Final version
11.4.03

The Plateau-Bézier problem

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Abstract. We study the Plateau problem restricted to polynomial surfaces using techniques coming from the theory of Computer Aided Geometric Design. The results can be used to obtain polynomial approximations to minimal surfaces. The relationship between harmonic Bézier surfaces and minimal surfaces with free boundaries is shown.

1 Introduction

This work deals with polynomial surfaces of minimal area. It seems to be a very simple question but we will try to study such kind of surfaces from a non usual point of view: the Bézier description of polynomial surfaces.

People working on minimal surface theory know very well that S. Bernstein was a prolific researcher on this subject at the beginning of the XXth century. The same people probably know that he found an alternative proof of the theorem of Weiertrass about the approximation of arbitrary functions with polynomial functions. What is possibly unknown is that the basis of polynomial functions he used in their proof, nowadays called Bernstein polynomials, is a fundamental component of CAD (Computer Aided Design).

From the very beginning of CAD, polynomial functions are considered the most easy way to construct curves and surfaces from the point of view of computer science. Nevertheless, the coefficients of a polynomial function in the usual basis of powers of the variable have no geometrical meaning. It is hard to control the shape of a polynomial curve or surface just from this set of coefficients.

The alternative basis of Bernstein polynomials solves this drawback because now the coefficients, called control points, have a very intuitive and clear geometric information. It is easy to control the shape of the designed objects just by variations of the control points.

In particular, the end points of a Bézier curve are two of the control points, and the border curves of a Bézier surface can be controlled by a subset of control points.

Like discrete surfaces (see [10]), Bézier surfaces have finite dimensional spaces of admissible variations, therefore the study of linear differential operators on the variation spaces reduces to the linear algebra of matrices.

We start by stating the corresponding Plateau problem for this kind of surfaces: Given the border, or equivalently, the boundary control points, of a Bézier surface, the Bézier-Plateau problem consists in finding the inner control points in such a way that the resulting Bézier surface be of minimal area among all other Bézier surfaces with the same boundary control points.

As it also happens in the theory of minimal surfaces, the area functional is highly nonlinear, so we start by studying instead the Dirichlet functional. We obtain the existence and uniqueness of the Dirichlet extremals and we show how to obtain a sequence of Dirichlet extremals whose areas converge to the area of a previously given minimal surface.

Nevertheless, the Bézier Dirichlet extremals are not harmonic charts. So, in a second part, we give the conditions that a Bézier surface must fulfill in order to be harmonic. The result is very surprising because it has a close relation with the theory of minimal surfaces with free boundaries. We show that harmonic Bézier surfaces are totally determined by the first and last rows of control points. As these two rows determine two of the boundary curves, what we get is that given two opposed boundary curves of a harmonic Bézier surface, the whole surface is fully determined.

2 The Dirichlet functional for Bézier surfaces

Let $\mathcal{P} = \{P_{ij}\}_{i,j=0}^{n,m}$ be the control net of a Bézier surface. Let us denote by $\vec{\mathbf{x}} : [0,1] \times [0,1] \to \mathbb{R}^3$, the chart of the Bézier surface.

$$\vec{\mathbf{x}}(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} B_i^m(u) B_j^n(v) P_{ij},$$

being $B_i^n(t)$ the ith-Bernstein polynomial of degree n

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}, \qquad \text{for } i \in \{0, \dots, n\},$$

otherwise $B_i^n(t) = 0$.

The area of the Bézier surface is

$$A(\mathcal{P}) = \int_R ||\vec{\mathbf{x}}_u \wedge \vec{\mathbf{x}}_v|| du\, dv = \int_R (EG - F^2)^{\frac{1}{2}} du\, dv,$$

where $R = [0,1] \times [0,1]$ and E, F, G are the coefficients of the first fundamental form of $\vec{\mathbf{x}}$.

Since the border of a Bézier surface is determined by the exterior control points we can state a kind of Plateau problem, that we will call the Bézier-Plateau problem: Given the exterior control points, $\{P_{ij}\}$ with $i = 0, n$ or $j = 0, m$, of a Bézier surface, find the inner ones in such a way that the area of the resulting Bézier surface be a minimum among all the areas of all Bézier surfaces with the same exterior control points.

The first non trivial example of polynomial minimal surface is the Enneper's surface. For its description as a bicubical Bézier surface, ie., its control net, we address the reader to the references [4] or [1]. Note that the Bézier surface defined by such control net is not an approximation of the Enneper's surface, like it happens with the discrete Enneper surface (see [10]), it is the same Enneper's surface.

