla_{X_1},\ldots,\nabla_{X_p})\right)_{(0)} \in C^\infty(M).$$

Note that this definition does not depend on the connection due to the following argument: The action of a graded form of algebraic derivations of degree 0 gives rise to elements of $\Gamma(\Lambda E)$ of degree greater than 0. Indeed, if $\alpha \in \Gamma(E^*)$ and $D_2,\ldots,D_p \in \mathrm{Der}(\Gamma(\Lambda E)))$ then $\lambda(i_\alpha, D_2,\ldots,D_p) \in \Gamma(\Lambda E)$. For a homogenous algebraic derivation of degree 0, $i_{\alpha\otimes\gamma}$, where $\alpha\otimes\gamma \in \Gamma(E^*\otimes E)$ we have that

$$\lambda(i_{\alpha\otimes\gamma}, D_2,\ldots,D_p) = \lambda(i_\alpha, D_2,\ldots,D_p)\gamma.$$

Thus $(\lambda(i_{\alpha\otimes\gamma}, D_2,\ldots,D_p))_{(0)} = 0$.

Obviously, if λ is a graded p-form of degree $k > 0$ then $\kappa\lambda = 0$.

Definition. The graded differential of a graded p-form, λ, is the graded $p+1$-form, $d^G\lambda$, defined by

$$(d^G\lambda)(D_1,\ldots,D_{p+1}) = \sum_{i=1}^{p+1}(-1)^{i-1+j_{i-1}|D_i|}D_i(\lambda(D_1,\ldots,\widehat{D_i},\ldots,D_{p+1}))$$
$$+\sum_{k<\ell}(-1)^{d_{k,\ell}}\lambda([D_k,D_\ell],D_1,\ldots,\widehat{D_k},\ldots,\widehat{D_\ell},\ldots,D_{p+1}),$$

where $j_i = \sum_{k=1}^{i}|D_k|$ and $d_{k,\ell} = |D_k|j_{k-1} + |D_\ell|j_{\ell-1} + |D_k||D_\ell| + k + \ell$.

In particular, for a graded 1-form, λ,

$$(d^G\lambda)(D_1,D_2) = D_1(\lambda(D_2)) - (-1)^{|D_1||D_2|}D_2(\lambda(D_1)) - \lambda([D_1,D_2]).$$

Note that the graded differential preserves the $\mathbb{Z}$-grading defined previously, i.e., if λ is a graded p-form of degree k, then $d^G\lambda$ is a graded $p+1$-form of the same degree k.

Is is easy to check that the graded differential commutes with κ, therefore κ is a cochain complex morphism. A fundamental result, [7, Theorem 4.7], states that κ induces an isomorphism in cohomology. As a corollary that will be used frequently we have

Corollary 2.1. *Every closed graded p-form of degree $k > 0$ is exact.*

3. Characterization of the graded 1-forms on $(M, \Gamma(\Lambda E))$

Graded 1-forms are $\Gamma(\Lambda E)$-linear homomorphisms from the module of graded vector fields into $\Gamma(\Lambda E)$. By Proposition 1.1 we have that a graded vector field, i.e., a derivation D, is uniquely determined by two objects: $\Psi \in \Gamma(TM\otimes\Lambda E)$ and $\Phi \in \Gamma(E^*\otimes\Lambda E)$.

Moreover, the maps from $\mathrm{Der}(\Gamma(\Lambda E))$ into $\Gamma(TM \otimes \Lambda E)$ and $\Gamma(E^* \otimes \Lambda E)$ defined by $D \to \Psi$ and $D \to \Phi$, respectively, are $\Gamma(\Lambda E)$-homomorphisms of $\Gamma(\Lambda E)$-modules. Indeed, if $s \in \Gamma(\Lambda E)$, $s\nabla_\Psi = \nabla_{s\Psi}$ and $si_\Phi = i_{s\Phi}$. Thus, a graded 1-form is completely determined by its action on the sets of derivations $\{\nabla_X\}$, where X is a vector field and $\{i_\alpha\}$ where $\alpha \in \Gamma(E^*)$.

Proposition 3.1. *Let ∇ be a linear connection on E. If λ is a graded 1-form of degree k, then λ is uniquely determined by two fields $L \in \Gamma(E \otimes \Lambda^{k-1}E)$ and $K \in \Gamma(T^*M \otimes \Lambda^k E)$.*

Proof. L and K are uniquely defined by

$$\begin{aligned}
\lambda(\nabla_X) &=: K(X;\cdot\,) \in \Gamma(\Lambda^k E), \qquad && \text{for all vector fields } X,\\
\lambda(i_\alpha) &=: L(\alpha;\cdot\,) \in \Gamma(\Lambda^{k-1} E), \qquad && \text{for all } \alpha \in \Gamma(E^*). \quad \square
\end{aligned}$$

Let us denote by $\lambda^\nabla_{(L,K)}$ the graded 1-form defined by $L \in \Gamma(E \otimes \Lambda^{k-1}E)$ and $K \in \Gamma(T^*M \otimes \Lambda^k E)$.

