NOTES ON MÖBIUS TRANSFORMATIONS

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1. Preliminary concepts

Let us work in the plane with all the machinery of the complex field $\mathbb{C}$. Then

Definition 1.1. A *Möbius transformation of the plane* $f : \mathbb{C} \to \mathbb{C}$ is a map given by

$$f(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ are such that $ad - bc \neq 0$.

We see that if $c \neq 0$ and $z = -\frac{d}{c}$, then $f(z)$ is not defined. Also, that a Möbius transformation has the form of a homography. In fact, the appropriate treatment of those transformations is to consider them as chart presentations of homographies of $P\mathbb{C}^1$. Let us detail this.

Definition 1.2. The *one-dimensional complex projective space* $P\mathbb{C}^1$ is the quotient of $\mathbb{C}^2\backslash\{0\}$ by the equivalence relation that makes equivalent $z, w \in \mathbb{C}^2\backslash\{0\}$ whenever there is some non-zero $\lambda \in \mathbb{C}$ such that $w = \lambda z$. We denote by $[z]$ or $[z_1, z_2]$ the equivalence class of $z = (z_1, z_2)$ and by $\pi : \mathbb{C}^2\backslash\{0\} \to P\mathbb{C}^1$ the map given by $\pi(z) = [z]$.

Let $U = \{(z_1, z_2) \in \mathbb{C}^2\backslash\{0\} : z_2 \neq 0\}$. Then, $\pi(U) = \{[z_1, z_2] \in P\mathbb{C}^1 : z_2 \neq 0\}$ and the map $p_1 : \pi(U) \to \mathbb{C}$ given by $p_1[z_1, z_2] = \frac{z_1}{z_2}$ is bijective, with inverse $p_1^{-1}(z) = [z, 1]$. Also, any non-vanishing point $(z_1, 0) \notin U$ is equivalent to $(1, 0)$. Hence, we can see $P\mathbb{C}^1$ as $\mathbb{C}$ completed with one point, $[1, 0]$, frequently called the *point at infinity.*

Definition 1.3. If $A \in Gl(2; \mathbb{C})$, that is $A : \mathbb{C}^2 \to \mathbb{C}^2$ is a $\mathbb{C}$-linear automorphism, it defines a map $\tilde{A} : P\mathbb{C}^1 \to P\mathbb{C}^1$ by

$$\tilde{A}[z] = [Az].$$

In fact, if $\lambda \neq 0$ and $z \in P\mathbb{C}^1$, we will have $[A(\lambda z)] = [\lambda Az] = [Az] = \tilde{A}[z]$, that is, $\tilde{A}$ is well defined and we have $\tilde{A} \circ \pi = \pi \circ A$. Such a map $\tilde{A}$ is called a *homography* or a *projective transformation of* $P\mathbb{C}^1$.

Thus, let $A \in Gl(2; \mathbb{C})$ be such that $Ae_i = e_j A_{ji}$, where (e_i, e_2) is the canonical basis of $\mathbb{C}^2$, and where we use the Einstein summation convention over repeated

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indexes. Then, if $z \in \mathbb{C}$, we have

$$\begin{aligned}(p_1 \circ \tilde{A} \circ p_1^{-1})(z) &= (p_1 \circ \tilde{A})[z, 1] = p_1[zAe_1 + Ae_2] \\ &= p_1[A_{11}z + A_{12}, A_{21}z + A_{22}] = \frac{A_{11}z + A_{12}}{A_{21}z + A_{22}}.\end{aligned}$$

Therefore, $(p_1 \circ \tilde{A} \circ p_1^{-1})(z)$ is a Möbius transformation.

Note that if $A \in Gl(2;\mathbb{C})$ and $\lambda \neq 0$, then $\lambda A \in Gl(2;\mathbb{C})$ defines the same homography $\tilde{A}$. Thus, it is frequent to normalize A such that its determinant be one.

