CURVES WITH CONSTANT CURVATURE RATIOS

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Abstract. Curves in $\mathbb{R}^n$ for which the ratios between two consecutive curvatures are constant are characterized by the fact that their tangent indicatrix is a geodesic in a flat torus. For $n = 3, 4$, spherical curves of this kind are also studied and compared with intrinsic helices in the sphere.

1. Introduction

The notion of a generalized helix in $\mathbb{R}^3$, a curve making a constant angle with a fixed direction, can be generalized to higher dimensions in many ways. In [7] the same definition is proposed but in $\mathbb{R}^n$. In [4] the definition is more restrictive: the fixed direction makes a constant angle with all the vectors of the Frenet frame. It is easy to check that this definition only works in the odd dimensional case. Moreover, in the same reference, it is proven that the definition is equivalent to the fact that the ratios $\frac{k_2}{k_1}, \frac{k_4}{k_3}, \dots,$ k_i being the curvatures, are constant. This statement is related with the Lancret Theorem for generalized helices in $\mathbb{R}^3$ (the ratio of torsion to curvature is constant). Finally, in [1] the author proposes a definition of a general helix in a 3-dimensional real-space-form substituting the fixed direction in the usual definition of generalized helix by a Killing vector field along the curve.

In this paper we study the curves in $\mathbb{R}^n$ for which all the ratios $\frac{k_2}{k_1}, \frac{k_3}{k_2}, \frac{k_4}{k_3}, \dots$ are constant. We call them curves with constant curvature ratios or ccr-curves. The main result is that, in the even dimensional case, a curve has constant curvature ratios if and only if its tangent indicatrix is a geodesic in the flat torus. In the odd case, a constant must be added as the new coordinate function.

In the last section we show that a ccr-curve in S^3 is a general helix in the sense of [1] if and only if it has constant curvatures. To achieve this result, we have obtained the characterization of spherical curves in $\mathbb{R}^4$ in terms of the curvatures. Moreover, we have also found explicit examples of spherical ccr-curves with non-constant curvatures.

2. Frenet's elements for a curve in $\mathbb{R}^n$

Let us recall from [5] the definition of the Frenet frame and curvatures.

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For C^{n-1} curves, α, which have linearly independent derivatives up to order $n-1$, the moving Frenet frame is constructed as it were in usual space using the Gram-Schmidt process. Orthonormal vectors $\{\overrightarrow{\mathbf{e_1}}, \overrightarrow{\mathbf{e_2}}, \dots, \overrightarrow{\mathbf{e_{n-1}}}\}$ are obtained and the last vector is added as the unit vector in $\mathbb{R}^n$ such that $\{\overrightarrow{\mathbf{e_1}}, \overrightarrow{\mathbf{e_2}}, \dots, \overrightarrow{\mathbf{e_n}}\}$ is an orthonormal basis with positive orientation.

The ith curvature is defined as

$$k_i = \frac{< \dot{\overrightarrow{\mathbf{e_i}}}, \overrightarrow{\mathbf{e_{i+1}}} >}{||\alpha'||},$$

for $i = 1, \dots, n-1$.

Frenet's formulae in n-space can be written as

(2.1)

$$\begin{pmatrix} \dot{\overrightarrow{\mathbf{e_1}}}(s) \\ \dot{\overrightarrow{\mathbf{e_2}}}(s) \\ \dot{\overrightarrow{\mathbf{e_3}}}(s) \\ \vdots \\ \dot{\overrightarrow{\mathbf{e_{n-1}}}}(s) \\ \dot{\overrightarrow{\mathbf{e_n}}}(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1 & 0 & 0 & \dots & 0 & 0 \\ -k_1 & 0 & k_2 & 0 & \dots & 0 & 0 \\ 0 & -k_2 & 0 & k_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & k_{n-1} \\ 0 & 0 & 0 & 0 & \dots & -k_{n-1} & 0 \end{pmatrix} \begin{pmatrix} \overrightarrow{\mathbf{e_1}}(s) \\ \overrightarrow{\mathbf{e_2}}(s) \\ \overrightarrow{\mathbf{e_3}}(s) \\ \vdots \\ \overrightarrow{\mathbf{e_{n-1}}}(s) \\ \overrightarrow{\mathbf{e_n}}(s) \end{pmatrix}.$$

In accordance with [7] we will say that a curve is twisted if its last curvature, k_{n-1} is not zero. Sometimes, we will also say that the curve is not regular.

3. CCR-CURVES

Instead of looking for curves making a constant angle with a fixed direction as in [4] or [7], we will study another way of generalizing the notion of helix.