Note. If $k = 0$ then λ is just defined by $K \in \Gamma(T^*M)$, that is, by a differential 1-form on M, indeed, $K = \kappa\lambda$.

Given (L, K), $\lambda^\nabla_{(L,K)}$ is not, in general, independent of ∇. But we have the following

Proposition 3.2. *Let $K \in \Gamma(T^*M \otimes \Lambda E)$, then the graded 1-form $\lambda^\nabla_{(0,K)}$ is independent of the linear connection ∇, and we will denote it by $\lambda_{(0,K)}$.*

Proof. For a given ∇, $\lambda^\nabla_{(0,K)}$ is defined by

$$\lambda^\nabla_{(0,K)}(\nabla_X) = K(X;\cdot\,) \in \Gamma(\Lambda E) \quad \text{and} \quad \lambda^\nabla_{(0,K)}(i_\alpha) = 0.$$

Let ∇' be another linear connection, then we have $\nabla_X = \nabla'_X + i_{A(X)}$. In order to check that $\lambda^\nabla_{(0,K)} = \lambda^{\nabla'}_{(0,K)}$ it is enough to check that they agree acting on the derivations ∇_X and i_α. Indeed

$$\begin{aligned}
\lambda^{\nabla'}_{(0,K)}(\nabla_X) &= \lambda^{\nabla'}_{(0,K)}(\nabla'_X + i_{A(X)})\\
&= \lambda^{\nabla'}_{(0,K)}(\nabla'_X) = K(X;\cdot\,) = \lambda^\nabla_{(0,K)}(\nabla_X). \quad \square
\end{aligned}$$

Closed graded 1-forms. Let $\lambda = \lambda^\nabla_{(L,K)}$ be a closed graded 1-form of degree $k > 0$, where $L \in \Gamma(E \otimes \Lambda^{k-1}E)$ and $K \in \Gamma(T^*M \otimes \Lambda^k E)$. By Corollary 2.1 we know that there exists a graded 0-form, $s \in \Gamma(\Lambda E)$, such that $\lambda = d^G s$. Since λ is of degree k, $s \in \Gamma(\Lambda^k E)$. $d^G s$ is the graded 1-form defined by

$$\begin{aligned}
d^G s(\nabla_X) &= \nabla_X s \in \Gamma(\Lambda^k E),\\
d^G s(i_\alpha) &= i_\alpha s \in \Gamma(\Lambda^{k-1} E).
\end{aligned}$$

Then, according with Proposition 3.1, $\lambda = \lambda^{\nabla}_{(s,d^{\nabla}s)}$. Therefore, $L = s \in \Gamma(\Lambda^k E) \subset \Gamma(E \otimes \Lambda^{k-1}E)$ and $K = d^{\nabla}s \in \Gamma(T^*M \otimes \Lambda^k E)$.

Then, we get the following

Proposition 3.3. *Let $\lambda = \lambda^{\nabla}_{(L,K)}$ be a graded 1-form of degree $k > 0$, where $L \in \Gamma(E \otimes \Lambda^{k-1}E)$ and $K \in \Gamma(T^*M \otimes \Lambda^k E)$. Then λ is closed if and only if $L \in \Gamma(\Lambda^k E)$ and $K = d^{\nabla}L$. If $\lambda = \lambda^{\nabla}_{(0,K)}$ is of degree 0, where $K \in \Gamma(T^*M)$, then it is closed if and only if $\kappa\lambda = K$ is closed.*

Remark. This can be also proved without using the fact that the natural morphism, κ, induces an isomorphism in cohomology. Indeed, if we write directly $(d^G\lambda)(\nabla_X, \nabla_Y) = 0$, $(d^G\lambda)(\nabla_X, i_\beta) = 0$ and $(d^G\lambda)(i_\alpha, i_\beta) = 0$ we obtain the same result.

The two fields L and K of the previous proposition are not unique. In order to choose a representative of the equivalence class we need to define the following map. Let $\varphi : \Gamma(E \otimes \Lambda E) \to \Gamma(\Lambda E)$ defined by

$$\varphi(L)(\alpha_1, \ldots, \alpha_k) = \frac{1}{k}\sum_{i=1}^{k}(-1)^{i-1}L(\alpha_i; \alpha_1 \ldots, \widehat{\alpha_i}, \ldots, \alpha_k),$$

if $L \in \Gamma(E \otimes \Lambda^{k-1}E)$.

Let us define $\Gamma^s(E \otimes \Lambda E)$ as the kernel of φ. Note that $\Gamma^s(E \otimes E) = \Gamma(S^2E)$.