In general, just a few configurations of the border points will produce a polynomial minimal surface. So, for arbitrary configurations we need to develop general methods for obtaining the extremal of the area functional. Nevertheless, as usual, we do not try to minimize directly the area functional due to its high nonlinearity. We shall work instead with the Dirichlet functional

$$D(\mathcal{P}) = \frac{1}{2}\int_R (||\vec{\mathbf{x}}_u||^2 + ||\vec{\mathbf{x}}_v||^2)du\ dv. \tag{1}$$

Let us recall the following fact relating the area and Dirichlet functionals:

$$(EG - F^2)^{\frac{1}{2}} \leq (EG)^{\frac{1}{2}} \leq \frac{E+G}{2}. \tag{2}$$

Therefore, for any control net, $\mathcal{P}$, $A(\mathcal{P}) \leq D(\mathcal{P})$. Moreover, equality in (2) can occur only if $E = G$ and $F = 0$, i.e., for isothermal charts.

Anyway, both functionals have a minimum in the Bézier case due to the following facts: first, they can be considered as continuous functions defined on $\mathbb{R}^{3(n-1)(m-1)}$. Indeed, the functions depend on the inner control points and its number is $(n-1) \times (m-1)$ and each inner control point belongs to $\mathbb{R}^3$. For example, if $n = m = 2$ there is just one free control point. So, the area functional, or the Dirichlet functional, are just real functions defined on $\mathbb{R}^3$.

Second, as a consequence of $E > 0, G > 0$ and $EG - F^2 > 0$, both functionals are bounded from below.

Third, the infima are attained: when looking for a minimum, we can restrict both functions to a suitable compact subset. If a control point goes far away, then the same happens with a portion of the surface and then, the area, and then the sum of E and G, increases. So, we can choose a compact subset such that, if one of the inner control points is outside the compact subset, then the area functional, and then the Dirichlet functional too, are greater than some bound. Finally, if we restrict both continuous functions to a compact subset we can affirm that the infima exist and they are attained.

2.1 Extremals of the Dirichlet functional

The next result translates the condition "a control net $\mathcal{P}$ is an extremal of the Dirichlet problem" into a system of linear equations in terms of the control points. Let us say that we are not computing the Euler-Lagrange equations of the Dirichlet functional. We will simply compute the points where the gradient of a real function defined on $\mathbb{R}^{3(n-1)(m-1)}$ vanishes.

Proposition 1. *A control net, $\mathcal{P} = \{P_{ij}\}_{i,j=0}^{n,m}$, is an extremal of the Dirichlet functional with prescribed border if and only if*

$$\begin{aligned} 0 = &\tfrac{n^2}{2(2n-2)m}\tbinom{n-1}{i}\tbinom{m}{j}\textstyle\sum_{k,\ell=0}^{n-1,m} A_{ni}^{k}\frac{\binom{m}{\ell}}{\binom{2m}{j+\ell}}\Delta^{10}P_{k\ell} \\ &+\tfrac{m^2}{2(2m-2)n}\tbinom{n}{i}\tbinom{m-1}{j}\textstyle\sum_{k,\ell=0}^{n,m-1} \frac{\binom{n}{k}}{\binom{2n}{i+k}}A_{mj}^{\ell}\Delta^{01}P_{k\ell}, \end{aligned} \tag{3}$$

for any $i \in \{1, \ldots, n-1\}$ *and* $j \in \{1, \ldots, m-1\}$ *where* A^k_{ni} *is defined by*

$$\frac{ni - nk - i}{(n-i)(2n-1-i-k)} \frac{\binom{n-1}{k}}{\binom{2n-2}{i+k-1}}.$$

Proof: Let us compute the gradient of the Dirichlet functional with respect to the coordinates of a control point $P_{ij} = (x^1_{ij}, x^2_{ij}, x^3_{ij})$. For any $a \in \{1, 2, 3\}$, $i \in \{1, \ldots, n-1\}$ and any $j \in \{1, \ldots, m-1\}$

$$\frac{\partial D(\mathcal{P})}{\partial x^a_{ij}} = \int_R (< \frac{\partial \vec{\mathbf{x}}_u}{\partial x^a_{ij}}, \vec{\mathbf{x}}_u > + < \frac{\partial \vec{\mathbf{x}}_v}{\partial x^a_{ij}}, \vec{\mathbf{x}}_v >) du\, dv$$