Definition 1. *A curve $\alpha : I \to \mathbb{R}^n$ is said to have constant curvature ratios (that is to say, it is a ccr-curve) if all the quotients $\frac{k_{i+1}}{k_i}$ are constant.*

As is well known, generalized helices in $\mathbb{R}^3$ are characterized by the fact that the quotient $\frac{\tau}{\kappa}$ is constant (Lancret's theorem). It is in this sense that ccr-curves are a generalization to $\mathbb{R}^n$ of generalized helices in $\mathbb{R}^3$.

In [4] the author defines a generalized helix in the n-dimensional space (n odd) as a curve satisfying that the ratios $\frac{k_2}{k_1}, \frac{k_4}{k_3}, \dots$ are constant. It is also proven that a curve is a generalized helix if and only if there exists a fixed direction which makes constant angles with all the vectors of the Frenet frame. Obviously, ccr-curves are a subset of generalized helices in the sense of [4].

3.1. **Examples.**

3.1.1. *Example with constant curvatures.* The subset of $\mathbb{R}^{2n}$ parametrized by $\overrightarrow{\mathbf{x}}(u_1, u_2, \dots, u_n) =$

$$= (r_1 \cos(u_1), r_1 \sin(u_1), r_2 \cos(u_2), r_2 \sin(u_2), \dots, r_n \cos(u_n), r_n \sin(u_n))$$

where $u_i \in \mathbb{R}$ is called a flat torus in $\mathbb{R}^{2n}$.

By analogy, the subset of $\mathbb{R}^{2n+1}$ parametrized by $\overrightarrow{\mathbf{x}}(u_1, u_2, \dots, u_n) =$

$$= (r_1 \cos(u_1), r_1 \sin(u_1), r_2 \cos(u_2), r_2 \sin(u_2), \dots, r_n \cos(u_n), r_n \sin(u_n), a)$$

where $u_i \in \mathbb{R}$ and a is a real constant, will be called a flat torus in $\mathbb{R}^{2n+1}$.

It is just a matter of computation to show that any curve in a flat torus of the kind

$$\alpha(t) = \overrightarrow{\mathbf{x}}(m_1 t, m_2 t, \dots, m_n t)$$

has all its curvatures constant (see [6]).

These curves are the geodesics of the flat tori, and it is proven in the cited paper that they are twisted curves if and only if the constants $m_i \neq m_j$ for all $i \neq j$.

3.1.2. *Example with non-constant curvatures.* Now, let $k(s)$ be a positive function. Let us define $g(s) = \int_0^s k(u)du$. If α is a curve parametrized by its arc-length and with constant curvatures, $a_1, a_2, \dots, a_{n-1}$, then the curve $\beta(s) = \int_0^s \overrightarrow{\mathbf{e_1}}^\alpha(g(u))du$ is a curve whose curvatures are $k_i(s) = a_i k(s)$.

Note that $\dot{\beta}(s) = \overrightarrow{\mathbf{e_1}}^\alpha(g(s))$. This implies that $\overrightarrow{\mathbf{e_1}}^\beta(s) = \overrightarrow{\mathbf{e_1}}^\alpha(g(s))$. Taking derivatives $k_1^\beta(s)\overrightarrow{\mathbf{e_2}}^\beta(s) = k_1^\alpha(g(s))\overrightarrow{\mathbf{e_2}}^\alpha(g(s))k(s)$. Therefore,

$$\overrightarrow{\mathbf{e_2}}^\beta(s) = \overrightarrow{\mathbf{e_2}}^\alpha(g(s)), \qquad \text{and} \qquad k_1^\beta(s) = a_1 k(s).$$

By similar arguments it is possible to show that $k_i^\beta(s) = a_i k(s)$ for any $i = 1, \dots, n-1$. Therefore, β is a ccr-curve with non-constant curvatures.

In the next section we will show that any ccr-curve is of this kind.

