HOLDITCH'S THEOREM IN 3D SPACE

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ABSTRACT. Holditch's theorem is an old result on areas generated by moving chords in closed planar curves. Some generalizations on this result have been given before, but none of these follows the same natural construction of the plane but done in the space. In this work, the notion of Holditch surface is defined, some properties of these surfaces are proved and they are used to generalize Holditch's theorem for closed space curves naturally. Moreover, an approximation for the area of interest is given. Finally, it is showed that the only minimal non-planar Holditch surface is the helicoid.

1. INTRODUCTION

Holditch's theorem, [5], is a classical geometrical result on areas of planar curves. Suppose that a constant length moving chord is allowed to rotate a full turn with its endpoints always lying in a closed convex planar curve α. A chosen point in the moving chord at a distance a from one end and at a distance b from the other will describe an inner closed curve H_α, the so-called Holditch curve of α. Holditch's theorem states that the difference between the areas of both curves is equal to $\pi\, a\, b$ (see Figure 1). At first, the most interesting point may be that this area only depends on the lengths in which the chosen point divides the chord and not on the shape or size of the initial curve.

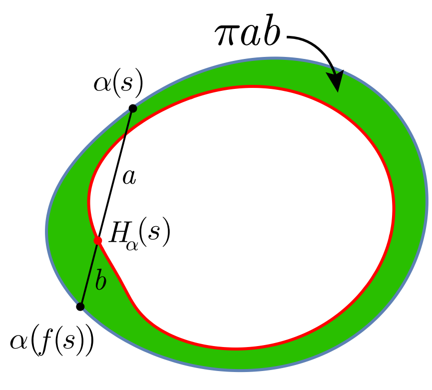

FIGURE 1. Representation of Holditch's theorem.

Some authors have studied Holditch's theorem among other generalizations in the plane. The interested reader can see, for instance, [2], [7], [8] or [9]. Also, the construction of Holditch curves can remind vaguely that of the De Casteljau's algorithm for Bézier curves. In the paper [4] written by a civil engineer, some applications of the theorem are exposed, such as regularization of railway lines or the division of an area into equal nested parts.

When the initial curve is a non-planar closed curve in the space, the generation of a new curve by the same procedure as in the plane is still possible. Therefore a natural problem that arises is to find an analogous of the Holditch's theorem for space curves, this is, a result relating lengths on the moving chord with the area of some surface between the initial curve and the generated one.

Some simple examples starting with non-planar polygonal curves allow to glimpse that in the spatial case the area will depend on the curve, maybe through its total curvature, in addition to the distances a and b.

Anyway, the first point is to determine from which surface one has to compute the area. A paper by Arnol'd, [1], dealing with caustic curves on the sphere gives us a possible candidate. There the author considers the notion of wave front, this is, a family of curves associated with a parameter.

Given a value $p \in [0,1]$, our candidate surface is the wave front made by all Holditch curves defined by a point which divides a chord of length ℓ into two pieces of lengths $p\,\ell$ and $(1-p)\,\ell$. The parameter of the wave front is the length ℓ. Thus, given $p \in [0,1]$, the p-Holditch surface for a chord of length L is the union of all the Holditch curves for lengths $\ell \in [0,L]$ divided according to the ratio $p : 1-p$. Notice that for different p values one gets different surfaces, all of them starting at the initial curve and having as tangent plane there the osculating plane of the initial curve (Proposition 3). Moreover, when the initial curve is planar, we recover the planar region of the classical Holditch's theorem and that allows to give a new proof for that result (Theorem 1).

The generalization of Holditch's theorem to non-planar closed curves states that given $p \in [0,1]$, if A_H is the area of the p-Holditch surface of a curve $\alpha : I \to \mathbb{R}^3$ up to some length, then the quantity

$$\frac{A_H}{p\,(1-p)}$$

is independent of p (Theorem 2). Moreover, with the first terms in the development of the area as a function of L (Theorem 3), we have:

$$\frac{A_H}{p\,(1-p)} = \frac{L^2}{2}\int_I \kappa(s)\,\mathrm{d}s - \frac{L^4}{96}\int_I \kappa(s)\,\tau^2(s)\,\mathrm{d}s + \mathcal{O}(L^6),$$

where κ and τ are the curvature and torsion, respectively, of the initial curve. Thus, up to third order, the area of the p-Holditch surface is

$$A_H \approx \frac{1}{2}\,(p\,L)\,\big((1-p)\,L\big)\int_I \kappa(s)\,\mathrm{d}s.$$

This is, two closed curves with the same total curvature will define the same Holditch area (up to third order). Torsion only appears from the fourth order term.

Holditch surfaces can also be built for non-closed curves. We have been able to show that the only minimal non-planar regular Holditch surface is the helicoid, which can be seen as the $\frac{1}{2}$-Holditch surface of the circular helix.

Finally, we would like to mention that this extension of Holditch's theorem might have similar applications as the planar one, for instance to generate smoother nested curves in a surface enclosing controlled areas.

2. Definition of Holditch surfaces

Throughout all the paper, $\alpha : I \to \mathbb{R}^n$ will be a planar ($n = 2$) or a space ($n = 3$) regular curve parameterized by arc-length. For $s \in I$, the tangent, normal and binormal vectors of α at the point $\alpha(s)$ will be denoted by $\mathbf{t}(s)$, $\mathbf{n}(s)$ and $\mathbf{b}(s)$, respectively. Moreover, we remark that torsion τ of α will be such that

$$\tau(s) = \big\langle \mathbf{b}'(s), \mathbf{n}(s)\big\rangle.$$

Henceforth, the value $p \in [0, 1]$ will determine the ratio $p : q$ in which a chord of constant length L is divided into two parts ($q = 1 - p$). If the chosen length of the chord is long enough, retrograde motion may appear. That happens when an endpoint of the chord pass through the same point more than once. In [6], a sufficient condition on L to avoid retrograde motion on convex planar curves was given (see also the discussion of [7]). In general, dealing with retrograde motion is a difficult problem. If there is no retrograde motion for a curve α and a length L, then we will say that the length L is admissible for α and its p-Holditch curve can be written as

$$H_\alpha(s) = (1-p)\,\alpha(s) + p\,\alpha\big(f(s)\big),$$

where $f : I \to I$ is the Holditch function, i.e., the continuous bijective function such that if $\alpha(s)$ is one endpoint of the chord, then $\alpha\big(f(s)\big)$ is the other endpoint (see Figure 1). Notice that f depends on α but also on L. If a different length of the chord is chosen, the Holditch function will vary. Let us write explicitly the dependence on the length L of the chord by naming it $f(s, L)$.