Let us compute now the partial derivatives

$$\begin{aligned} \tfrac{\partial \vec{\mathbf{x}}_u}{\partial x^a_{ij}} &= \tfrac{\partial}{\partial x^a_{ij}} \tfrac{\partial}{\partial u} \vec{\mathbf{x}} = \tfrac{\partial}{\partial u} \tfrac{\partial}{\partial x^a_{ij}} \vec{\mathbf{x}} \\ &= \tfrac{\partial}{\partial u} B^n_i(u) B^m_j(v) e^a = n(B^{n-1}_{i-1}(u) - B^{n-1}_i(u)) B^m_j(v) e^a, \end{aligned}$$

where e^a denotes the a-th vector of the canonical basis, i.e, $e^1 = (1,0,0), e^2 = (0,1,0), e^3 = (0,0,1)$. Analogously

$$\frac{\partial \vec{\mathbf{x}}_v}{\partial x^a_{ij}} = m B^n_i(u)(B^{m-1}_{j-1}(v) - B^{m-1}_j(v)) e^a.$$

Therefore

$$\begin{aligned} \frac{\partial D(\mathcal{P})}{\partial x^a_{ij}} &= \int_R \big(n(B^{n-1}_{i-1}(u) - B^{n-1}_i(u)) B^m_j(v) < e^a, \vec{\mathbf{x}}_u > \\ &\quad + m B^n_i(u)(B^{m-1}_{j-1}(v) - B^{m-1}_j(v)) < e^a, \vec{\mathbf{x}}_v > \big)\, dudv \end{aligned}$$

$$= \int_R (n(B^{n-1}_{i-1}(u) - B^{n-1}_i(u)) B^m_j(v) < e^a, n \sum_{k,\ell=0}^{n-1,m} B^{n-1}_k(u) B^m_\ell(v) \Delta^{10} P_{k\ell} >$$

$$+ m B^n_i(u)(B^{m-1}_{j-1}(v) - B^{m-1}_j(v)) < e^a, m \sum_{k,\ell=0}^{n,m-1} B^n_k(u) B^{m-1}_\ell(v) \Delta^{01} P_{k\ell} >) dudv.$$

Applying now that for any $n \in \mathbb{N}$ and for any $i = 0, \ldots, n$, $\int_0^1 B^n_i(t) dt = \frac{1}{n+1}$, we get

$$\begin{aligned} \tfrac{\partial D(\mathcal{P})}{\partial x^a_{ij}} &= \tfrac{n^2}{2(2n-2)m} \textstyle\sum_{k,\ell=0}^{n-1,m} \left(\frac{\binom{n-1}{i-1}\binom{n-1}{k}}{\binom{2n-2}{i+k-1}} - \frac{\binom{n-1}{i}\binom{n-1}{k}}{\binom{2n-2}{i+k}} \right) \frac{\binom{m}{\ell}\binom{m}{j}}{\binom{2m}{j+\ell}} < e^a, \Delta^{10} P_{k\ell} > \\ &\quad + \tfrac{m^2}{2(2m-2)n} \textstyle\sum_{k,\ell=0}^{n,m-1} \frac{\binom{n}{k}\binom{n}{i}}{\binom{2n}{i+k}} \left(\frac{\binom{m-1}{j-1}\binom{m-1}{\ell}}{\binom{2m-2}{j+\ell-1}} - \frac{\binom{m-1}{j}\binom{m-1}{\ell}}{\binom{2m-2}{j+\ell}} \right) < e^a, \Delta^{01} P_{k\ell} > \\ &= \tfrac{n^2}{2(2n-2)m} \binom{n-1}{i} \binom{m}{j} \textstyle\sum_{k,\ell=0}^{n-1,m} A^k_{ni} \frac{\binom{m}{\ell}}{\binom{2m}{j+\ell}} < e^a, \Delta^{10} P_{k\ell} > \\ &\quad + \tfrac{m^2}{2(2m-2)n} \binom{n}{i} \binom{m-1}{j} \textstyle\sum_{k,\ell=0}^{n,m-1} \frac{\binom{n}{k}}{\binom{2n}{i+k}} A^\ell_{mj} < e^a, \Delta^{01} P_{k\ell} > . \square \end{aligned}$$

Remark 1. A simple computation shows that the second derivatives $\frac{\partial^2 D(\mathcal{P})}{\partial x^a_{ij} x^b_{k\ell}}$ are constant. Indeed, it is a consequence of the fact that the chart depends linearly on the control points.

Let us recall that the Weierstrass approach to the theory of minimal surfaces points out that any minimal surface is related with some complex functions on a variable $z = u + \mathbf{i}v$, being u, v the parameters of the surface. When dealing with polynomial complex functions, the degrees of the resulting minimal surface in the variables u and v are the same. So, this seems to indicate that Bézier surfaces with a squared control net are more suitable in this setting. In the squared case, equations (3) are simpler.