2. Determination of a Möbius transformation

We want to prove here the following well known proposition(see [1] and [2]):

Proposition 2.1. *Let $z_1, z_2, z_3 \in \mathbb{C}^2 \backslash \{0\}$ be such that the $[z_i]$ are different from each other, and let $w_1, w_2, w_3 \in \mathbb{C}^2 \backslash \{0\}$ be such that the $[w_i]$ are different from each other. Then, there is a unique homography $\tilde{A} : P\mathbb{C}^1 \to P\mathbb{C}^1$ such that $\tilde{A}[z_i] = [w_i],\ i = 1, 2, 3$.*

Proof. Let $z_i = (u_i, v_i),\ i = 1, \ldots, 3$, and suppose that there are $a, b \in \mathbb{C}$, not both zero, such that $az_1 + bz_2 = 0$. Then, if for instance $a \neq 0$, we have $z_1 = -\frac{b}{a} z_2$, whence $[z_1] = [z_2]$, against the hypotheses. Therefore, (z_1, z_2) is a basis of $\mathbb{C}^2$. Hence, we can put $z_3 = az_1 + bz_2$, and, as before, $a \neq 0$ and $b \neq 0$. Hence, let us put

$$\tilde{z}_1 = az_1, \quad \tilde{z}_2 = bz_2, \quad \tilde{z}_3 = z_3.$$

Then we have $\tilde{z}_3 = \tilde{z}_1 + \tilde{z}_2$ and $[\tilde{z}_i] = [z_i],\ i = 1, 2, 3$. Note that $(\tilde{z}_1, \tilde{z}_2)$ is also a basis because $a \neq 0$ and $b \neq 0$. We do the same with the w_i to get $\tilde{w}_i$ with the same respective properties. Now, since $(\tilde{z}_1, \tilde{z}_2)$ and $(\tilde{w}_1, \tilde{w}_2)$ are bases of $\mathbb{C}^2$, there is a unique automorphism A of $\mathbb{C}^2$ such that $A\tilde{z}_i = \tilde{w}_i,\ i = 1, 2$, and, as a consequence, $A\tilde{z}_3 = A(\tilde{z}_i + \tilde{z}_2) = \tilde{w}_1 + \tilde{w}_2 = \tilde{w}_3$. Therefore

$$\tilde{A}[z_i] = \tilde{A}[\tilde{z}_i] = [\tilde{w}_i] = [w_i], \quad i = 1, 2, 3.$$

Now, suppose that $B \in Gl(2;\mathbb{C})$ is such that $\tilde{B}[z_i] = [Bz_i] = [w_i]$. Then $Bz_i = \lambda_i w_i,\ i = 1, \ldots, 3$, for some non-zero $\lambda_i,\ i = 1, \ldots, 3$. Since the same occurs with A, we see that there are three non-zero numbers μ_i such that $Az_i = \mu_i Bz_i$. Since $z_3 = az_1 + bz_2$, we have $Az_3 = \mu_3 Bz_3 = a\mu_1 Bz_1 + b\mu_2 Bz_2 = a\mu_3 Bz_1 + b\mu_3 Bz_2$. Since (Bz_1, Bz_2) is a basis, we get $a\mu_1 = a\mu_3$, and $b\mu_2 = b\mu_3$, and since a and b do not vanish we conclude that $\mu_1 = \mu_2 = \mu_3$, and from this that $\tilde{A} = \tilde{B}$. □

Let us examine now how the calculation of A can be done. For $i = 1, \ldots, 3$, let us put $z_i = (u_i, v_i)$ with $u_i, v_i \in \mathbb{C}$, and denote by $d_i = u_j v_k - u_k v_j$ the determinant of (z_j, z_k), where $j := i \mod 3 + 1$ and $k := j \mod 3 + 1$. Then, we determine $a, b \in \mathbb{C}$ defined by $z_3 = az_1 + bz_2$,

$$a = -\frac{d_1}{d_3}, \qquad b = -\frac{d_2}{d_3}.$$

Let $\tilde{z}_1 := (\tilde{u}_1, \tilde{v}_1) = az_1,\ \tilde{z}_2 := (\tilde{u}_2, \tilde{v}_2) = bz_2$. Then, we have

$$\begin{aligned}
A\tilde{z}_1 &= A(\tilde{u}_1 e_1 + \tilde{v}_1 e_2) = \tilde{u}_1(A_{11}e_1 + A_{21}e_2) + \tilde{v}_1(A_{12}e_1 + A_{22}e_2)\\
&= (\tilde{u}_1 A_{11} + \tilde{v}_1 A_{12})e_1 + (\tilde{u}_1 A_{21} + \tilde{v}_1 A_{22})e_2,\\
A\tilde{z}_2 &= A(\tilde{u}_2 e_1 + \tilde{v}_2 e_2) = \tilde{u}_2(A_{11}e_1 + A_{21}e_2) + \tilde{v}_2(A_{12}e_1 + A_{22}e_2)\\
&= (\tilde{u}_2 A_{11} + \tilde{v}_2 A_{12})e_1 + (\tilde{u}_2 A_{21} + \tilde{v}_2 A_{22})e_2.
\end{aligned}$$