So, given a fixed p, a natural parametric surface can be defined by varying the length of the moving chord. It is formed by all the generated p-Holditch curves of α for each length ℓ of the chord up to some higher fixed value

L. That parametric surface, say $\mathbf{h}^p : I \times]0, L[\to \mathbb{R}^n$, will be called the p-Holditch surface of α up to a length L, and it is defined by

$$\mathbf{h}^p(s, \ell) = (1 - p)\, \alpha(s) + p\, \alpha\big(f(s, \ell)\big). \tag{1}$$

Coordinate curves $s \mapsto \mathbf{h}^p(s, \ell_0)$, with ℓ_0 constant, are Holditch curves of α.

Note that we can reparameterize the surface by the change $u = f(s, \ell) - s$ to simplify its expression as we state in Definition 1. Define

$$m(s, L) := f(s, L) - s.$$

Definition 1 (Holditch surface)**.** Given an admissible length $L > 0$, the p-*Holditch surface of* α *up to* L is the parametric surface $\mathbf{x}^p : I \times]0, m(s, L)[\to \mathbb{R}^n$ defined by

$$\mathbf{x}^p(s, u) = (1 - p)\, \alpha(s) + p\, \alpha(s + u). \tag{2}$$

If there is no ambiguity in α, p and L, we will call it simply the Holditch surface.

Now coordinate curves $s \mapsto \mathbf{x}^p(s, u_0)$, with u_0 constant, are no longer Holditch curves. Also, notice that

$$\lim_{u \to 0} \mathbf{x}^p(s, u) = \alpha(s).$$

The definition of Holditch surface works to any curve, but we will focus later in closed curves in order to generalize Holditch's theorem for space closed curves. Look at Figure 2 to see some examples of Holditch surfaces for closed curves.

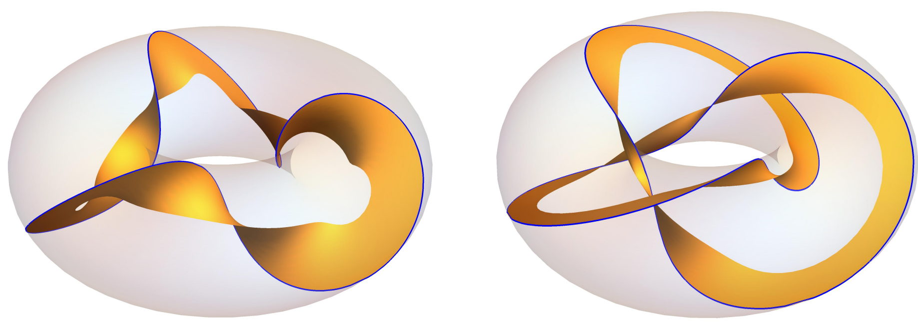

FIGURE 2. Two curves in a torus and their 1/2-Holditch surfaces for $L = 5/2$.

Define $\phi(s, u)$ as the angle between $\mathbf{t}(s)$ and $\mathbf{t}(s + u)$. It is easy to show that there exists some value $\tilde{L}$ such that for lengths $0 < L < \tilde{L}$, the angle $\phi(s, u)$ is not zero for $u \in]0, m(s, L)[$. To ensure the regularity of Holditch surfaces, this restriction on the length of the chord must be set additionally to the property of being admissible. Thus, if both conditions are satisfied, we will say that the length L is admissible for regularity.

Proposition 1. *For any $p \in [0,1]$, the p-Holditch surface of α up to an admissible for regularity length is regular in its domain.*

Proof. The derivatives of (2) with respect to s and u are

$$\begin{aligned} \mathbf{x}^p_s(s,u) &= (1-p)\,\mathbf{t}(s) + p\,\mathbf{t}(s+u), \\ \mathbf{x}^p_u(s,u) &= p\,\mathbf{t}(s+u). \end{aligned}$$

Therefore, their cross product is

$$(\mathbf{x}^p_s \wedge \mathbf{x}^p_u)(s,u) = (1-p)\,p\,\mathbf{t}(s) \wedge \mathbf{t}(s+u)$$

and its norm:

$$\big\|(\mathbf{x}^p_s \wedge \mathbf{x}^p_u)(s,u)\big\| = (1-p)\,p\,\sin\phi(s,u). \tag{3}$$

Since $\phi(s,u) \neq 0$ for the length L and $u \neq 0$, we have that regularity is ensured. □

3. Some properties of Holditch surfaces

With the previous calculations we have the following.

Proposition 2. *The normal vector of a p-Holditch surface of a curve up to an admissible for regularity length is given by*

$$N^{\mathbf{x}^p}(s,u) = \frac{(\mathbf{x}^p_s \wedge \mathbf{x}^p_u)(s,u)}{\big\|(\mathbf{x}^p_s \wedge \mathbf{x}^p_u)(s,u)\big\|} = \frac{\mathbf{t}(s) \wedge \mathbf{t}(s+u)}{\sin\phi(s,u)},$$

which is independent of p.

Immediately, next result is obtained.

Proposition 3. *Let $L > 0$ be an admissible for regularity length. The tangent planes of the sequence of the p-Holditch surfaces of α up to ℓ, for $0 < \ell < L$, tend to the osculating plane of α at the same point.*

Proof. It is sufficient to show that

$$\lim_{u\to 0} N^{\mathbf{x}^p}(s,u) = \mathbf{b}(s).$$

That can be seen as a limiting case from the point of view of discrete geometry, since $N^{\mathbf{x}^p}$ represents a discrete version of the binormal vector of α. Another possibility is to compute the limit using the L'Hôpital's rule. Indeed,

$$\lim_{u\to 0} N^{\mathbf{x}^p}(s,u) = \lim_{u\to 0} \frac{\kappa(s+u)\,\mathbf{t}(s) \wedge \mathbf{n}(s+u)}{\cos\phi(s,u)\,\phi_u(s,u)} = \mathbf{b}(s),$$

since $\phi(s,0) = 0$ and $\phi_u(s,0) = \kappa(s)$ (see Lemma 2 below). □

With the coefficients of the first and second fundamental forms we can get the Gauss and the mean curvature of Holditch surfaces.