Corollary 1. *A squared control net, $\mathcal{P} = \{P_{ij}\}_{i,j=0}^{n,n}$, is an extremal of the Dirichlet functional with prescribed border if and only if*

$$0 = \sum_{k,\ell=0}^{n-1,n} \frac{\binom{n}{\ell}}{\binom{2n}{j+\ell}} C^k_{ni} \Delta^{10} P_{k\ell} + \sum_{k,\ell=0}^{n,n-1} \frac{\binom{n}{k}}{\binom{2n}{i+k}} C^\ell_{mj} \Delta^{01} P_{k\ell}, \tag{4}$$

for any $i, j = 1, \ldots, n-1$, where $C^k_{ni} = \frac{(n-1)i-nk}{i+k} \frac{\binom{n-1}{k}}{\binom{2n-2}{i+k}}$.

Let us recall that, as we have said before, a minimum of the Dirichlet functional with prescribed border always exists. So, fixing the exterior control points and taking as unknowns the inner control points, the linear system (3) and, in particular, the linear system (4), always is compatible and it can be solved in terms of the exterior control points.

2.2 Uniqueness of the Dirichlet extremal

We have seen that the extremals of the Dirichlet functional always exists. Let us now prove the uniqueness.

Theorem 1. *The Dirichlet extremal is unique.*

Proof. We know that Dirichlet extremals are computed as solutions of the linear system (1). Let us write it as

$$A \cdot P = B. \tag{5}$$

where B is a column vector computed through the boundary control points, P is the column vector of the inner control points, and A is a square matrix whose entries are independent of the control points, they just depend on the dimensions of the control net.

Let us check that the rank of A is maximal, i.e., the linear system has a unique solution.

A well known theorem of Bernstein ([9], page 38) affirms that the only minimal surface being the graph of a function is a plane. So, let us choose the control net

$$\mathcal{P} := \{(\frac{i}{n}, \frac{j}{m}, 0)\}_{i,j=0}^{n,m}.$$

The associated Bézier chart is $\vec{\mathbf{x}}(u,v) = (u,v,0)$. This is the so called linear precision property of the Bézier surfaces. It is minimal and isothermal, therefore, $\mathcal{P}$ is a Dirichlet extremal for the same boundary conditions.

Let us check that it is the unique Dirichlet extremal. Any other configuration $\mathcal{P}_0$ of the inner control points with at least a control point with nonzero third coordinate will produce a non planar associated Bézier surface, and then

$$\mathcal{D}(\mathcal{P}) = \mathcal{A}(\mathcal{P}) < \mathcal{A}(\mathcal{P}_0) \leq \mathcal{D}(\mathcal{P}_0).$$

Any other configuration of the inner control points with zero third coordinate will produce a planar surface but a non isothermal parametrization, and then

$$\mathcal{D}(\mathcal{P}) = \mathcal{A}(\mathcal{P}) = \mathcal{A}(\mathcal{P}_0) < \mathcal{D}(\mathcal{P}_0).$$

Therefore, $\mathcal{P}$ is the only Dirichlet extremal. This implies that Eq. (5) has an unique solution for the boundary conditions given by $\mathcal{P}$. Therefore the matrix A is of maximal rank. □

Note that the situation in the Bézier case, and for the Dirichlet functional, is rather different than in the discrete surface case for the area functional. In this case, as in the differentiable surface case, given a prescribed border, there could exist more than just one surface with minimal area.

3 Convergence results

In this section, we will study how to reach the minimal area with prescribed boundary by a sequence of Bézier surfaces which are Dirichlet extremals.

Theorem 2. *Let $\vec{\mathbf{x}} : [0,1] \times [0,1] \to \mathbb{R}^3$ be an isothermal chart of a surface of minimal area among all surfaces with the same boundary.*

Let $\vec{\mathbf{y}}_n$ be the Dirichlet extremal of degree n with boundary defined by the exterior control points of the control net $\mathcal{P}_n = \{\vec{\mathbf{x}}(\frac{i}{n}, \frac{j}{n})\}_{i,j=0}^{n}$.