We do the same with the $w_i = (p_i, q_i)$. Therefore, we must solve the equations:

$$\begin{aligned}
\tilde{u}_1 A_{11} + \tilde{v}_1 A_{12} = \tilde{p}_1, &\quad \tilde{u}_1 A_{21} + \tilde{v}_1 A_{22} = \tilde{q}_1,\\
\tilde{u}_2 A_{11} + \tilde{v}_2 A_{12} = \tilde{p}_2, &\quad \tilde{u}_2 A_{21} + \tilde{v}_2 A_{22} = \tilde{q}_2.
\end{aligned}$$

These equations can be written in matrix form as follows:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \tilde{u}_1 & \tilde{u}_2 \\ \tilde{v}_1 & \tilde{v}_2 \end{pmatrix} = \begin{pmatrix} \tilde{p}_1 & \tilde{p}_2 \\ \tilde{q}_1 & \tilde{q}_2 \end{pmatrix}.$$

Therefore we have:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \frac{1}{\tilde{u}_1\tilde{v}_2 - \tilde{u}_2\tilde{v}_1} \begin{pmatrix} \tilde{v}_2 & -\tilde{u}_2 \\ -\tilde{v}_1 & \tilde{u}_1 \end{pmatrix} \begin{pmatrix} \tilde{p}_1 & \tilde{p}_2 \\ \tilde{q}_1 & \tilde{q}_2 \end{pmatrix}.$$

Thus, we have found A. Since the multiplication by a nonvanishing factor leaves $\tilde{A}$ invariant, we may dispense the division by the denominator in the right hand side of the above expression.

3. Stereographic projection and its inverse. Elementary approach

For the computer visualization of stereographic projections of $\mathbb{C}$ to spheres, we will use maps from $\mathbb{C}$ to $\mathbb{R}^2 \times \{-1, 0, +1\}$ that will be compositions of three maps. The first one is a stereographic projection of $\mathbb{C}$ into a sphere of radius r centered at the point $c \in \mathbb{C} \subset \mathbb{R}^3$ with respect to its south pole, that is the point $(c, -r) \in \mathbb{R}^3$, where the inclusion of $\mathbb{C}$ into $\mathbb{R}^3$ is given by $z \mapsto (z, 0)$. The next is a translation of $\mathbb{R}^3$ given by the addition of $(-c, 0)$ followed by a dilation of scale s. Finally, we use an orthonormal basis of $\mathbb{R}^3$, (h, v, n), and we project $\mathbb{R}^3$ to $\mathbb{R}^2 \times \{-1, 0, +1\}$ by putting $p(x) = (x \cdot h, x \cdot v, \mathrm{sg}(x \cdot n))$, where $\mathrm{sg}(a)$ is equal to -1 if $a < 0$, to 0 if $a = 0$, and to $+1$ if $a > 0$.

We want to express this composition and its inverse $\mathbb{R}^2 \times \{-1, 0, +1\} \to \mathbb{C}$.

Let us consider the line

$$t \mapsto \alpha(t) = (c, -r) + t\big((z, 0) - (c, -r)\big) = \big(c + t(z - c), r(t - 1)\big).$$

The condition for the point $\alpha(t)$ to belong to the sphere is:

$$\big|\big(c + t(z - c), r(t - 1)\big) - (c, 0)\big|^2 - r^2 = t^2|z - c|^2 + r^2t^2 - 2r^2t = 0.$$

The value $t = 0$ means that $\alpha(t)$ is the south pole, and we discard it. Thus we have the desired values of t and of $t - 1$.

$$t = \frac{2r^2}{r^2 + |z - c|^2}, \qquad t - 1 = \frac{r^2 - |z - c|^2}{r^2 + |z - c|^2}.$$