4. Solving the natural equations for ccr-curves

The Frenet formulae can be explicitly integrated only for some particular cases. Ccr-curves are one of these. In fact, Frenet's formulae are

$$\begin{pmatrix} \dot{\overrightarrow{\mathbf{e_1}}}(s) \\ \dot{\overrightarrow{\mathbf{e_2}}}(s) \\ \dot{\overrightarrow{\mathbf{e_3}}}(s) \\ \vdots \\ \dot{\overrightarrow{\mathbf{e_{n-1}}}}(s) \\ \dot{\overrightarrow{\mathbf{e_n}}}(s) \end{pmatrix} = k_1(s) \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & c_2 & 0 & \dots & 0 & 0 \\ 0 & -c_2 & 0 & c_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & c_{n-1} \\ 0 & 0 & 0 & 0 & \dots & -c_{n-1} & 0 \end{pmatrix} \begin{pmatrix} \overrightarrow{\mathbf{e_1}}(s) \\ \overrightarrow{\mathbf{e_2}}(s) \\ \overrightarrow{\mathbf{e_3}}(s) \\ \vdots \\ \overrightarrow{\mathbf{e_{n-1}}}(s) \\ \overrightarrow{\mathbf{e_n}}(s) \end{pmatrix},$$

for some constants, $c_2, \dots, c_{n-1}$.

Reparametrization of the curve allows that system to be reduced to an easier one. The reparametrization is given by the inverse function of

$$g(s) = \int_0^s k_1(u)du.$$

Note that $t = g(s)$ is a reparametrization because k_1 is a positive function. The reparametrization we need is the inverse function $s = g^{-1}(t)$. It is a simple matter to verify that, with respect to parameter t, the Frenet's formulae are reduced to a linear system of first order differential equations with constant coefficients

$$\begin{pmatrix} \overrightarrow{\mathbf{e_1}}'(t) \\ \overrightarrow{\mathbf{e_2}}'(t) \\ \overrightarrow{\mathbf{e_3}}'(t) \\ \vdots \\ \overrightarrow{\mathbf{e_{n-1}}}'(t) \\ \overrightarrow{\mathbf{e_n}}'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 0 & c_2 & 0 & \dots & 0 & 0 \\ 0 & -c_2 & 0 & c_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & c_{n-1} \\ 0 & 0 & 0 & 0 & \dots & -c_{n-1} & 0 \end{pmatrix} \begin{pmatrix} \overrightarrow{\mathbf{e_1}}(t) \\ \overrightarrow{\mathbf{e_2}}(t) \\ \overrightarrow{\mathbf{e_3}}(t) \\ \vdots \\ \overrightarrow{\mathbf{e_{n-1}}}(t) \\ \overrightarrow{\mathbf{e_n}}(t) \end{pmatrix}. \tag{4.1}$$

We can apply the well-known methods of integration of systems of linear equations with constant coefficients. Let F_n be the matrix of constant coefficients of this system.

4.1. **Eigenvalues and their multiplicity.** The first thing we have to do is to compute the eigenvalues of the coefficient matrix.

Due to the skewsymmetry of the matrix, it can have not real eigenvalues other than zero. Due to the fact that the determinant of F_n vanishes only for odd n, we can say that for odd dimensions, 0 is an eigenvalue, whereas for even dimensions, 0 is an eigenvalue only if $k_{n-1} = 0$.

By definition, we have that constants $c_2, c_3, \dots, c_{n-2}$ are not zero. If the last constant, c_{n-1}, vanishes, then the same happens with the last curvature function k_{n-1}. In this case the curve is included in a hyperspace, so we can consider it to be a curve in an $n-1$ dimensional space.

Therefore, from now on, we shall consider that all the curvatures, and then all the constants c_i, are not zero.

Note that, in this case, for any $x \in \mathbb{C}$, the rank (in $\mathbb{C}$) of the matrix

$$\begin{pmatrix} x & 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & x & c_2 & 0 & \dots & 0 & 0 \\ 0 & -c_2 & x & c_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & x & c_{n-1} \\ 0 & 0 & 0 & 0 & \dots & -c_{n-1} & x \end{pmatrix}$$

is at least $n-1$. Therefore, their eigenvalues are all of multiplicity 1.

4.2. **Canonical Jordan form.** Let $a_\ell \pm \mathbf{i} b_\ell$, $\ell = 1, \dots, [\frac{n}{2}]$, with $a_\ell, b_\ell \in \mathbb{R}$, be the non-zero eigenvalues of the coefficient matrix. Therefore, for $n = 2k$, the associated canonical Jordan form is of the kind

$$\begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_k \end{pmatrix}$$

where $J_\ell = \begin{pmatrix} a_\ell & -b_\ell \\ b_\ell & a_\ell \end{pmatrix}$.

The matrix can be diagonalized because all the eigenvalues are of multiplicity one. Therefore, there is a orthogonal matrix, S, such that if C is the matrix of constant coefficients, then

$$C = S^{-1} J S.$$

Therefore, the general solution of the system for the first vector is

$$\overrightarrow{\mathbf{e}}_1(u) := \sum_{\ell=1}^{k} \overrightarrow{A_\ell}\ e^{a_\ell u} \cos(b_\ell\ u) + \overrightarrow{B_\ell}\ e^{a_\ell u} \sin(b_\ell\ u),$$

where $\{\overrightarrow{A_\ell}, \overrightarrow{B_\ell}\}_{\ell=1}^{k}$ is a family of orthogonal vectors.