Proposition 4. *For any $p \in [0,1]$, the p-Holditch surface of α up to an admissible for regularity length has Gauss curvature*

$$K(s,u) = -\frac{\kappa(s)\,\kappa(s+u)}{p\,q\sin^4\phi(s,u)}\,\big\langle \mathbf{t}(s+u),\mathbf{b}(s)\big\rangle\,\big\langle \mathbf{b}(s+u),\mathbf{t}(s)\big\rangle$$

and mean curvature

$$H(s,u) = \frac{1}{2\sin^3\phi(s,u)}\left(\frac{\kappa(s+u)}{p}\,\big\langle \mathbf{b}(s+u),\mathbf{t}(s)\big\rangle - \frac{\kappa(s)}{q}\,\big\langle \mathbf{t}(s+u),\mathbf{b}(s)\big\rangle\right).$$

Proof. Let $\mathbf{x}^p$ be the p-Holditch surface of α up to L as in (2). The coefficients of the first fundamental form are given by

$$\begin{aligned} E(s,u) &= p^2+q^2+2\,p\,q\cos\phi(s,u),\\ F(s,u) &= p^2+p\,q\,\cos\phi(s,u),\\ G(s,u) &= p^2. \end{aligned}$$

Therefore,

$$\big(E\,G-F^2\big)(s,u) = p^2\,q^2\sin^2\phi(s,u). \tag{4}$$

Now, since

$$\mathbf{x}^p_{ss}(s,u) = q\,\kappa(s)\,\mathbf{n}(s) + p\,\kappa(s+u)\,\mathbf{n}(s+u),$$

and

$$\mathbf{x}^p_{su}(s,u) = \mathbf{x}^p_{uu}(s,u) = p\,\kappa(s+u)\,\mathbf{n}(s+u),$$

by using Proposition 2 the coefficients of the second fundamental form are easily obtained:

$$\begin{aligned} e(s,u) &= \frac{p\,\kappa(s+u)}{\sin\phi(s,u)}\,\big\langle \mathbf{b}(s+u),\mathbf{t}(s)\big\rangle - \frac{q\,\kappa(s)}{\sin\phi(s,u)}\,\big\langle \mathbf{t}(s+u),\mathbf{b}(s)\big\rangle,\\ f(s,u) &= \frac{p\,\kappa(s+u)}{\sin\phi(s,u)}\,\big\langle \mathbf{b}(s+u),\mathbf{t}(s)\big\rangle = g(s,u). \end{aligned}$$

Hence,

$$\big(e\,g-f^2\big)(s,u) = -\frac{p\,q\,\kappa(s)\,\kappa(s+u)}{\sin^2\phi(s,u)}\,\big\langle \mathbf{t}(s+u),\mathbf{b}(s)\big\rangle\,\big\langle \mathbf{b}(s+u),\mathbf{t}(s)\big\rangle. \tag{5}$$

The expression of the Gauss curvature of the statement is obtained by dividing (5) by (4). Also, dividing

$$\begin{aligned} &(e\,G-2\,f\,F+g\,E)(s,u)\\ &\qquad= \frac{p\,q^2\,\kappa(s+u)}{\sin\phi(s,u)}\,\big\langle \mathbf{b}(s+u),\mathbf{t}(s)\big\rangle - \frac{p^2\,q\,\kappa(s)}{\sin\phi(s,u)}\,\big\langle \mathbf{t}(s+u),\mathbf{b}(s)\big\rangle \end{aligned}$$

by (4) and multiplying by $1/2$, the expression of the mean curvature is deduced. □

Immediately, from (3) we get the expression for the area of Holditch surfaces.

Proposition 5. *Given $p \in [0,1]$, the area of the p-Holditch surface of α up to an admissible for regularity length $L > 0$ is*

$$p\,(1-p) \int_I \int_0^{m(s,L)} \sin\phi(s,u)\ du\ ds.$$

4. Holditch surfaces of planar curves

The aim of this section is to give a new proof of Holditch's theorem in the plane by using Holditch surfaces. The first point is to see that the Holditch surface of a planar curve describes the Holditch region, i.e., the region determined by the initial curve and its Holditch curve.

Proposition 6. *Any p-Holditch surface of a planar curve α up to some admissible for regularity length $L > 0$ is the planar region determined by α and the p-Holditch curve of α for the length L.*

Proof. The p-Holditch surface up to L consists on all the p-Holditch curves for lengths ℓ from 0 to L. We know that the p-Holditch curves of a planar curve are also planar in the same plane and that they are nested and do not intersect to each other given two different lengths. That is consequence of the regularity of the Holditch surface (Proposition 1)—its coordinate lines with (1) are Holditch curves—and of being the change $f(s,\ell) = s + u$ injective. Therefore, the region determined by the Holditch surface is the region between α and the p-Holditch curve for the length L. □

By Proposition 6, overlapping area portions will not appear in planar Holditch surfaces for admissible for regularity lengths. Thus, the area of the region determined by a Holditch surface in the plane agrees with the notion of Holditch area. With that, a new proof of the classical Holditch's theorem is possible using Holditch surfaces. Before that, we state an useful observation.

Lemma 1. *Suppose the angle ϕ to be defined by a chord of an admissible length. If α is a strictly convex planar curve, then*

$$\phi(s,u) = \int_s^{s+u} \kappa(t)\ dt,$$

where κ is the curvature function of α.

Proof. Let $\sigma : I \to \mathbb{R}$ be the positively oriented angle from the OX axis to $\mathbf{t}(s)$. By definition of ϕ, we have that

$$\phi(s,u) = \sigma(s+u) - \sigma(s) = \int_s^{s+u} \sigma'(t)\ \mathrm{d}t.$$

The result is obtained since $\sigma' = \kappa$. □

Next theorem shows the proof of the classical Holditch's theorem by describing the Holditch region with a Holditch surface.