Let $\lambda = \lambda^{\nabla}_{(L,K)}$ be a graded 1-form of degree $k > 0$, where $L \in \Gamma(E \otimes \Lambda^{k-1}E)$ and $K \in \Gamma(T^*M \otimes \Lambda^k E)$. Let us define $\bar{L} = L - \varphi(L) \in \Gamma^s(E \otimes \Lambda^{k-1}E)$, and $\bar{K} = K - d^{\nabla}\varphi(L) \in \Gamma(T^*M \otimes \Lambda^k E)$. Then $d^G\lambda^{\nabla}_{(L_1,K_1)} = d^G\lambda^{\nabla}_{(L_2,K_2)}$ if and only if $\bar{L_1} = \bar{L_2}$ and $\bar{K_1} = \bar{K_2}$. We get then the following

Proposition 3.4. *Each class of the quotient of graded forms modulo closed forms is uniquely determined by two fields $L \in \Gamma^s(E \otimes \Lambda E)$ and $K \in \Gamma(T^*M \otimes \Lambda E)$.*

4. Characterization of graded symplectic forms on $(M, \Gamma(\Lambda E))$

As we have done for graded 1-forms, we can characterize graded 2-forms by means of some tensors.

Proposition 4.1. *Let ∇ be a fixed connection on E. If ω is a graded 2-form of degree k, then ω is uniquely determined by three fields $L \in \Gamma(S^2E \otimes \Lambda^{k-2}E)$, $K \in \Gamma(T^*M \otimes E \otimes \Lambda^{k-1}E)$, and $J \in \Gamma(\Lambda^2 T^*M \otimes \Lambda^k E)$.*

Proof. L, K and J are uniquely defined by

$$\begin{aligned}
\omega(\nabla_X, \nabla_Y) &= J(X, Y; \cdot\,) \in \Gamma(\Lambda^k E),\\
\omega(\nabla_X, i_\beta) &= K(X; \beta; \cdot\,) \in \Gamma(\Lambda^{k-1}E),\\
\omega(i_\alpha, i_\beta) &= L(\alpha, \beta; \cdot\,) \in \Gamma(\Lambda^{k-2}E).
\end{aligned}$$

Note that since $\omega(i_\alpha, i_\beta) = \omega(i_\beta, i_\alpha)$ we have $L(\alpha,\beta;\cdot\,) = L(\beta,\alpha;\cdot\,)$, and that since $\omega(\nabla_X, \nabla_Y) = -\omega(\nabla_Y, \nabla_X)$ we have $J(X,Y;\cdot\,) = -J(Y,X;\cdot\,)$. $\square$

Let us denote by $\omega^\nabla_{(L,K,J)}$ the graded 2-form of degree k defined by L, K and J.

The next lemma will be useful to characterize symplectic forms. We need first the following definitions. Let $R^\nabla \in \Gamma(\Lambda^2 T^*M \otimes E^* \otimes E)$ be the curvature of the connection ∇, defined by

$$[\nabla_X, \nabla_Y] = i_{R^\nabla(X,Y)} + \nabla_{[X,Y]}.$$

Given $\bar{L} \in \Gamma^s(E \otimes \Lambda^{k-1}E)$, we denote by $\bar{L} \circ R^\nabla$ the element of $\Gamma(\Lambda^2 T^*M \otimes \Lambda^k E)$ defined by

$$\bar{L} \circ R^\nabla(X,Y;\alpha_1,\dots,\alpha_k) = \sum_{i=1}^{k} (-1)^{i-1} \bar{L}(R^\nabla(X,Y)\alpha_i;\alpha_1,\dots,\widehat{\alpha_i},\dots,\alpha_k).$$

And we denote by $\bar{L}^s$ the element of $\Gamma(S^2 E \otimes \Lambda^{k-2}E)$ defined by

$$\bar{L}^s(\alpha,\beta;\cdot\,) = \bar{L}(\alpha;\beta,\cdot\,) + \bar{L}(\beta;\alpha,\cdot\,).$$

Note that if $\bar{L}^s = 0$ then $\bar{L} = \varphi(\bar{L}) = 0$ because $\bar{L}$ is in the kernel of φ.

Lemma 4.2. *Let $\omega = \omega^\nabla_{(L,K,J)}$ be a graded 2-form of degree $k > 0$. Then ω is closed if and only if there exist $\bar{L} \in \Gamma^s(E \otimes \Lambda^{k-1}E)$, and $\bar{K} \in \Gamma(T^*M \otimes \Lambda^k E)$ such that*

$$L = \bar{L}^s, \quad K = d^\nabla \bar{L} - \bar{K}, \quad J = d^\nabla \bar{K} - \bar{L} \circ R^\nabla.$$

(If $k = 1$ then $\bar{L} = 0$ because $\Gamma^s(E) = 0$.)

Proof. If ω is closed, then, by Corollary 2.1, it is exact. Let $\lambda = \lambda^\nabla_{(\bar{L},\bar{K})}$ be a graded 1-form of degree k such that $d^G\lambda = \omega$. By Proposition 3.4, we can suposse that $\bar{L} \in \Gamma^s(E \otimes \Lambda^{k-1}E)$, and $\bar{K} \in \Gamma(T^*M \otimes \Lambda^k E)$.