Then,

$$\lim_{n\to\infty} \mathcal{A}(\vec{\mathbf{y}}_n) = \mathcal{A}(\vec{\mathbf{x}}).$$

Proof: Let $\vec{\mathbf{x}}_n$ be the associated Bézier chart to the control net $\mathcal{P}_n$. The Bernstein's proof of the Weiertrass theorem indicates that the sequence $\{\vec{\mathbf{x}}_n\}_{n=1}^{\infty}$ is uniformly convergent to $\vec{\mathbf{x}}$. Moreover,

$$\lim_{n\to\infty} \mathcal{A}(\vec{\mathbf{x}}_n) = \mathcal{A}(\vec{\mathbf{x}}) = \mathcal{D}(\vec{\mathbf{x}}) = \lim_{n\to\infty} \mathcal{D}(\vec{\mathbf{x}}_n), \tag{6}$$

where the equality $\mathcal{A}(\vec{\mathbf{x}}) = \mathcal{D}(\vec{\mathbf{x}})$ is a consequence of the fact that the chart is minimal and isothermal.

Let $\overrightarrow{\mathbf{z}}_n$ be a chart of the surface with minimal area and with the same boundary than $\overrightarrow{\mathbf{y}}_n$. Therefore, $\mathcal{A}(\overrightarrow{\mathbf{z}}_n) \leq \mathcal{A}(\overrightarrow{\mathbf{y}}_n)$.

Moreover, due to the fact that area functional is always lesser than the Dirichlet functional, we have that for any $n \in \mathbb{N}$, $\mathcal{A}(\overrightarrow{\mathbf{y}}_n) \leq \mathcal{D}(\overrightarrow{\mathbf{y}}_n)$.

And, recalling that $\overrightarrow{\mathbf{y}}_n$ is the Dirichlet extremal of degree n and that it has the same boundary than the polynomial chart $\overrightarrow{\mathbf{x}}_n$, we have that $\mathcal{D}(\overrightarrow{\mathbf{y}}_n) \leq \mathcal{D}(\overrightarrow{\mathbf{x}}_n)$.

Resuming, we have for any $n \in \mathbb{N}$

$$\mathcal{A}(\overrightarrow{\mathbf{z}}_n) \leq \mathcal{A}(\overrightarrow{\mathbf{y}}_n) \leq \mathcal{D}(\overrightarrow{\mathbf{y}}_n) \leq \mathcal{D}(\overrightarrow{\mathbf{x}}_n). \tag{7}$$

Now, according to the results on the boundary behaviour of minimal surfaces (see [8], paragraph 327), as the boundary of $\overrightarrow{\mathbf{z}}_n$ converges uniformly to the boundary of $\overrightarrow{\mathbf{x}}$ then

$$\lim_{n\to\infty} \mathcal{A}(\overrightarrow{\mathbf{z}}_n) = \mathcal{A}(\overrightarrow{\mathbf{x}}). \tag{8}$$

On the other hand, due to the fact that $\overrightarrow{\mathbf{x}}_n$ C^1-converges to $\overrightarrow{\mathbf{x}}$, (see [6], Th. 1.8.1) then

$$\lim_{n\to\infty} \mathcal{D}(\overrightarrow{\mathbf{z}}_n) = \mathcal{D}(\overrightarrow{\mathbf{x}}) = \mathcal{A}(\overrightarrow{\mathbf{x}}). \tag{9}$$

Therefore, the result follows from Eqs. (7), (8) and (9). □

Nevertheless, the previous result is not useful to obtain good approximations of low degree. The reason of this is a consequence of a property of Bézier curves that has a correspondence in Bézier surfaces. For example, if we take the control points on a circle, the resulting Bézier curve is a bad approximation to the circle. In order to obtain better approximations with the same degree one has to solve a least square problem.

A good approximation to the catenoid can be obtained with a degree 7 Bézier surface solution of the Dirichlet problem. In Figure I, the depicted Bézier surface has an area exceeding the area of the corresponding half catenoid in 0.05%.

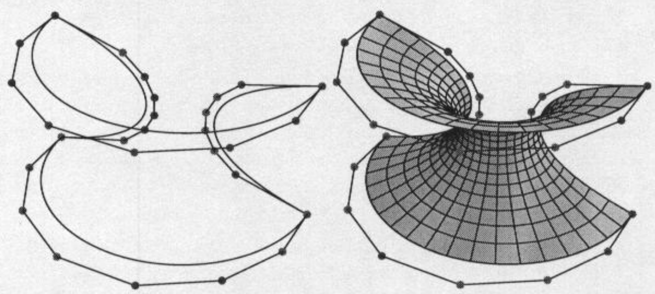

Figure I: If we take suitable control points such that the corresponding Bézier curves are a good approximation of the border of a half catenoid, then the resulting Bézier surface obtained as the minimum of the Dirichlet problem is a good approximation to catenoid. Left, the border conditions. Right, the degree 7 Bézier surface.

A degree 8×8 Bézier surface can be built resembling the minimal surface obtained by Schwarz (see [8], page 75) by placing the exterior control points on some of the edges of a cube (Fig. II).