Theorem 1. *The p-Holditch surface of a closed strictly convex planar curve up to an admissible for regularity length $L > 0$ has planar area equal to the Holditch area: $\pi\, p\,(1-p)\, L^2$.*

Proof. According to Proposition 6, the p-Holditch surface will be the planar region between the initial curve α and its p-Holditch curve for the length L. That region will have an area

$$A_H = p\,(1-p) \int_I \int_0^{m(s,L)} \sin\phi(s,u)\ \mathrm{d}u\ \mathrm{d}s$$

by Proposition 5.

Recall that given the curvature κ of a planar curve $\big(x(s), y(s)\big)$, we can recover

$$x'(s) = \cos \int \kappa(s)\ \mathrm{d}s, \quad y'(s) = \sin \int \kappa(s)\ \mathrm{d}s,$$

up to a rotation (given by the integration constant in the chosen primitive of κ). Therefore, using Lemma 1,

$$\begin{aligned}\sin\phi(s,u) &= \sin \int_s^{s+u} \kappa(t)\ \mathrm{d}t = \sin\left(\int_0^{s+u} \kappa(t)\ \mathrm{d}t - \int_0^{s} \kappa(t)\ \mathrm{d}t\right)\\ &= \sin \int_0^{s+u} \kappa(t)\ \mathrm{d}t\, \cos \int_0^{s} \kappa(t)\ \mathrm{d}t - \cos \int_0^{s+u} \kappa(t)\ \mathrm{d}t\, \sin \int_0^{s} \kappa(t)\ \mathrm{d}t\\ &= y'(s+u)\, x'(s) - x'(s+u)\, y'(s).\end{aligned}$$

Thus,

$$\begin{aligned}&\int_0^{m(s,L)} \sin\phi(s,u)\ \mathrm{d}u\\ &\qquad = x'(s) \int_0^{m(s,L)} y'(s+u)\ \mathrm{d}u - y'(s) \int_0^{m(s,L)} x'(s+u)\ \mathrm{d}u\\ &\qquad = x'(s)\, y\big(s+m(s,L)\big) - y'(s)\, x\big(s+m(s,L)\big)\\ &\qquad\qquad - x'(s) y(s) + y'(s)\, x(s).\end{aligned}$$

The area of α is

$$A = \int_I x(s)\, y'(s)\ \mathrm{d}s = -\int_I x'(s)\, y(s)\ \mathrm{d}s.$$

Hence,

$$\frac{A_H}{p\,(1-p)} = 2\,A + \int_I \Big(x'(s)\, y\big(s+m(s,L)\big) - y'(s)\, x\big(s+m(s,L)\big)\Big)\ \mathrm{d}s. \tag{6}$$

We know that the curve $\alpha\big(s+m(s,L)\big)-\alpha(s)$ is a circle of radius L. Its area, $\pi\, L^2$, could be computed with

$$\begin{aligned}
&\int_I \Big(x\big(s+m(s,L)\big)-x(s)\Big)\Big(\Big(y\big(s+m(s,L)\big)\Big)'-y'(s)\Big)\,\mathrm{d}s\\
&\quad=\int_I x\big(s+m(s,L)\big)\Big(y\big(s+m(s,L)\big)\Big)'\,\mathrm{d}s-\int_I x\big(s+m(s,L)\big)y'(s)\,\mathrm{d}s\\
&\qquad-\int_I x(s)\Big(y\big(s+m(s,L)\big)\Big)'\,\mathrm{d}s+\int_I x(s)\,y'(s)\,\mathrm{d}s\\
&\quad=2\,A+\int_I\Big(x'(s)\,y\big(s+m(s,L)\big)-y'(s)\,x\big(s+m(s,L)\big)\Big)\,\mathrm{d}s.
\end{aligned}$$

Finally, using this on (6) we get

$$A_H=\pi\,p\,(1-p)\,L^2,$$

which is the Holditch area. □

In Figure 3 two examples of planar curves are represented. The planar region determined by the Holditch surface coincides with the Holditch region.

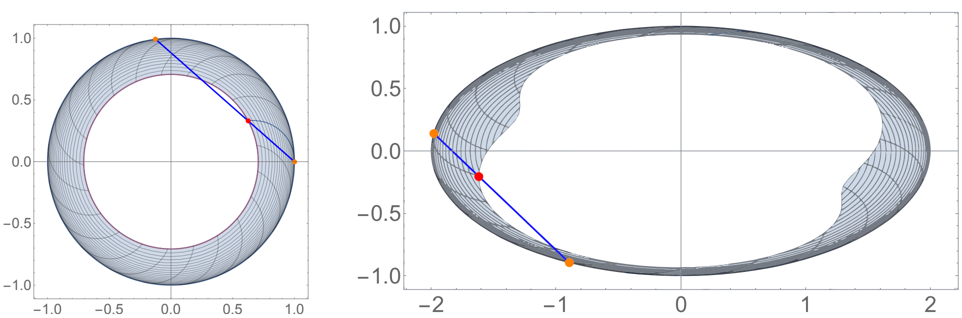

FIGURE 3. The 1/3-Holditch surfaces for $L=3/2$ in a unit circle and in an ellipse with semiaxes 2 and 1. The region determined by such a surface has area equal to the Holditch area.

5. Holditch surfaces of space curves

In this section we will deal with space curves. The immediate result which generalizes Holditch's theorem to the space is stated next. It follows directly from Proposition 5.

Theorem 2. *Given $p \in [0,1]$, if A_H is the area of the p-Holditch surface of a curve α up to an admissible for regularity length, then the quantity*

$$\frac{A_H}{p\,(1-p)}$$

is independent of p.

In the planar case, the quotient of Theorem 2 is independent of p and α. In the space case, the situation is different, since although it does not depend on p, the dependence on α is not avoided. In fact, in the known generalizations of Holditch's theorem for constant curvature surfaces (taking geodesic moving chords instead of line chords), the dependence of α in the full Holditch formula is not avoided either (see for example [10]).

The following lemma is just a matter of computation. It will be used later to find an approximation for the area of a Holditch surface in the space case.