For $n = 2k + 1$, the associated canonical Jordan form is of the kind

$$\begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & J_1 & 0 & \dots & 0 \\ 0 & 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & J_k \end{pmatrix}$$

Now, the general solution of the system for the first vector is

$$\overrightarrow{\mathbf{e}}_1(u) := \overrightarrow{A_0} + \sum_{\ell=1}^{k} \overrightarrow{A_\ell}\ e^{a_\ell u} \cos(b_\ell\ u) + \overrightarrow{B_\ell}\ e^{a_\ell u} \sin(b_\ell\ u),$$

where $\{\overrightarrow{A_0}\} \cup \{\overrightarrow{A_\ell}, \overrightarrow{B_\ell}\}_{\ell=1}^{k}$ is a family of orthogonal vectors.

4.3. **The eigenvalues are pure imaginaries.** Condition $||\overrightarrow{\mathbf{e}}_1(u)|| = 1$ for all u implies that all the real parts of the eigenvalues are zero. Indeed, if, for example, $a_1 \neq 0$, then let m be a non-zero coordinate of $\overrightarrow{A_1}$.

Bearing in mind that

$$|m|\ e^{a_1 u}\ |\cos(b_1 u)| \leq ||\overrightarrow{\mathbf{e}}_1(u)||,$$

and that the left-hand member is an unbounded function, then $||\overrightarrow{\mathbf{e}}_1(u)|| \neq 1$.

Therefore, all the real parts of the eigenvalues are zero and the general solution (in the even case) of the system for the first vector is

$$\overrightarrow{\mathbf{e}}_1(u) := \sum_{\ell=1}^{k} \overrightarrow{A_\ell}\cos(b_\ell\ u) + \overrightarrow{B_\ell}\sin(b_\ell\ u).$$

Analogously for the odd case.

Moreover, let us recall that the vectors $\{\overrightarrow{A_i}, \overrightarrow{B_i}\}_{i=1}^k$ are an orthogonal base of $\mathbb{R}^n$ associated to the canonical Jordan form.

4.4. **The main result.** Finally, an isometry of $\mathbb{R}^n$ allows us to state the next result.

Theorem 1. *A curve has constant curvature ratios if and only if its tangent indicatrix is a twisted geodesic on a flat torus.*

Note that in the odd dimensional case this result implies that the last coordinate of the tangent indicatrix is a constant. So, there is a direction making a constant angle with the curve. Nevertheless, this is not the case in the even dimensional case. There are no fixed directions making a constant angle with the tangent vector.

When all the curvatures are constant, then the curve is also a ccr-curve and its tangent indicatrix is of the kind described in the previous statement. Moreover, the reparametrization $g(s) = \int_0^s k_1(u)du$ is just the product by a constant.

Since the integration of a geodesic on a flat torus in $\mathbb{R}^{2k}$ with respect to its parameter is again a curve of the same kind, we get the following corollary:

Corollary 1. *A curve has constant curvatures if and only if it is*

1. *a twisted geodesic on a flat torus, in the even dimensional case, or*
2. *a twisted geodesic on a flat torus times a linear function of the parameter, in the odd dimensional case.*

4.5. $n = 3$**.** The eigenvalues of the matrix of coefficients are 0 and $\pm\sqrt{1+c^2}\,\mathbf{i}$ ($c = c_2$, to simplify).

Therefore, the general solution of the system for the first vector is

$$\overrightarrow{\mathbf{e_1}}(u) = \overrightarrow{A_1} + \overrightarrow{A_2}\cos(\sqrt{1+c^2}u) + \overrightarrow{A_3}\sin(\sqrt{1+c^2}u),$$

where $\overrightarrow{A_i}, i = 1, 2, 3$ are constant vectors.