Now, it is just a matter of computation to find the three tensors that define the graded 2-form $d^G\lambda$.

$$\begin{aligned}
&L(\alpha,\beta;\cdot\,) = d^G\lambda(i_\alpha, i_\beta) = \bar{L}^s(\alpha,\beta;\cdot\,).\\
&K(X;\beta;\cdot\,) = d^G\lambda(\nabla_X, i_\beta) = \nabla_X(\bar{L}(\beta;\cdot\,)) - i_\beta \bar{K}(X;\cdot\,) - \bar{L}(\nabla_X\beta;\cdot\,)\\
&\qquad = (d^\nabla \bar{L})(X;\beta;\cdot\,) - \bar{K}(X;\beta,\cdot\,).\\
&J(X,Y;\cdot\,) = d^G\lambda(\nabla_X, \nabla_Y)\\
&\qquad = \nabla_X(\bar{K}(Y;\cdot\,)) - \nabla_Y(\bar{K}(X;\cdot\,)) - \bar{K}([X,Y];\cdot\,) - \lambda(i_{R^\nabla(X,Y)})\\
&\qquad = (d^\nabla \bar{K} - \bar{L} \circ R^\nabla)(X,Y;\cdot\,). \quad \square
\end{aligned}$$

Note. $K = 0$ implies that $d^\nabla \bar{L} = 0$ and $\bar{K} = 0$.

Definition. A closed graded 2-form is said to be a symplectic form if it is non-singular, i.e., if the $\Gamma(\Lambda E)$-linear map $\operatorname{Der}\Gamma(\Lambda E) \to \Omega^1(M, \Gamma(\Lambda E))$, $D \to i(D)\omega$, is an isomorphism.

Theorem 4.3. *Let ∇ be a given linear connection on E and let ω be a closed graded 2-form on $(M, \Gamma(\Lambda E))$. Then ω is uniquely determined by the following fields*

$$\tilde{\omega} \in A^2(M), \text{ closed,}$$
$$\bar{K} = \sum_{k=1}^{n} \bar{K}_k \in \sum_{k=1}^{n} \Gamma(T^*M \otimes \Lambda^k E)$$
$$\bar{L} = \sum_{k=2}^{n} \bar{L}_k \in \sum_{k=1}^{n} \Gamma^s(E \otimes \Lambda^{k-1} E).$$

The fields $\tilde{\omega}, K_1$ and L_2 are independent of the connection ∇.
Moreover, let $\rho : TM \oplus E^* \to T^*M \oplus E$ defined by

$$\rho(X) = \tilde{\omega}(X, \cdot\,) + K_1(X; \cdot\,),\ \rho(\alpha) = -K_1(\,\cdot\,; \alpha) + L_2(\alpha, \cdot\,).$$

Then, ω is symplectic if and only if ρ is an isomorphism.

Proof. Let us decompose ω as the sum of its homogeneous parts $\sum_{k=0}^{n} \omega_{(k)}$, where $\omega_{(k)}$ is a graded 2-form of degree k.

$\omega_{(0)}$ is uniquely determined by $\tilde{\omega} := \kappa\omega$, and $\tilde{\omega}$ is closed because the graded differential commutes with κ.

For the rest of the $\omega_{(k)}$ apply Lemma 4.2.

L_2 is defined by $\omega_{(2)}(i_\alpha, i_\beta) = L_2(\alpha, \beta)$, thus L_2 is obviously independent of ∇.

K_1 is defined by $\omega_{(1)}(\nabla_X, i_\beta) = K_1(X; \beta)$. If ∇' is another linear connection, then $\nabla_X = \nabla'_X + i_{A(X)}$ where $A(X) \in \Gamma(E^* \otimes E)$, then

$$\omega_{(1)}(\nabla'_X + i_{A(X)}, i_\beta) = \omega_{(1)}(\nabla'_X, i_\beta)$$

because $\omega_{(1)}(i_\alpha, i_\beta) = 0$ for all $\alpha, \beta \in \Gamma(E^*)$. Therefore K_1 does not depend on ∇.

ω is symplectic if and only if for any local basis of TM, $\{X_j\}_{j=1}^m$, and any local basis of E, $\{e^j\}_{j=1}^n$, with dual basis $\{\alpha_j\}_{j=1}^n$, the determinant

$$\det \begin{pmatrix} (\omega(\nabla_{X_j}, \nabla_{X_k}))_{(0)} & (\omega(i_{\alpha_j}, \nabla_{X_k}))_{(0)} \\ (\omega(\nabla_{X_j}, i_{\alpha_k}))_{(0)} & (\omega(i_{\alpha_j}, i_{\alpha_k}))_{(0)} \end{pmatrix}$$
$$= \det \begin{pmatrix} \tilde{\omega}(X_j, X_k) & -K_1(X_k; \alpha_j) \\ K_1(X_j; \alpha_k) & L_2(\alpha_j, \alpha_k) \end{pmatrix} \neq 0,$$

and this is equivalent to say that ρ is an isomorphism. □

Corollary 4.4. (1) *If ω is an even graded symplectic form then $\tilde{\omega}$ is a symplectic form on M, $K_1 = 0$ and L_2 is a metric on E^*.*

(2) *If ω is an odd graded symplectic form then $\tilde{\omega} = 0 = L_2$ and K_1 defines an isomorphism between TM and E.*

Corollary 4.5. *Let $K_1 \in \Gamma(T^*M \otimes \Lambda^1 E)$ be such that it defines an isomorphism between TM and E. Then the graded 2-form $d^G(\lambda^{\nabla}_{(0,K)})$ is an odd symplectic form and it is independent of the linear connection ∇. We shall denote it by ω_K. Thus we have a canonical lifting from isomorphisms between TM and E to odd symplectic forms.*

Proof. $d^G(\lambda^{\nabla}_{(0,K)})$ is independent of ∇ because $\lambda^{\nabla}_{(0,K)}$ is. (See Proposition 3.2).