Lemma 2. *The three first derivatives of $m(s,L)$ with respect to L at $L=0$ are*

$$m_L(s,0)=1, \quad m_{LL}(s,0)=0, \quad m_{LLL}(s,0)=\frac{1}{4}\,\kappa^2(s),$$

and the same of $\phi(s,u)$ with respect to u at $u=0$ are

$$\phi_u(s,0)=\kappa(s), \quad \phi_{uu}(s,0)=\kappa'(s), \quad \phi_{uuu}(s,0)=\kappa''(s)-\frac{1}{4}\,\kappa(s)\,\tau^2(s),$$

where κ and τ are the curvature and torsion, respectively, of a curve α with non-vanishing curvature.

Proof. We have two conditions from which we can compute the derivatives of m with respect to L and of ϕ with respect to u. These are

$$\left\|\alpha\big(s+m(s,L)\big)-\alpha(s)\right\|^2 = L^2 \tag{7}$$

and

$$\cos\phi(s,u) = \big\langle \mathbf{t}(s), \mathbf{t}(s+u)\big\rangle. \tag{8}$$

Let us begin with the derivatives of m. Differentiating (7) with respect to L, we get

$$m_L(s,L)\left\langle \mathbf{t}\big(s+m(s,L)\big), \alpha\big(s+m(s,L)\big)-\alpha(s)\right\rangle = L.$$

Differentiating again,

$$\begin{aligned} &m_{LL}\,\langle \mathbf{t}(s+m), \alpha(s+m)-\alpha(s)\rangle \\ &\qquad + m_L\left(\kappa(s+m)\,m_L\,\langle \mathbf{n}(s+m), \alpha(s+m)-\alpha(s)\rangle + m_L\right) = 1. \end{aligned}$$

Evaluating at $L=0$, since $m(s,0)=0$, we find that

$$m_L(s,0)=1.$$

The third derivative of (7) yields

$$\begin{aligned}&\big(m_{LLL} - \kappa^2(s+m)\, m_L^3\big)\,\big\langle \mathbf{t}(s+m), \alpha(s+m) - \alpha(s)\big\rangle\\ &\quad + \big(3\, m_L\, m_{LL}\, \kappa(s+m) + m_L^3\, \kappa'(s+m)\big)\,\big\langle \mathbf{n}(s+m), \alpha(s+m) - \alpha(s)\big\rangle\\ &\quad - m_L^3\, \kappa(s+m)\, \tau(s+m)\,\big\langle \mathbf{b}(s+m), \alpha(s+m) - \alpha(s)\big\rangle + 3\, m_L\, m_{LL} = 0.\end{aligned}$$

Evaluating at $L = 0$, since $m_L(s, 0) = 1$, we get $m_{LL}(s, 0) = 0$.

Hence, the fourth derivative evaluated at $L = 0$ gives

$$4\, m_{LLL}(s, 0) - \kappa^2(s) = 0,$$

so that $m_{LLL}(s, 0) = \frac{1}{4}\,\kappa^2(s)$.

Now, let us compute the derivatives of ϕ. Differentiating (8) with respect to u, we get

$$-\sin\phi(s, u)\,\phi_u(s, u) = \kappa(s+u)\,\big\langle \mathbf{t}(s), \mathbf{n}(s+u)\big\rangle.$$

Differentiating again,

$$\begin{aligned}&\cos\phi(s, u)\,\big(\kappa^2(s+u) - \phi_u^2(s, u)\big) - \sin\phi(s, u)\,\phi_{uu}(s, u)\\ &\qquad = \kappa'(s+u)\,\big\langle \mathbf{t}(s), \mathbf{n}(s+u)\big\rangle - \kappa(s+u)\,\tau(s+u)\,\big\langle \mathbf{t}(s), \mathbf{b}(s+u)\big\rangle.\end{aligned}$$

Evaluating at $u = 0$, since $\phi(s, 0) = 0$, we obtain $\phi_u^2(s, 0) = \kappa^2(s)$, so that $\phi_u(s, 0) = \kappa(s)$. The third derivative is

$$\begin{aligned}&3\cos(\phi)\,\big(\kappa'(s+u)\,\kappa(s+u) - \phi_u\,\phi_{uu}\big) - \sin(\phi)\,\big(\kappa^2(s+u)\,\phi_u - \phi_u^3 + \phi_{uuu}\big)\\ &= \big(\kappa''(s+u) - \kappa(s+u)\,\tau^2(s+u)\big)\,\big\langle \mathbf{t}(s), \mathbf{n}(s+u)\big\rangle\\ &- \big(2\,\kappa'(s+u)\,\tau(s+u) + \kappa(s+u)\,\tau'(s+u)\big)\,\big\langle \mathbf{t}(s), \mathbf{b}(s+u)\big\rangle.\end{aligned}$$

Evaluating at $u = 0$, it reduces to

$$\kappa'(s)\,\kappa(s) - \phi_u(s, 0)\,\phi_{uu}(s, 0) = 0.$$

Since $\phi_u(s, 0) = \kappa(s) \neq 0$, we deduce $\phi_{uu}(s, 0) = \kappa'(s)$.

Finally, the fourth derivative of (8) evaluated at $u = 0$ gives

$$4\,\kappa(s)\,\kappa''(s) - 4\,\kappa(s)\,\phi_{uuu}(s, 0) = \kappa^2(s)\,\tau^2(s).$$

Since $\kappa(s) \neq 0$, we deduce

$$\phi_{uuu}(s, 0) = \kappa''(s) - \frac{1}{4}\,\kappa(s)\,\tau^2(s).$$

$\square$

Remark 1. From Lemma 2, notice that a third order approximation for the functions m and ϕ can be written:

$$m(s, L) = L + \frac{1}{24}\,\kappa^2(s)\,L^3 + \mathcal{O}\big(L^4\big).$$

and

$$\phi(s, u) = \kappa(s)\,u + \frac{\kappa'(s)}{2}\,u^2 + \frac{1}{6}\bigg(\kappa''(s) - \frac{1}{4}\,\kappa(s)\,\tau^2(s)\bigg)u^3 + \mathcal{O}\big(u^4\big).$$

Now an easy approximation for the area of a p-Holditch surface can be found in the general case using Lemma 2.