Once we have the tangent vector, we only have to undo the reparametrization and to integrate to obtain the curve

$$\alpha(s) = x_0 + \overrightarrow{c_1}s + \overrightarrow{c_2}\int_0^s \cos(\sqrt{1+c^2}g(v))dv + \overrightarrow{c_3}\int_0^s \sin(\sqrt{1+c^2}g(v))dv.$$

4.6. $n = 4$. The eigenvalues are

$$\pm\frac{\mathbf{i}}{\sqrt{2}}\sqrt{1 + c_2^2 + c_3^2 \pm \sqrt{(1 + c_2^2 + c_3^2)^2 - 4c_3^2}}.$$

Therefore, the general solution of the system for the first vector is

$$\overrightarrow{\mathbf{e}}_1(u) := \overrightarrow{A_1}\cos(m_+u) + \overrightarrow{B_1}\sin(m_+u) + \overrightarrow{A_2}\cos(m_-u) + \overrightarrow{B_2}\sin(m_-u),$$

where

$$m_\pm = \frac{1}{\sqrt{2}}\sqrt{1 + c_2^2 + c_3^2 \pm \sqrt{(1 + c_2^2 + c_3^2)^2 - 4c_3^2}}$$

and where $\overrightarrow{A_i}, \overrightarrow{B_i}, i = 1, 2$ are constant vectors.

5. Spherical ccr-curves

In order to compare ccr-curves with the definition of generalized helices given in [1], we will try to determine which ccr-curves are included in a sphere.

Lemma 1. *A curve $\alpha : I \to \mathbb{R}^4$ is spherical, i.e., it is contained in a sphere of radius R, if and only if*

$$\frac{1}{k_1^2} + \left(\frac{\dot{k_1}}{k_1^2k_2}\right)^2 + \frac{1}{k_3^2}\left(\left(\frac{\dot{k_1}}{k_1^2k_2}\right)^{\cdot} - \frac{k_2}{k_1}\right)^2 = R^2. \tag{5.1}$$

Proof. The proof here is similar to that for spherical curves in $\mathbb{R}^3$. It consists in obtaining information thanks to successive derivatives of the expression $< \alpha(s) - m, \alpha(s) - m >= R^2$, where m is the center of the sphere. In particular, what can be proven is that spherical curves can be decomposed as

$$\alpha(s) = m - \frac{R}{k_1}\overrightarrow{\mathbf{e_2}}(s) + R\frac{\dot{k_1}}{k_1^2k_2}\overrightarrow{\mathbf{e_3}}(s) + R\frac{1}{k_3}\left(\left(\frac{\dot{k_1}}{k_1^2k_2}\right)^{\cdot} + \frac{k_2}{k_1}\right)\overrightarrow{\mathbf{e_4}}(s). \tag{5.2}$$

□

As a corollary we obtain the classical result for spherical three-dimensional curves:

Corollary 2. *A curve $\alpha : I \to \mathbb{R}^3$ is spherical, i.e., it is contained in a sphere of radius R, if and only if*

$$\frac{1}{k_1^2} + \left(\frac{\dot{k_1}}{k_1^2k_2}\right)^2 = R^2. \tag{5.3}$$

From now on, we shall suppose that $m = 0$ and $R = 1$.

5.1. **Spherical ccr-curves in $\mathbb{R}^3$.** In this case, we can rewrite Eq. 5.3 in terms of curvature, $k_1 = \kappa$, and torsion $k_2 = \tau = c\kappa$, c being a constant.

$$\frac{\dot{\kappa}}{\kappa^2\sqrt{\kappa^2 - 1}} = \pm c.$$

Let us consider just the positive sign. This differential equation can be integrated and the solution is

$$\kappa(s) = \frac{1}{\sqrt{1 - (cs + s_0)^2}}.$$

Thanks to a shift of the parameter we get that the curvature and torsion of a spherical generalized helix are given by

$$\kappa(s) = \frac{1}{\sqrt{1 - c^2 s^2}}, \qquad \tau(s) = \frac{c}{\sqrt{1 - c^2 s^2}}.$$

We now need to compute the reparametrization

$$u = g(s) = \int_0^s \kappa(t)dt = \frac{1}{c}\arcsin(cs).$$

With the appropriate initial conditions, the generalized spherical helix is

$$\begin{aligned}\alpha_c(s) \quad = \quad & (\sqrt{1 - c^2 s^2}\cos(\tfrac{\sqrt{1+c^2}\arcsin(cs)}{c}) + \tfrac{c^2 s}{\sqrt{1+c^2}}\sin(\tfrac{\sqrt{1+c^2}\arcsin(cs)}{c}),\\ & -\sqrt{1 - c^2 s^2}\sin(\tfrac{\sqrt{1+c^2}\arcsin(cs)}{c}) + \tfrac{c^2 s}{\sqrt{1+c^2}}\cos(\tfrac{\sqrt{1+c^2}\arcsin(cs)}{c}),\\ & \tfrac{cs}{\sqrt{1+c^2}})\end{aligned}$$