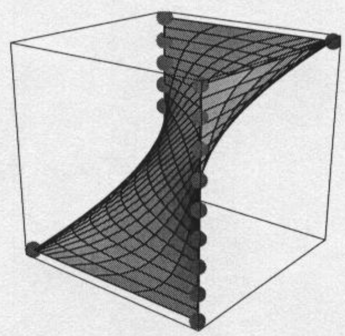

Figure II: An approximation to a Schwarz's surface. Note that the control points $(0,0,0)$ and $(1,1,1)$ are repeated 7 times.

4 Bézier harmonic charts

The Euler-Lagrange equations defined by the Dirichlet functional (1) are $\Delta \overrightarrow{\mathbf{x}} = 0$. So, in the unrestricted case, extremals of the Dirichlet functional are harmonic charts. But harmonic charts are not polynomial in general and then, they can not be solutions of the Bézier-Plateau problem.

The *Dirichlet principle* for domains bounded by a Jordan curve says that harmonic functions are the infima of the Dirichlet functional among all functions defined on the same domain and with the same values on the border. (See [8], paragraph 229). In our case the Jordan curve is the square $[0,1] \times [0,1]$, therefore, if a polynomial chart is harmonic, then it is an extremal of the Dirichlet functional for Bézier surfaces with prescribed border.

A natural question is then to ask for the conditions that a Bézier surface must fulfill in order to be harmonic. To answer this question, we will compute the Laplacian of a Bézier chart .

Theorem 3. *Given a control net in* $\mathbb{R}^3$, $\{P_{ij}\}_{i,j=0}^{n,m}$, *the associated Bézier surface,* $\overrightarrow{\mathbf{x}} : [0,1] \times [0,1] \to \mathbb{R}^3$, *is harmonic, i.e,* $\Delta \overrightarrow{\mathbf{x}} = 0$ *if and only if for any*

$i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, n\}$

$$\begin{aligned}
0 = \ & n(n-1)(P_{i+2,j}a_{in} + P_{i+1,j}(b_{i-1,n} - 2a_{in}) \\
& \qquad\qquad + P_{i-1,j}(b_{i-1,n} - 2c_{i-2,n}) + P_{i-2,j}c_{i-2,n}) \\
& + m(m-1)(P_{i,j+2}a_{jm} + P_{i,j+1}(b_{j-1,m} - 2a_{jm}) \\
& \qquad\qquad + P_{i,j-1}(b_{j-1,m} - 2c_{j-2,m}) + P_{i,j-2}c_{j-2,m}) \\
& + P_{ij}((a_{in} - 2b_{i-1,n} + c_{i-2,n})n(n-1) \\
& \qquad\qquad + (a_{jm} - 2b_{j-1,m} + c_{j-2,m})m(m-1)),
\end{aligned}$$

where, for $i \in \{0, \ldots, n-2\}$

$$a_{in} = (n-i)(n-i-1), \quad b_{in} = 2(i+1)(n-i-1), \quad c_{in} = (i+1)(i+2),$$

and $a_{in} = b_{in} = c_{in} = 0$ otherwise.

For the proof, see [1].

In the case of a quadratic net $(n = m)$ we can state the following corollary

Corollary 2. *Given a quadratic net of points in $\mathbb{R}^3$, $\{P_{ij}\}_{i,j=0}^n$, the associated Bézier surface, $\overrightarrow{\mathbf{x}} : [0,1] \times [0,1] \to \mathbb{R}^3$, is harmonic, i.e, $\Delta \overrightarrow{\mathbf{x}} = 0$ if and only if for any $i \in \{1, \ldots, n\}$*

$$\begin{aligned}
0 = \ & P_{i+2,j}a_{in} + P_{i+1,j}(b_{i-1,n} - 2a_{in}) + P_{i-1,j}(b_{i-1,n} - 2c_{i-2,n}) \\
& + P_{i-2,j}c_{i-2,n} + P_{i,j+2}a_{jn} + P_{i,j+1}(b_{j-1,n} - 2a_{jn}) \\
& + P_{i,j-1}(b_{j-1,n} - 2c_{j-2,n}) + P_{i,j-2}c_{j-2,n} \\
& + P_{ij}(a_{in} - 2b_{i-1,n} + c_{i-2,n} + a_{jn} - 2b_{j-1,n} + c_{j-2,n}).
\end{aligned} \tag{10}$$

An analysis of Equations (10) for degree $n = 3$ (the control net has sixteen points) shows that all the equations can be reduced to a system of just eight independent linear equations. Moreover, it is possible to show that the linear system can be solved by expressing the eight control points in the two middle rows as functions of the other eight control points in the first and last rows. This was done in [1]. So, our aim in the rest of the section is to show that this is true for any dimension, i.e., that the first and last rows of control points fully determine the harmonic Bézier surface.