Theorem 3. *Let α be a regular closed curve with non-vanishing curvature κ. Given $p \in [0,1]$, if A_H is the area of the p-Holditch surface of α up to an admissible for regularity length $L > 0$, then*

$$\frac{A_H}{p\,(1-p)} \approx \frac{L^2}{2} \int_I \kappa(s)\ ds - \frac{L^4}{96} \int_I \kappa(s)\,\tau^2(s)\ ds.$$

Proof. We are going to expand the double integral for the area of the considered p-Holditch surface,

$$F(L) = \int_I \int_0^{m(s,L)} \sin\phi(s,u)\ \mathrm{d}u\ \mathrm{d}s,$$

by Taylor series at $L = 0$. The desired expansion will have the form

$$F(L) = F(0) + L\,F'(0) + \frac{L^2}{2}\,F''(0) + \frac{L^3}{6}\,F'''(0) + \frac{L^4}{24}\,F^{(4)}(0) + \dots$$

First, note that $\phi(s,0) = 0$ and $m(s,0) = 0$, so $F(0) = 0$. Now, define

$$G_s(L) := \int_0^{m(s,L)} \sin\phi(s,u)\ \mathrm{d}u.$$

The first derivative of G_s is

$$G_s'(L) = \sin\phi\big(s, m(s,L)\big)\, m_L(s,L).$$

Since $G_s'(0) = 0$, we have that $F'(0) = 0$.

The second derivative of G_s is

$$\begin{aligned} G_s''(L) = {} & \cos\phi\big(s, m(s,L)\big)\ \phi_u\big(s, m(s,L)\big)\, m_L^2(s,L) \\ & + \sin\phi\big(s, m(s,L)\big)\, m_{LL}(s,L). \end{aligned}$$

Evaluating at $L = 0$, we find

$$G_s''(0) = \phi_u(s,0)\, m_L^2(s,0).$$

Since $m_L(s,0) = 1$ and $\phi_u(s,0) = \kappa(s)$ by Lemma 2, we have $G_s''(0) = \kappa(s)$, so that

$$F''(0) = \int_I \kappa(s)\ \mathrm{d}s.$$

The third derivative of G_s is

$$\begin{aligned} G_s'''(L) = {} & m_L\ \cos(\phi)\ \big(\phi_{uu}\, m_L^2 + 3\,\phi_u\, m_{LL}\big) \\ & + \sin(\phi)\ \big(m_{LLL} - \phi_u^2\, m_L^3\big). \end{aligned}$$

Evaluated at $L = 0$ yields

$$G_s'''(0) = \phi_{uu}(s,0) + 3\,\kappa(s)\, m_{LL}(s,0).$$

Again, by Lemma 2, $\phi_{uu}(s,0) = \kappa'(s)$ and $m_{LL}(s,0) = 0$. Thus, $G_s''(0) = \kappa'(s)$ and

$$F'''(0) = \int_I \kappa'(s)\, \mathrm{d}s.$$

Since α is a closed curve, κ is periodic on I and we deduce $F'''(0) = 0$.

The fourth derivative of G_s evaluated at $L = 0$ is

$$G_s^{(4)}(0) = \big(\phi_{uuu}(s,0) + 3\,\kappa(s)\, m_{LLL}(s,0)\big) + \kappa(s)\left(m_{LLL}(s,0) - \kappa^2(s)\right).$$

By Lemma 2, $m_{LLL}(s,0) = \frac{1}{4}\,\kappa^2(s)$ and $\phi_{uuu}(s,0) = \kappa''(s) - \frac{1}{4}\,\kappa(s)\,\tau^2(s)$, so simplifying,

$$G_s^{(4)}(0) = \kappa''(s) - \frac{1}{4}\,\kappa(s)\,\tau^2(s).$$

Hence,

$$F^{(4)}(0) = -\frac{1}{4}\int_I \kappa(s)\,\tau^2(s)\, \mathrm{d}s,$$

where we have used that $\int_I \kappa''(s)\, \mathrm{d}s = 0$ because it is the integral of the derivative of a periodic function.

□

Remark 2. Following the same idea as in Lemma 2, it can be computed explicitly that

$$m_{LLLL}(s,0) = \kappa(s)\,\kappa'(s)$$

and

$$\phi^{(4)}(s,0) = -\frac{1}{2}\,\tau^2(s)\,\kappa'(s) - \kappa(s)\,\tau(s)\,\tau'(s) + \kappa^{(3)}(s).$$

With that, also

$$G_s^{(5)}(0) = \frac{3}{2}\,\kappa^2(s)\,\kappa'(s) - \frac{1}{2}\tau^2(s)\,\kappa'(s) - \kappa(s)\,\tau(s)\,\tau'(s) + \kappa^{(3)}(s).$$

Now, notice that

$$-\frac{1}{2}\left(\kappa(s)\,\tau^2(s)\right)' = -\frac{1}{2}\tau^2(s)\,\kappa'(s) - \kappa(s)\,\tau(s)\,\tau'(s)$$

and

$$\frac{1}{2}\left(\kappa^3(s)\right)' = \frac{3}{2}\,\kappa^2(s)\,\kappa'(s).$$

Therefore, $G_s^{(5)}(0)$ can be written as a sum of three exact derivatives of periodic functions, which means that $F^{(5)}(0) = 0$ if α is closed. Hence, the expression given in Theorem 3 has 5th order of approximation:

$$\frac{A_H}{p\,(1-p)} = \frac{L^2}{2}\int_I \kappa(s)\, \mathrm{d}s - \frac{L^4}{96}\int_I \kappa(s)\,\tau^2(s)\, \mathrm{d}s + \mathcal{O}(L^6).$$

Remark 3. The planar Holditch's theorem (also for non-closed curves) can be stated by means of the area swept out by a piece of the moving chord. In such a case, the Holditch area is equal to

$$\frac{1}{2}\,(p\,L)\,\big((1-p)\,L\big)\int_I \kappa(s)\,\mathrm{d}s,$$

where notice that the integral of the curvature measures the total angle swept out by the moving chord seen as an indicatrix (see [7] for a further discussion on this).

In this case, taking only the first term of the approximation given in Theorem 3, the exact value of the Holditch area is obtained.

Let us compute the Holditch area in an example.