Note that the curve α_c is defined in the interval $]-\frac{1}{c}, \frac{1}{c}[$. If we change the parameter in accordance with $s = \frac{1}{c}\sin t$, the spherical helix is now parametrized as

$$\begin{aligned}\beta_c(t) \quad = \quad & (\cos t\cos(\tfrac{\sqrt{1+c^2}}{c}t) + \tfrac{c}{\sqrt{1+c^2}}\sin t\sin(\tfrac{\sqrt{1+c^2}}{c}t),\\ & -\cos t\sin(\tfrac{\sqrt{1+c^2}}{c}t) + \tfrac{c}{\sqrt{1+c^2}}\sin t\cos(\tfrac{\sqrt{1+c^2}}{c}t), \tfrac{\sin t}{\sqrt{1+c^2}})\end{aligned}$$

Now, it is clear that the projection of these curves on the plane xy are arcs of epicycloids. This result was known by W. Blaschke, as is mentioned in [8], where it is also proven by different methods.

5.2. **Spherical ccr-curves in $\mathbb{R}^4$.**

5.2.1. *The constant curvatures case.* The curve

$$\alpha(s) = \frac{1}{\sqrt{r_1^2 + r_2^2}}(\frac{r_1}{m_1}\sin(m_1 s), -\frac{r_1}{m_1}\cos(m_1 s), \frac{r_2}{m_2}\sin(m_2 s), -\frac{r_2}{m_2}\cos(m_2 s))$$

is a spherical curve (with radius 1), if and only if

$$r_1^2 m_2^2 + r_2^2 m_1^2 = m_1^2 m_2^2 (r_1^2 + r_2^2).$$

5.2.2. *The non-constant case.* In this case, we can rewrite Eq. 5.1 in terms of curvature, k_1, $k_2 = c_2k_1$ and $k_3 = c_3k_1$, where c_2, c_3 are constants.

$$\frac{1}{k_1^2} + \left(\frac{\dot{k_1}}{c_2k_1^3}\right)^2 + \frac{1}{c_3^2k_1^2}\left(\left(\frac{\dot{k_1}}{c_2k_1^3}\right)^{\cdot} + c_2\right)^2 = 1. \tag{5.4}$$

By changing $f = \frac{1}{k_1^2}$ the equation is reduced to

$$f + \frac{1}{4c_2^2}\dot{f}^2 + \frac{1}{c_3^2}f(-\frac{1}{2c_2}\ddot{f} + c_2)^2 = 1. \tag{5.5}$$

Computation of the general solution seems to be a difficult task. Instead, we can try to compute some particular solutions.

For instance, the constant solution $f(s) = \frac{c_3^2}{c_2^2+c_3^2}$ or the polynomial solutions of degree 2

$$f(s) = \frac{-2c_2^2 + c_3^2 - c_3\sqrt{-8c_2^2 + c_3^2}}{2(c_2^2 + c_3^2)} + \frac{1}{2}\left(2c_2^2 - c_3^2 - c_3\sqrt{-8c_2^2 + c_3^2}\right)s^2,$$

$$f(s) = 2c_2s + \frac{1}{2}\left(2c_2^2 - c_3^2 - c_3\sqrt{-8c_2^2 + c_3^2}\right)s^2.$$

For these three particular solutions the reparametrization g, where $g(s) = \int_0^s k_1(t)dt = \int_0^s \frac{1}{\sqrt{f(t)}}dt$, can be computed explicitly. We can thus obtain explicit examples of ccr-curves in S^3 with non-constant curvatures.

A particular case. With $c_2 = \frac{1}{2}, c_3 := \frac{\sqrt{3}}{2}$, then $m_1 = \sqrt{\frac{3}{2}}, m_2 = \frac{1}{\sqrt{2}}$ and $r_1 = r_2 = \frac{1}{\sqrt{2}}$. The function $f(s) = \frac{1}{2} - 2s^2$ is a solution of Eq. 5.5. Therefore, $k_1(s) = \frac{2}{\sqrt{1-4s^2}}$, and $g(s) = \int_0^s \frac{2}{\sqrt{1-4t^2}}dt = \arcsin(2s)$.

If

$$\overrightarrow{\mathbf{e_1}}(t) = \frac{1}{\sqrt{2}}(\cos(\sqrt{\frac{3}{2}}t), \sin(\sqrt{\frac{3}{2}}t), \cos(\frac{1}{\sqrt{2}}t), \sin(\frac{1}{\sqrt{2}}t)),$$

then

$$\alpha(s) = (0, -\frac{\sqrt{3}}{2}, 0, \frac{1}{2}) + \int_0^s \overrightarrow{\mathbf{e_1}}(\arcsin(2u))du, \quad s \in \,]-\frac{1}{2}, \frac{1}{2}[$$

is a spherical ccr-curve with center at the origin of coordinates, with radius 1 and with non-constant curvatures.