5. The odd symplectic form associated to a Riemannian manifold

Let (M,g) be a Riemannian, or pseudo-Riemannian, manifold. Let us consider the graded manifold $(M, A(M))$ whose dimension is (m,m), in this case $E = T^*M$. $g \in \Gamma(S^2T^*M)$ can be considered as an element, K_1, of $\Gamma(T^*M \otimes T^*M)$ and it defines an isomorphism between TM and T^*M.

As we have seen before, $\tilde{\omega} = 0, L = 0$ and $K = K_1$ define a unique odd symplectic form that we shall call $\omega_g := d^G(\lambda_{(0,K_1)})$.

By Lemma 4.2, the action of ω_g on pairs of derivations is given by

$$\begin{aligned}
\omega_g(\nabla_X, \nabla_Y) &= (d^{\nabla} g)(X,Y) \\
&= \nabla_X(g(Y,\cdot)) - \nabla_Y(g(X,\cdot)) - g([X,Y],\cdot) \in A^1(M), \\
\omega_g(\nabla_X, i_Y) &= -g(X,Y) \in C^\infty(M), \\
\omega_g(i_X, i_Y) &= 0.
\end{aligned}$$

We want to study now the super-Poisson structure induced by the odd symplectic form ω_g. We need to compute first the Hamiltonian vector fields.

Given $\alpha \in A(M)$ there is a unique $D_\alpha \in \operatorname{Der} A(M)$ such that

$$\omega(D_\alpha, D) = (d^G\alpha)(D) = D(\alpha).$$

D_α is called the Hamiltonian graded vector field defined by α.

In order to make computations easier, let us suppose that ∇ is the Levi-Civita connection. Thus, $\omega_g(\nabla_X, \nabla_Y) = 0$.

Proposition 5.1. (1) *If $f \in C^\infty(M)$, then $D_f = i_{\operatorname{grad} f}$.*
(2) *If $f \in C^\infty(M)$, then $D_{df} = -\mathcal{L}_{\operatorname{grad} f}$.*

Proof. (1) Simply compute $\omega_g(i_{\operatorname{grad} f}, i_X) = 0 = i_X f$, and

$$\omega_g(i_{\operatorname{grad} f}, \nabla_X) = g(X, \operatorname{grad} f) = X(f) = \nabla_X f.$$

(2) By Lemma 1.2 we have $\mathcal{L}_X = \nabla_X + i_{d^{\nabla}X}$, thus

$$\omega_g(-\mathcal{L}_{\operatorname{grad} f}, i_X) = g(\operatorname{grad} f, X) = X(f) = i_X(df),$$

and, if Z is a vector field

$$\begin{aligned}\omega_g(-\mathcal{L}_{\operatorname{grad} f}, \nabla_X)(Z) &= \omega_g(\nabla_X, i_{\operatorname{grad} f})(Z)\\ &= g(X, \nabla_Z \operatorname{grad} f)\\ &= Z(X(f)) - \nabla_Z X(f)\\ &= X(Z(f)) - \nabla_X Z(f)\\ &= (\nabla_X(df))(Z). \quad \square\end{aligned}$$

The definition of the super-Poisson structure is given by, see [7],

$$\{\alpha, \beta\} = \omega_g(D_\alpha, D_\beta).$$

See also [3].

Proposition 5.2. *Let $f, h \in C^\infty(M)$, and let $\Omega \in A^m(M)$ be the Riemannian volume element then*

(1) $\{f, h\} = 0$,
(2) $\{f, dh\} = -\operatorname{grad} h(f)$,
(3) $\{df, dh\} = -d(\operatorname{grad} h(f))$,
(4) $\{\Omega, f\} = i_{\operatorname{grad} f}\Omega$ *and*
(5) $\{\Omega, df\} = (\Delta f)\Omega$, *where Δ denotes the Laplacian.*

Proof. Apply Proposition 5.1.

6. The even symplectic form associated to a symplectic form and a Riemannian metric

We want to define a canonical lifting from symplectic forms and metrics on M to even graded symplectic forms on the graded manifold $(M, A(M))$. As we have seen before, a symplectic form $\tilde{\omega}$ on M defines a unique closed graded 2-form of degree 0, ω_0. Our purpose now is to define a closed graded 2-form of degree 2, which is the graded differential of a graded 1-form of degree 2.