Example 1. Consider the closed curve

$$\alpha(t) = \big(\cos t,\, \sin t,\, \sin^2 t\big), \quad t \in [0, 2\pi]\,.$$

The p-Holditch surface of α up to a length L is given by

$$\begin{aligned}\mathbf{x}^p(s,u) = \big(&p\cos(t+u) + (1-p)\cos(t),\, p\sin(t+u) + (1-p)\sin(t),\\ &p\sin^2(t+u) + (1-p)\sin^2(t)\big),\end{aligned}$$

for $t \in [0, 2\pi]$ and $u \in\,]0, m(t,L)[$. In Figure 4, two examples of these surfaces are represented for different choices of p. We have computed the function $m(t,L)$ numerically.

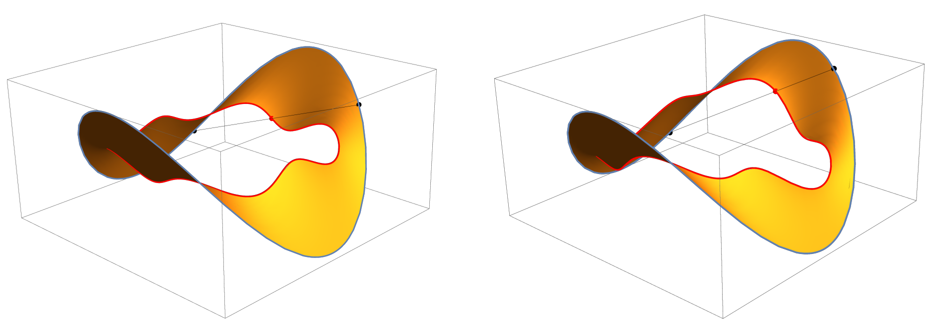

Figure 4. The p-Holditch surface of α up to a length $L = 3/2$. On left, $p = 1/2$; on right, $p = 1/3$.

For example, the area of the Holditch surface for $L = 1$ and $p = 1/2$ computed as the area of a surface is $A_H \approx 1.05461$. Using the expression of Theorem 3, we get 1.05497. If we cut the same expression to third order, the approximation is 1.06503. Using a higher approximation order, a more accurate answer is given. For instance, the value obtained for a sixth order approximation is 1.05453. See Figure 5 to compare, for each L, the value of Holditch area in the surface with the given approximations.

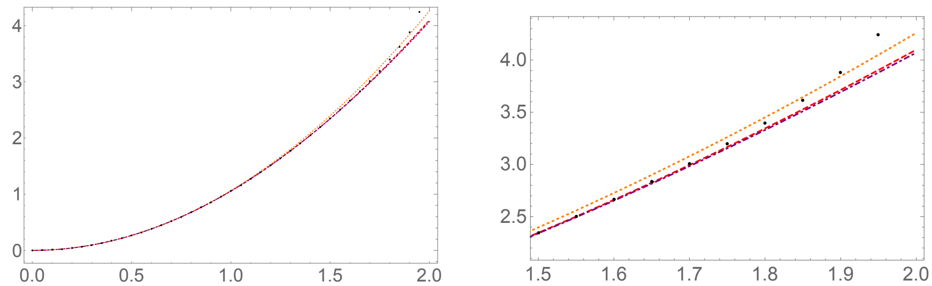

FIGURE 5. For $p = 1/2$, the big black points represent the shape of the Holditch area function when varying the length L of the chord. We represent in orange-dotted, the third order approximation function; in red-dashed, the fifth order and in purple-dot-dashed a sixth order approximation. For small values of L, they are very similar. For bigger L, little differences can be seen (right).

The maximum admissible length in this example is $L = 2$, for which $m(t, 2) = \pi$ for all $t \in [0, 2\pi]$. The $\frac{1}{2}$-Holditch curve of α for $L = 2$ is just a segment of double points. Moreover, the $\frac{1}{2}$-Holditch surface of α up to $L = 2$ has, at the ends of that segment, two Whitney umbrella type singularities (see Figure 6).

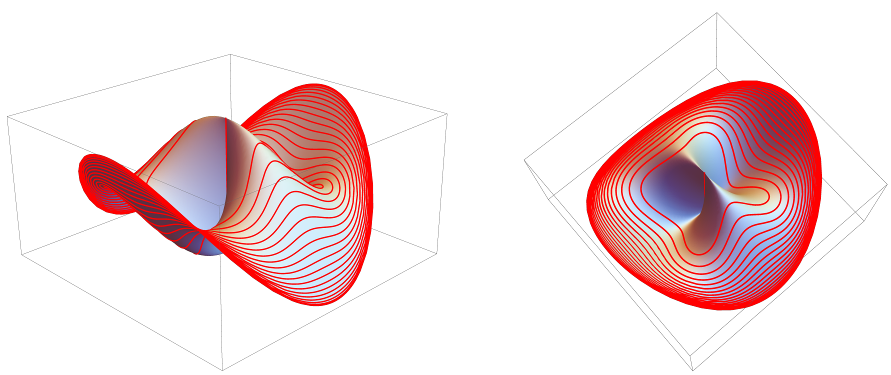

FIGURE 6. The $\frac{1}{2}$-Holditch surface of α up to a length $L = 2$ exhibit two Whitney umbrella type singularities.

Let us present now another example of Holditch surface but for a non-closed space curve: the circular helix.

Example 2. Let $a > 0$, $b \neq 0$ and $c = \sqrt{a^2 + b^2}$. Consider as initial curve a circular helix parameterized by arc-length:

$$\alpha_{a,b}(s) = \left(a\cos\left(\frac{s}{c}\right),\, a\sin\left(\frac{s}{c}\right),\, \frac{b\,s}{c} \right).$$

In Figure 7, some p-Holditch surfaces of a circular helix are represented.