6. Intrinsic generalized helices

In [1] the author proposes a definition of general helix on a 3-dimensional real-space-form substituting the fixed direction in the usual definition of generalized helix by a Killing vector field along the curve.

Let $\alpha : I \to M$ be an immersed curve in a 3-dimensional real-space-form M. Let us denote the intrinsic Frenet frame by $\{\overrightarrow{\mathbf{t}}, \overrightarrow{\mathbf{n}}, \overrightarrow{\mathbf{b}}\}$. The intrinsic

Frenet's formulae are

$$\begin{cases} \nabla_{\overrightarrow{\mathbf{t}}} \overrightarrow{\mathbf{t}} = \kappa \overrightarrow{\mathbf{n}}, \\ \nabla_{\overrightarrow{\mathbf{t}}} \overrightarrow{\mathbf{n}} = -\kappa \overrightarrow{\mathbf{t}} + \tau \overrightarrow{\mathbf{b}}, \\ \nabla_{\overrightarrow{\mathbf{t}}} \overrightarrow{\mathbf{b}} = -\tau \overrightarrow{\mathbf{n}}, \end{cases} \tag{6.1}$$

where ∇ is the Levi-Civita connection of M and where κ and τ are called the intrinsic curvature and torsion functions of curve α, respectively.

From now on we shall suppose that $M = S^3$. Therefore, any curve on S^3 can also be considered to be a curve in $\mathbb{R}^4$. We shall try to obtain the relationship between the Frenet elements, $\{\overrightarrow{\mathbf{e_1}}, \overrightarrow{\mathbf{e_2}}, \overrightarrow{\mathbf{e_3}}, \overrightarrow{\mathbf{e_4}}, k_1, k_2, k_3\}$, of the curve as a curve in 4-dimensional Euclidian space and the intrinsic Frenet elements $\{\overrightarrow{\mathbf{t}}, \overrightarrow{\mathbf{n}}, \overrightarrow{\mathbf{b}}, \kappa, \tau\}$. Note first that $\overrightarrow{\mathbf{t}} = \overrightarrow{\mathbf{e_1}}$. Then

$$\nabla_{\overrightarrow{\mathbf{t}}} \overrightarrow{\mathbf{t}} = \dot{\overrightarrow{\mathbf{e_1}}} - < \dot{\overrightarrow{\mathbf{e_1}}}, \alpha > \alpha = k_1(\overrightarrow{\mathbf{e_2}} - < \overrightarrow{\mathbf{e_2}}, \alpha > \alpha),$$

where we have used as the Gauss map of the sphere the identity map.

Therefore

$$\overrightarrow{\mathbf{n}} = \frac{\nabla_{\overrightarrow{\mathbf{t}}} \overrightarrow{\mathbf{t}}}{||\nabla_{\overrightarrow{\mathbf{t}}} \overrightarrow{\mathbf{t}}||} = \frac{1}{\sqrt{1 - < \overrightarrow{\mathbf{e_2}}, \alpha >^2}} (\overrightarrow{\mathbf{e_2}} - < \overrightarrow{\mathbf{e_2}}, \alpha > \alpha), \tag{6.2}$$

and

$$\kappa = < \nabla_{\overrightarrow{\mathbf{t}}} \overrightarrow{\mathbf{t}}, \overrightarrow{\mathbf{n}} > = k_1 \sqrt{1 - < \overrightarrow{\mathbf{e_2}}, \alpha >^2} = \sqrt{k_1^2 - 1},$$

which were obtained using Eq. 5.2.

The intrinsic binormal vector is the only vector such that $\{\overrightarrow{\mathbf{t}}, \overrightarrow{\mathbf{n}}, \overrightarrow{\mathbf{b}}, \alpha\}$ is an orthonormal basis of $\mathbb{R}^4$ with positive orientation. Then

$$\overrightarrow{\mathbf{b}} = \alpha \wedge \overrightarrow{\mathbf{t}} \wedge \overrightarrow{\mathbf{n}}.$$