In order to do that, it is better to come back to the usual basis of polynomials.

Lemma 1. *Let $f(u,v) = \sum_{k,\ell=0}^n a_{k\ell}u^k v^\ell$ be a harmonic polynomial function of degree $n \geq 2$, then,*

1. *If n is odd, then all coefficients $\{a_{k\ell}\}_{k=2,\ell=0}^n$ are totally determined by the coefficients $\{a_{0\ell}, a_{1\ell}\}_{\ell=0}^n$.*
2. *If n is even, then all coefficients $\{a_{k\ell}\}_{k=2,\ell=0}^n$ and also the coefficient a_{1n} are totally determined by the coefficients $\{a_{0\ell}\}_{\ell=0}^n$ and $\{a_{1\ell}\}_{\ell=0}^{n-1}$.*

Proof: The harmonic condition $\Delta f = 0$ can be translated into a system of linear equations in terms of the coefficients $\{a_{k\ell}\}_{k,\ell=0}^{n}$

$$(k+2)(k+1)a_{k+2,\ell} + (\ell+2)(\ell+1)a_{k,\ell+2} = 0, \qquad k,\ell = 0,\ldots,n,$$

but with the convention $a_{n+1,\ell} = a_{n+2,\ell} = a_{n,\ell+2} = a_{n,\ell+1} = 0$.

This means that any coefficient $a_{k\ell}$ with $k > 1$ can be related with $a_{k-2,\ell+2}$ and so on until the first subindex is 0 or 1, or until the second subindex is greater than n. In this second case, $a_{k\ell}$ is directly 0. Indeed, if $\ell + 2k > n$ then $a_{2k,\ell} = a_{2k+1,\ell} = 0$, otherwise

$$a_{2k,\ell} = (-1)^k \binom{2k+\ell}{\ell} a_{0,2k+\ell}, \; a_{2k+1,\ell} = (-1)^k \frac{1}{2k+1}\binom{2k+\ell}{\ell} a_{1,2k+\ell}. \quad (11)$$

So, when n is odd, the result is proved. When n is even, we have that, in addition, coefficient a_{1n} vanishes. $\square$

As we have said before, the next result was conjectured and checked for low dimensions in [1]. We can now give the general result.

Proposition 2. *Let $\vec{\mathbf{x}}(u,v) = \sum_{k,\ell=0}^{n} B_k^n(u)B_\ell^n(v)P_{k\ell}$ be a harmonic Bézier chart of degree n with control net $\{P_{k\ell}\}_{k,\ell=0}^{n}$, then*

1. *If n is odd, control points in the inner rows $\{P_{k\ell}\}_{k=1,\ell=0}^{n-1,n}$ are determined by the control points in the first and last rows, $\{P_{0\ell}\}_{\ell=0}^{n}$ and $\{P_{n\ell}\}_{\ell=0}^{n}$.*
2. *If n is even, control points in the inner rows $\{P_{k\ell}\}_{k=1,\ell=0}^{n-1,n}$ and also the corner control point P_{nn} are determined by the control points in the first and last rows, $\{P_{0\ell}\}_{\ell=0}^{n}$ and $\{P_{n\ell}\}_{\ell=0}^{n-1}$.*

Proof: Let us write the Bézier chart in the usual basis of polynomials

$$\vec{\mathbf{x}}(u,v) = \sum_{k,\ell=0}^{n} u^k v^\ell (a_{k\ell}, b_{k\ell}, c_{k\ell}).$$

Let us consider the case n odd. Note that the first and last rows of control points determine the two opposed border curves $\vec{\mathbf{x}}(0,v)$, $\vec{\mathbf{x}}(1,v)$, $v \in [0,1]$. The first border curve is

$$\vec{\mathbf{x}}(0,v) = \sum_{\ell=0}^{n} v^\ell (a_{0\ell}, b_{0\ell}, c_{0\ell}), \quad (12)$$

and the second one is

$$\vec{\mathbf{x}}(1,v) = \sum_{\ell=0}^{n} v^\ell \sum_{k=0}^{n} (a_{k\ell}, b_{k\ell}, c_{k\ell}). \quad (13)$$