Therefore the stereographic map sends z to

$$\left(c + \frac{2r^2(z - c)}{r^2 + |z - c|^2}, \frac{r(r^2 - |z - c|^2)}{r^2 + |z - c|^2}\right).$$

The composition of this with the translation by $(-c, 0)$ and dilation by s gives

$$\phi(z) = \frac{s\,r\big(2r(z-c),\ r^2 - |z-c|^2\big)}{r^2 + |z-c|^2}.$$

Finally, the full composition ψ is given by

$$\psi(z) = \big(h \cdot \phi(z),\ v \cdot \phi(z), \mathrm{sg}(n \cdot \phi(z))\big).$$

We note that $\phi(z) \cdot \phi(z) = s^2 r^2$. Therefore, if we write $\psi(z) = (p, q, \epsilon)$, we will have $p^2+q^2 = \phi(z)\cdot\phi(z) - (n\cdot\phi(z))^2 = s^2r^2 - (n\cdot\phi(z))^2$, that is $(n\cdot\phi(z))^2 = s^2r^2 - p^2 - q^2$. Thus we see that (p, q) belongs to the closed disk $D(sr)$ of radius sr centered at the origin or $\mathbb{R}^2$, and that if $p^2 + q^2 = s^2r^2$, that is if (p, q) belongs to the sphere $S(sr)$ of radius sr centered at the origin of $\mathbb{R}^2$, then $\epsilon = 0$.

Let us compute $\psi^{-1}(p, q, \epsilon)$, where $p, q \in D(sr)$ and $\epsilon \in \{-1, 0, +1\}$ is zero iff $(p, q) \in S(sr)$. We put

$$y = p\,h + q\,v + \epsilon\sqrt{s^2r^2 - p^2 - q^2}\ n.$$

and it is clear that it is the unique point of the sphere of radius sr in $\mathbb{R}^3$ that by the third map projects upon (p, q, ϵ). In other words, $y = \phi(z)$. Thus, if the components of y in the canonical basis of $\mathbb{R}^3$ are (y_1, y_2, y_3) and we put $\tilde{y} = (y_1, y_2) \in \mathbb{R}^2 = \mathbb{C}$, we can write

$$\phi(z) = (\tilde{y}, y_3) = s(t(z-c), r(t-1)).$$

From this we conclude that $sr(t-1) = y_3$, that is $t = \frac{y_3 + s\,r}{s\,r}$ and

$$\tilde{y} = s(z-c)t = \frac{(z-c)(y_3 + s\,r)}{r},$$

and from this we get finally the inverse.

$$\psi^{-1}(p, q, \epsilon) = c + \frac{r\tilde{y}}{s\,r + y_3}$$

Therefore, we have formulas that enable us for passing between a point in screen and the corresponding point in $\mathbb{C}$ and also its stereographic projection upon the sphere.

4. Direct definition of the stereographic representation of $P\mathbb{C}^1$

We will use the following definition. An Hermitian form H in $\mathbb{C}^2$ is a map $H : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ which is bilinear for the sum and such that $H(\lambda z, \mu w) = \overline{\lambda}\mu H(z, w)$, $H(z, w) = \overline{H(w, z)}$ for any $z, w \in \mathbb{C}^2$ and $\lambda, \mu \in \mathbb{C}$. From this we get $H(z, w) = H(iz, iw)$. We denote by $h : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{R}$ the $\mathbb{R}$-bilinear symmetric form given by $h(z, w) = \frac{1}{2}(H(z, w) + H(w, z)) = \mathrm{Re}(H(z, w))$. Then, we have obviously $h(iz, iw) = h(z, w)$. Also

$$\begin{aligned} H(z, w) &= \frac{1}{2}(H(z, w) + H(w, z)) + \frac{1}{2}(H(z, w) - H(w, z)) \\ &= h(z, w) - \frac{i}{2}(iH(z, w) - iH(w, z)) = h(z, w) - \frac{i}{2}(H(z, iw) + H(iw, z)), \\ &= h(z, w) - ih(z, iw). \end{aligned}$$

We shall put

$$K(H) = \{z \in \mathbb{C}^2 \backslash \{0\}\ :\ H(z, z) = 0\}.$$

If we add to the definition of H the property $H(z, z) = h(z, z) > 0$ if $z \neq 0$, we say that H and h are positive definite (or negative definite if we change the inequality sense). Of course, then $K(H) = \{0\}$. In the following we will assume that we have a positive definite Hermitian form H in $\mathbb{C}^2$. It could be the canonical one, that is

$$H(x,y) = \overline{x}_1 y_1 + \overline{x}_2 y_2, \quad h(x,y) = \frac{1}{2}(x_1\overline{y}_1 + \overline{x}_1 y_1 + x_2\overline{y}_2 + \overline{x}_2 y_2),$$

for elements $x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{C}^2$. However we will not require this in what follows.