Let $g \in \Gamma(S^2T^*M)$ be a metric on M. Let us define $L = g \in \Gamma^s(T^*M \otimes \Lambda^1T^*M)$ and $K^\nabla \in \Gamma(T^*M \otimes \Lambda^2T^*M)$ by

$$\begin{aligned}K^\nabla(X; Y, Z) &= (\nabla_Y g)(X, Z) - (\nabla_Z g)(X, Y)\\ &\quad + g(X, T^\nabla(Z, Y)) + g(Y, T^\nabla(Z, X)) + g(Z, T^\nabla(X, Y)).\end{aligned}$$

Proposition 6.1. *The 1-form $\lambda^\nabla_{(L,K^\nabla)}$ does not depend on ∇, therefore its graded differential ω_2 is a graded 2-form of degree 2 independent of the linear connection.*

Proof. Let ∇' be another linear connection, then we have $\nabla_X = \nabla'_X + i_{A(X)}$. It is obvious that the action of $\lambda^\nabla_{(L,K^\nabla)}$ on algebraic derivations does not depend on ∇. Let us check now that

$$\lambda^{\nabla}_{(L,K^{\nabla})}(\nabla_X) = \lambda^{\nabla'}_{(L,K^{\nabla'})}(\nabla_X).$$

Let Y, Z be vector fields, then

$$\begin{aligned}
&(\lambda^{\nabla}_{(L,K^{\nabla})}(\nabla_X) - \lambda^{\nabla'}_{(L,K^{\nabla'})}(\nabla_X))(Y,Z)\\
&\quad = (\lambda^{\nabla}_{(L,K^{\nabla})}(\nabla_X) - \lambda^{\nabla'}_{(L,K^{\nabla'})}(\nabla'_X + i_{A(X)})(Y,Z)\\
&\quad = K^{\nabla}(X;Y,Z) - K^{\nabla'}(X;Y,Z) - g(A(X)Y,Z) + g(A(X)Z,Y)\\
&\quad = 0,
\end{aligned}$$

as a straightforward computation shows. □

Thus, the map that assigns to the symplectic form $\tilde{\omega}$ and to the Riemannian metric g the even symplectic form $\omega_0 + \omega_2$ is independent of the linear connection used to construct it. We shall denote this even symplectic form by $\omega_{(\tilde{\omega},g)}$

If ∇ is the Levi-Civita connection of g, then $K^{\nabla} = 0$ and the action of the symplectic form on pairs of derivations is given, after Lemma 4.2, by

$$\begin{aligned}
&\omega_{(\tilde{\omega},g)}(\nabla_X, \nabla_Y) = \tilde{\omega}(X,Y) - g \circ R^{\nabla}(X,Y),\\
&\omega_{(\tilde{\omega},g)}(\nabla_X, i_Y) = 0,\\
&\omega_{(\tilde{\omega},g)}(i_X, i_Y) = 2g(X,Y).
\end{aligned}$$

Note that in this case, $g \circ R^{\nabla}(X,Y)(Z,W) = 2g(R^{\nabla}(X,Y)Z,W) = -2R(X,Y,Z,W)$ in the notation of [6].

Naturality. Let $f : M \to N$ be a local diffeomorphism between differentiable manifolds. Let $f^* : A(N) \to A(M)$ be its pull-back, thus $F = (f, f^*) : (M, A(M)) \to (N, A(N))$ is a local diffeomorphism between graded manifolds.

Let $D \in \operatorname{Der} A(M)$, we define $F_* D \in \operatorname{Der} A(N)$ as

$$(F_* D)\alpha := (f^{-1})^* D f^* \alpha \in A(N), \qquad \text{for all } \alpha \in A(N).$$

It is easy to check that if X is a vector field on M then

$$F_*(i_X) = i_{f_* X} \quad \text{and} \quad F_*(\nabla_X) = \nabla^*_{f_* X},$$

where ∇ is a linear connection in M and ∇^* is the linear connection in N induced by the local diffeomorphism f.

Note that if ∇ is the Levi-Civita connection of a metric, g, on M, then ∇^* is the Levi-Civita connection of the metric $(f^{-1})^* g$.

Let λ be a graded p-form on the graded manifold $(N, A(N))$, we define $F^*\lambda$, a graded p-form on $(M, A(M))$, as

$$F^*\lambda(D_1, \ldots, D_p) = f^*(\lambda(F_* D_1, \ldots, F_* D_p)) \in A(M),$$

for all $D_1, \dots, D_p \in \mathrm{Der}\, A(M)$.

It is easy to check that $d^G F^*\lambda = F^* d^G \lambda$.