From Eq. (12) we can obtain coefficients $(a_{0\ell}, b_{0\ell}, c_{0\ell})$ for $\ell = 0,\ldots,n$. By the previous lemma, all coefficients $(a_{k\ell}, b_{k\ell}, c_{k\ell})$ are determined by the coefficients $(a_{0\ell}, b_{0\ell}, c_{0\ell})$ and $(a_{1\ell}, b_{1\ell}, c_{1\ell})$. In particular, thanks to Eq. (11), we can reduce Eq. (13) to just a system of linear equations involving the coefficients $(a_{1\ell}, b_{1\ell}, c_{1\ell})$. Moreover, the matrix of coefficients of this system is triangular and with the unit in the diagonal entries. Therefore, the knowledge of the first and last rows of control points, implies the knowledge of the coefficients $(a_{0\ell}, b_{0\ell}, c_{0\ell})$ and $(a_{1\ell}, b_{1\ell}, c_{1\ell})$ and then, the knowledge of all the coefficients, i.e., of the whole harmonic chart, or equivalently, of the whole control net.

For the even case, the arguments are similar. □

5 The Gergonne problem revisited

The result shown in Proposition 2 is analogous to what happens with problems about minimal surfaces with free boundaries: To find minimal surfaces the boundary of which (or part of it) is left free on supporting manifolds. With Bézier surfaces we have seen that given two disjoint border curves, i.e., given two border lines of control points, the other lines are determined thanks to Eqs. (10), and then, the whole Bézier surface is determined.

A typical problem of minimal surfaces with free boundaries is the well known Gergonne problem giving raise to a surface (see [8], page 79, or [5]) that should be no confused with another surface called the Gergonne surface. The original problem was stated as follows: "Couper un cube en deux parties, de telle manière que la section vienne se terminer aux diagonales inverses de deux faces opposées, et que l'aire de cette section, terminée à la surface du cube, soit un minimum".

The solution was finally found by Schwarz in 1872 (see Fig. III, right). What is remarkable is that given the inverse diagonals of two opposed faces of a cube, the Gergonne surface is fully determined.

In the Bézier case, given two opposed lines of border control points, the harmonic Bézier surface is fully determined. A degree 6 harmonic Bézier approximation of the this surface can be seen in Fig. III, left.

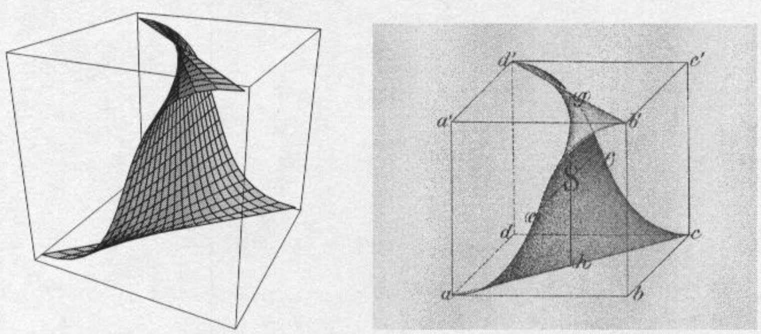

Figure III: Right, the Schwarz solution to the Gergonne problem. Left, an approximation found as a harmonic Bézier surface of degree 6×6.

The border control points to generate such a surface has been chosen as follows:

The bottom row is

$$P_{00} = (0,0,0), \qquad P_{01} = (a,a,0), \qquad P_{02} = (b,b,0),$$
$$P_{03} = (1-b,1-b,0), P_{04} = (1-a,1-a,0), P_{05} = (1,1,0),$$

and the top row,

$$P_{50} = (1,0,1), \qquad P_{51} = (1-a,a,1), P_{52} = (1-b,b,1),$$
$$P_{53} = (b,1-b,1), P_{54} = (a,1-a,1), P_{55} = (0,1,1).$$

The choice of the parameters a and b can be made according to different principles. For example, we can ask for a uniform distribution of the control points by taking $a = \frac{1}{5}, b = \frac{2}{5}$. The resulting Bézier surface is a hyperbolic paraboloid and its area is 1.28079. Or we can ask for isothermality of the Bézier surface on the corners. Then the values are ($a = \frac{\sqrt{257}-9}{32} \sim 0.22, b = \frac{32a+9}{32} \sim 0.40.$) and the resulting Bézier surface is an approximation to a portion of helycoid. The area of the restriction of the helycoid to the cube $[0,1]^3$ is 1.25364.

But there is a choice of the parameters a and b minimizing the area of surface inside the cube. An approximation of these values is $a = 0.41, b = 0.51$ and the resulting Bézier surface, plotted in Figure IV, left, shows a shape resembling the Gergonne surface. The area is 1.24294.

6 Acknowledgments

This work has been partially supported by DGICYT grant BFM2002-00770.

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