Now, by replacing the intrinsic tangent and normal with $\overrightarrow{\mathbf{t}} = \overrightarrow{\mathbf{e_1}}$ and 6.2, we get

$$\overrightarrow{\mathbf{b}} = \frac{k_1}{\sqrt{k_1^2 - 1}} \, \alpha \wedge \overrightarrow{\mathbf{e_1}} \wedge \overrightarrow{\mathbf{e_2}} = \frac{1}{\sqrt{1 - (\frac{1}{k_1})^2}} \, \alpha \wedge \overrightarrow{\mathbf{e_1}} \wedge \overrightarrow{\mathbf{e_2}}.$$

Therefore

$$\dot{\overrightarrow{\mathbf{b}}} = \left(\frac{1}{\sqrt{1 - (\frac{1}{k_1})^2}} \right)^{\cdot} \alpha \wedge \overrightarrow{\mathbf{e_1}} \wedge \overrightarrow{\mathbf{e_2}} + \frac{1}{\sqrt{1 - (\frac{1}{k_1})^2}} \, \alpha \wedge \overrightarrow{\mathbf{e_1}} \wedge k_2 \overrightarrow{\mathbf{e_3}}.$$

A consequence of this computation is that $<\dot{\overrightarrow{\mathbf{b}}},\alpha>=0$, and therefore, $\nabla_{\overrightarrow{\mathbf{t}}}\overrightarrow{\mathbf{b}}=\dot{\overrightarrow{\mathbf{b}}}$. Finally,

$$\begin{aligned}\tau &= -<\nabla_{\overrightarrow{\mathbf{t}}}\overrightarrow{\mathbf{b}},\overrightarrow{\mathbf{n}}>\\ &= -<\tfrac{1}{\sqrt{1-(\frac{1}{k_1})^2}}\,\alpha\wedge\overrightarrow{\mathbf{e_1}}\wedge k_2\overrightarrow{\mathbf{e_3}},\tfrac{1}{\sqrt{1-(\frac{1}{k_1})^2}}\overrightarrow{\mathbf{e_2}}>\\ &= -\tfrac{k_2}{1-(\frac{1}{k_1})^2}<\alpha\wedge\overrightarrow{\mathbf{e_1}}\wedge\overrightarrow{\mathbf{e_3}},\overrightarrow{\mathbf{e_2}}>=\tfrac{k_2}{1-(\frac{1}{k_1})^2}=\tfrac{k_1^2k_2}{\kappa^2}.\end{aligned}$$

Proposition 1. *The only* 4*-dimensional spherical non-trivial ccr-curves which are also intrinsic generalized helices of* S^3 *are helices, i.e., curves with all curvatures constant.*

Proof. As it is proven in [1], a curve in S^3 is an intrinsic helix if and only if $\tau=0$ or there exists a constant b such that $\tau=b\kappa\pm 1$.

The case $\tau=0$ implies that $k_1k_2=0$ and we get a non-regular curve.

In the other case, if the curve is also a ccr-curve (with $k_2=ck_1$), then

$$\frac{ck_1^3}{\kappa^2}=b\kappa\pm 1.$$

Equivalently

$$\left(\frac{ck_1^3}{k_1^2-1}\mp 1\right)^2=b(k_1^2-1).$$

That is, the function k_1 is the solution of a polynomial equation with constant coefficients; and, therefore, the function k_1 is constant, and so the other two curvatures k_2 and k_3 are also constant. The same happens with κ and τ. We are then in the presence of a helix according to the designation in [1], or a geodesic in a flat torus in $\mathbb{R}^4$ according to [6]. □

References

[1] M. Barros, *General helices and a theorem of Lancret*, Proceedings of the Am. Math. Soc., vol. 125, 1503-1509 (1997).
[2] do Carmo, M. P., *Differential Geometry of curves and surfaces*, Prentice-Hall Inc., (1976).
[3] A. Ferrández, A. Giménez, P. Lucas, *Null generalized helices in Lorentz-Minkowski spaces*, J. of Physics, A: Math. and general, vol 35, 8243-8251 (2002).
[4] H. A. Hayden, *On a generalized helix in a Riemannian n-space*, Proc. London Math. Soc., vol 32, 37-45 (1931).
[5] W. Klingenberg, *A course in Differential Geometry*, Springer-Verlag (1978).
[6] S. Rodrigues Costa, *On closed twisted curves*, Proceedings of the Am. Math. Soc., vol. 109, 205-214 (1990).
[7] M.C. Romero-Fuster, E. Sanabria-Codesal, *Generalized helices, twistings and flattenings of curves in n-space*, Matemática Contemporânea, vol. 17, 267-280 (1999).
[8] Struick, D. J.; *Lectures on Classical Differential Geometry*, Dover, New-York, (1988).

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