With this definitions, a straightforward computation shows that, if ∇ is the Levi-Civita connection of a metric g on M then

$$F^*(\lambda_{((f^{-1})^*g,0)})(\nabla_X) = 0 = \lambda_{(g,0)}(\nabla_X),$$
$$F^*(\lambda_{((f^{-1})^*g,0)})(i_X) = g(X,\cdot\,) = \lambda_{(g,0)}(i_X).$$

Thus $F^*(\lambda_{((f^{-1})^*g,0)}) = \lambda_{(g,0)}$ and then the same happens with the graded differentials

$$F^*(\omega^2_{(f^{-1})^*g}) = d^G F^*(\lambda_{((f^{-1})^*g,0)}) = d^G\lambda_{(g,0)} = \omega^2_g.$$

Moreover, if $\omega^0_{\tilde\omega}$ is the graded 2-form of degree 0 on $(M, A(M))$ defined by

$$\omega(\nabla_X, \nabla_Y) = \tilde\omega(X,Y), \quad \omega(\nabla_X, i_Y) = 0, \quad \omega(i_X, i_Y) = 0,$$

then

$$F^*(\omega^0_{(f^{-1})^*\tilde\omega}) = \omega^0_{\tilde\omega}.$$

Finally, we get the naturality of the even symplectic form $\omega_{(\tilde\omega,g)} = \omega^0_{\tilde\omega} + \omega^2_g$ because

$$F^*(\omega_{((f^{-1})^*\tilde\omega,(f^{-1})^*g)}) = \omega_{(\tilde\omega,g)}.$$

The odd symplectic form defined in Section 5 is also natural following this sense.

The associated Poisson bracket. The Poisson bracket defined by this graded symplectic form is a bracket defined on $A(M)$ and constructed by means of a symplectic form and a metric on M. A first attempt to define a Poisson bracket constructed just by means of a symplectic form can be found in [11].

We shall see that the Poisson bracket associated to the even symplectic form $\omega = \omega_{(\tilde\omega,g)}$ is, in a certain sense, an extension to differential forms of the Poisson bracket on functions associated to the symplectic form $\tilde\omega$.

Let ∇ be the Levi-Civita connection defined by g. Let us compute the Hamiltonian graded vector field, D_f, associated to a function f. Since D_f is a derivation on $A(M)$ we have, by Proposition 1.1, that there exist $X_f, Y_f \in \Gamma(\Lambda T^*M \otimes TM)$ such that $D_f = \nabla_{X_f} + i_{Y_f}$.

By definition of D_f, $\omega(D_f, i_X) = i_X f = 0$ for all vector field on M, X. Therefore $Y_f = 0$. Since ω is an even symplectic form we have that D_f is an even derivation. Let us decompose X_f as sum of homogenous components $X^0_f + X^2_f + \cdots$, where $X^{2k}_f \in \Gamma(\Lambda^{2k}T^*M \otimes TM)$.

By definition of D_f, $\omega(D_f, \nabla_X) = \nabla_X f = X(f) = df(X) = \tilde\omega(H_f, X)$, where H_f denotes the classical Hamiltonian vector field associated to f. On the other hand, by definition of the even symplectic form $\omega = \omega_{(\tilde\omega,g)}$

$$\omega(D_f, \nabla_X) = \tilde\omega(X^0_f, X) - g\circ R(X^0_f, X) + \tilde\omega(X^2_f, X) - \cdots,$$

where the dots denote terms of degree 4 or higher and where $\tilde\omega(X^2_f, X)$ is defined by the following condition: If $K^2 = \alpha\otimes Y \in \Gamma(\Lambda^2 T^*M \otimes TM)$ then $\tilde\omega(K^2, X) = \alpha\tilde\omega(Y, X)$.

Therefore, equating terms of the same degree we get $X_f^0 = H_f$ and $\tilde{\omega}(X_f^2, X) = g \circ R(X_f^0, X)$ for all vector field X. $\tilde{\omega}$ defines an isomorphism between T^*M and TM. This isomorphism can be obviously extended to an isomorphism between $\Lambda^k T^*M \otimes T^*M$ and $\Lambda^k T^*M \otimes TM$. This isomorphism assures us that X_f^2 is uniquely defined by the condition $\tilde{\omega}(X_f^2, X) = g \circ R(X_f^0, X)$. Analogously for X_f^{2k+2}.

Let us now go to the Poisson bracket defined by ω, $\{\cdot,\cdot\}$. Let us denote by $\{\cdot,\cdot\}_{\tilde{\omega}}$ the classical Poisson bracket defined by $\tilde{\omega}$.

The relation between both brackets is given by $\{f,h\} = \{f,h\}_{\tilde{\omega}} + \cdots$, where $f, h \in C^\infty(M)$ and the dots denote terms of degree higher than 0. This is why we call $\{\cdot,\cdot\}$ an extension of the classical Poisson bracket $\{\cdot,\cdot\}_{\tilde{\omega}}$.

We will say that the extension is strict if $\{f,h\} = \{f,h\}_{\tilde{\omega}}$.

Proposition 6.2. *The even symplectic form $\omega = \omega_{(\tilde{\omega},g)}$ gives rise to an strict extension of the Poisson bracket associated to $\tilde{\omega}$ if and only if the curvature of the metric g vanishes.*

Proof. If the curvature vanishes then $D_f = \nabla_{H_f}$ and we get thus an strict extension. Reciprocally, if the Poisson bracket defined by ω is an strict extension then $\{f,h\} = \omega(D_f, D_h) \in C^\infty(M)$. Since $D_f = \nabla_{H_f} + \nabla_{X_f^2} + \cdots$ we have

$$\omega(D_f, D_h) = \tilde{\omega}(H_f, H_h) - g \circ R(H_f, H_h) + \tilde{\omega}(X_f^2, H_h) + \tilde{\omega}(H_f, X_h^2) + \cdots$$

where dots denote terms of degree 4 or higher.