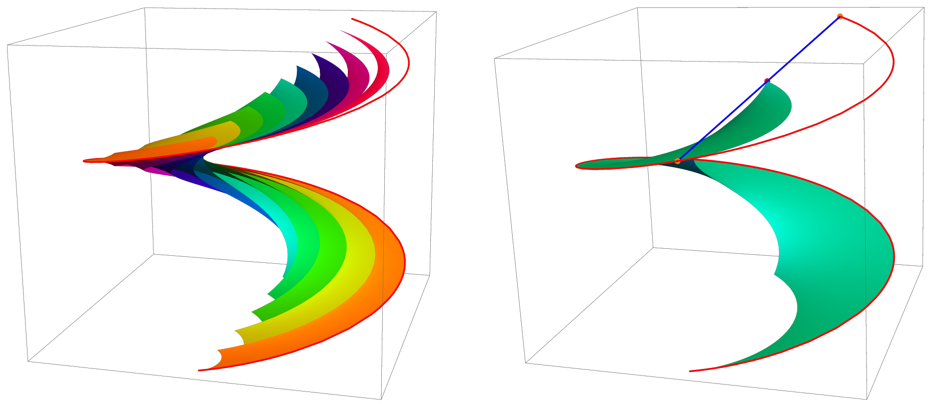

FIGURE 7. On left, for different choices of p, some p-Holditch surfaces of $\alpha_{1,1/5}$ up to a length $L = 2$ are plotted. On right, it is only represented the $\frac{1}{2}$-Holditch surface, which is a helicoid.

We know that $\kappa(s) = \frac{a}{a^2+b^2}$ and $\tau(s) = -\frac{b}{a^2+b^2}$ are constant. In this case,

$$\phi(s,u) = \arccos\left(\frac{b^2 + a^2\cos\left(\frac{u}{c}\right)}{c^2}\right).$$

By Proposition 4, the curvatures of the Holditch surfaces of $\alpha_{a,b}$ can be easily computed. The Gauss curvature is always negative:

$$K(s,u) = -\frac{b^2}{p\,q\left(a^2 + 2\,b^2 + a^2\cos\left(\frac{u}{c}\right)\right)^2}.$$

And the mean curvature

$$H(s,u) = \frac{b\,c^2\,(q-p)\csc\left(\frac{u}{2c}\right)}{2\sqrt{2}\,a\,p\,q\left(a^2 + 2\,b^2 + a^2\,\cos\left(\frac{u}{c}\right)\right)^2}$$

is zero if and only if $p = q = \frac{1}{2}$.

Let us study separately the case $p = \frac{1}{2}$. The parameterization of the $\frac{1}{2}$-Holditch surface of $\alpha_{a,b}$ up to an admissible for regularity length L is

$$\mathbf{x}_{a,b}(s,u) = \left(\frac{a}{2}\left(\cos\left(\frac{s}{c}\right) + \cos\left(\frac{s+u}{c}\right)\right),\right.$$
$$\left.\frac{a}{2}\left(\sin\left(\frac{s}{c}\right) + \sin\left(\frac{s+u}{c}\right)\right), \frac{b\left(s+\frac{u}{2}\right)}{c}\right).$$

We are going to prove that $\mathbf{x}_{a,b}$ is a helicoid. A helicoid has parameterization

$$\mathbf{y}_\beta(s,u) = \big(u\cos(s),\ u\sin(s), \beta s\big).$$

Since the parameter change

$$g_{a,b}(s,u) = \left(\frac{2\,s+u}{2\,c},\ a\cos\!\left(\frac{u}{2\,c}\right)\right)$$

verifies that

$$\mathbf{y}_b\big(g_{a,b}(s,u)\big) = \mathbf{x}_{a,b}(s,u),$$

then we have that the Holditch surface $\mathbf{x}_{a,b}$ is a helicoid.

Recall that minimal surfaces are those that locally minimize their areas. Equivalently, surfaces with zero mean curvature. Therefore, we have found that the helicoid is an example of minimal Holditch surface.

From the previous example, now we ask for all the minimal Holditch surfaces. It turns out that the unique non-planar regular minimal Holditch surface is the one seen above: the helicoid.

Theorem 4. *Let α be a regular curve with non-vanishing curvature and torsion. The only minimal non-planar regular Holditch surface is the helicoid.*

Proof. Using the local canonical form of α in a neighborhood of $u=0$ up to third order (see [3], page 27), from Proposition 4, we obtain the first term of the Laurent series at $u=0$ for the mean curvature H:

$$\frac{(p-q)\,\tau(s)}{4\,p\,q\,\kappa(s)}\,\frac{1}{u}.$$

If the Holditch surface is minimal, then $H=0$. That implies $p=q=\frac{1}{2}$. Thus, the Taylor series of H at $u=0$ reduces to

$$\begin{aligned} H(s,u) = &-\frac{2\,\kappa'(s)\,\tau(s)+\kappa(s)\,\tau'(s)}{6\,\kappa^2(s)} \\ &+\kappa'(s)\,\frac{4\,\kappa'(s)\,\tau(s)-\kappa(s)\,\tau'(s)}{12\,\kappa^3(s)}\,u+\mathcal{O}\big(u^2\big). \end{aligned}$$

Notice that it can be rewritten as

$$H(s,u) = -\frac{\big(\kappa^2(s)\,\tau(s)\big)'}{6\,\kappa^3(s)}+\frac{\kappa'(s)\,\tau^2(s)}{12\,\kappa^6(s)}\left(\frac{\kappa^4(s)}{\tau(s)}\right)'u+\mathcal{O}\big(u^2\big).$$

Because of being $H=0$, from the term in u we deduce that one, $\kappa(s)$ or $\kappa^4(s)/\tau(s)$, is constant. In the first case, if κ is constant, from the independent term, τ must be constant. In the second case, if $\kappa^4(s)/\tau(s)$ is constant, since the independent term implies $\kappa^2(s)\,\tau(s)$ constant, the same conclusion is deduced: both κ and τ are constant. Therefore α must be a circular helix.

Since we have seen in the Example 2 that the $\frac{1}{2}$-Holditch surface of a circular helix is a helicoid, which is minimal, the rest of the terms in the Taylor series of H will be automatically zero and satisfied with the condition of being κ and τ constant. □

Remark 4. From the example of the circular helix, one can ask if the generated Holditch surfaces (or the helicoid, in particular) can be extended on the "outside" part of the cylinder which contains the helix. In other words, is there any family of surfaces naturally defined such that connect well with the initial curve and the defined Holditch surfaces? The answer to this question is affirmative but it may be worth a detailed full study in a separate work.

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