By definition of X_f^2 we have that $\tilde{\omega}(X_f^2, H_h) = g \circ R(H_f, H_h)$, and then

$$\begin{aligned}\omega(D_f, D_h) &= \tilde{\omega}(H_f, H_h) + \tilde{\omega}(H_f, X_h^2) + \cdots \\ &= \tilde{\omega}(H_f, H_h) + g \circ R(H_f, H_h) + \cdots.\end{aligned}$$

Therefore, if the extension is strict we have that $g \circ R(H_f, H_h)$ for all functions f, h. Since there exists a local basis of Hamiltonian vector fields, we have that $R = 0$. $\square$

Relation between the Berezinian and the curvature. When the differentiable manifold M is a surface is $\mathbb{R}^3$ a curious relation between the Berezinian of $\omega_{(\tilde{\omega},g)}$ and the Gauss curvature can be shown. For the definition of the Berezinian see [10] or [18].

First note that any volume form in M can be seen as a sympletic form, $\tilde{\omega}$. Let us denote by g the Riemannian metric induced by the euclidean metric in $\mathbb{R}^3$. Let $\omega_{(\tilde{\omega},g)}$ be the graded symplectic form canonically defined by $\tilde{\omega}$ and g. We shall compute the Berezinian of such form in the following local basis of graded vector fields $\{\nabla_{\partial/\partial x_1}, \nabla_{\partial/\partial x_2}, i_{\partial/\partial x_1}, i_{\partial/\partial x_2}\}$, where $\{x_1, x_2\}$ are Darboux coordinates for $\tilde{\omega}$, and ∇ is the Levi-Civita connection.

The matrix of $\omega_{(\tilde{\omega},g)}$ in such basis is

$$\begin{pmatrix} A & O \\ 0 & G \end{pmatrix} = \begin{pmatrix} 0 & -1 - 2R_{1212}\, dx_1 \wedge dx_2 & 0 & 0 \\ 1 + 2R_{1212}\, dx_1 \wedge dx_2 & 0 & 0 & 0 \\ 0 & 0 & g_{11} & g_{12} \\ 0 & 0 & g_{12} & g_{22} \end{pmatrix}$$

where

$$R_{1212} = g\left(R^{\nabla}\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\right) \quad \text{and} \quad g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right).$$

Then

$$\operatorname{Ber}\omega_{(\tilde{\omega},g)} = \frac{\det A}{\det G} = \frac{1 + 4R_{1212}\, dx_1 \wedge dx_2}{g_{11}g_{22} - g_{12}^2} = \frac{1}{\det G} + 4K\tilde{\omega},$$

where K is the Gauss curvature of the surface M. Note that since M is a surface then $\det G \neq 0$ and thus the Berezinian has sense.

7. The Schouten-Nijenhuis bracket

As a final example, we shall see that the Schouten-Nijuenhuis bracket can be expressed as the Poisson bracket of an odd symplectic structure. This was first observed in [9]. A dual construction on differential forms was constructed in [8]. For a detailed study of this bracket see [12].

Now, E is the tangent bundle, TM. Thus, the graded manifold is $(M, V(M) := \Gamma(\Lambda TM))$. The elements of $V(M)$ are called multivectors. Let us choose as $K_1 \in \Gamma(TM \otimes T^*M)$ the identity isomorphism from TM into itself.

Let λ be the graded 1-form $\lambda^{\nabla}_{(0,K_1)}$ and let $\omega = d\lambda$ be the associated odd symplectic form. As it has been seen, ω does not depend on ∇.

It it easy to check that, by Lemma 4.2,

$$\begin{aligned}
&\omega(\nabla_X, \nabla_Y) = T^{\nabla}(X, Y),\\
&\omega(\nabla_X, i_\alpha) = -\alpha(X),\\
&\omega(i_\alpha, i_\beta) = 0.
\end{aligned}$$

Theorem 7.1. *The Poisson bracket on multivectors induced by ω is the Schouten-Nijenhuis bracket.*

Proof. Let $X \in \Gamma(TM)$. The Hamiltonian graded vector field associated to X is the derivation D_X defined by

$$\omega(D_X, D) = D(X), \qquad \text{for all } D \in \operatorname{Der} V(M).$$

It is easy to check that $D_X = -(\nabla_X - i_{d^{\nabla}X + T^{\nabla}(X,\ )})$ and, by Lemma 1.2, this is equal to $-\mathcal{L}_X$.

Let $X, Y \in V^1(M)$ and $f, h \in C^\infty(M)$. The Poisson bracket is given by

$$\begin{aligned}
&\{X, Y\} = \omega(D_X, D_Y) = D_Y(X) = -\mathcal{L}_Y X = [X, Y],\\
&\{f, X\} = \omega(D_f, D_X) = -X(f),
\end{aligned}$$

then, $\{X, f\} = -\{f, X\} = X(f)$. Finally, $\{f, h\} = 0$, because it is an element of degree -1.

By the known properties of both brackets, they agree on multivectors. □

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