Let $n \in \mathbb{C}^2$ be a h-unit vector. Let T be the h-orthogonal complement of the complex line generated by n. Then, T is also the H-orthogonal complement of that complex line and is a complex line of $\mathbb{C}^2$.

We consider the real hyperplane $P_n = \mathbb{R}n \oplus T \subset \mathbb{C}^2$ and in it the two-dimensional sphere S_n^2 of radius 1 centered at n. More precisely, $S_1^2 = \{z \in P_n : H(z,z) = 1\}$. That sphere passes obviously by the origin of $\mathbb{C}^2$. Also, $z \in P_n$ iff $h(z, in) = 0$.

Proposition 4.1. *Let $z \in \mathbb{C}^2$. Then, if $z \in T$ we have that $\mathbb{C}\, z \cap S_n^2 = \{0\}$; and if $z \notin T$ then $\mathbb{C}\, z \cap S_n^2$ consists of the origin and another point that we shall denote by $p_n(z)$, and call the stereographic projection of $[z] \in P\mathbb{C}^1$ to S_n^2. We shall put $p_n(z) = 0$ if $z \in T$. Thus, the map $p_n : P\mathbb{C}^1 \to S_n^2$ is a bijection that we shall call the* stereographic projection.

Proof. Since the origin belongs to T and to S_n^2, and T is h-orthogonal to n, it is clear that $T \cap S_n^2 = \{0\}$. So, let $z \notin T$. First of all, we prove that if $\lambda = H(z,n)$, then $h(\lambda z, in) = 0$. In fact, if $\lambda = a + ib$, we have

$$H(\lambda z, in) = iH(\lambda z, n) = i(a - ib)H(z,n) = i(a - ib)(a + ib) = i(a^2 + b^2).$$

Therefore $h(\lambda z, in) = \mathrm{Re}(H(\lambda z, in)) = 0$. Thus, we only need to find a real number p such that $H(p\lambda z - n, p\lambda z - n) - 1 = 0$. But

$$\begin{aligned} H(p\lambda z - n, p\lambda z - n) - 1 &= p^2\lambda\overline{\lambda}H(z,z) - p\,H(\lambda z, n) - pH(n, \lambda z) \\ &= p^2\lambda\overline{\lambda}H(z,z) - p\overline{\lambda}H(z,n) - p\lambda H(n,z) = p^2\lambda\overline{\lambda}H(z,z) - 2p\overline{\lambda}\lambda \\ &= \lambda\overline{\lambda}p(pH(z,z) - 2). \end{aligned}$$

Note that if $\lambda\overline{\lambda} = 0$, then $H(z,n) = 0$, so that $z \in T$ against our hypothesis. Therefore we have two solutions. The first one, $p = 0$, gives $p\lambda z = 0$ and the other is

$$p = \frac{2}{H(z,z)}$$

that gives

$$p\lambda z = \frac{2H(z,n)}{H(z,z)} z.$$

Note that $H(z,z) > 0$ because $z \neq 0$. Therefore the proposition is proved. □

Corollary 4.2. *The stereographic projection $p_n : P\mathbb{C}^1 \to S_n^2$ is given by*

$$p_n([z]) = \frac{2}{h(z,z)}\big(h(z,n) - ih(z,in)\big)z$$

or equivalently

$$p_n([z]) = \frac{2H(z,n)}{H(z,z)} z,$$

for any $[z] \in P\mathbb{C}^1$:

Note that p_n is defined here by a map from $\mathbb{C}^2\backslash\{0\}$ to S^2_n that is invariant along complex lines. We will denote that map also by p_n.

This formula allows also to compute directly the action of a Möbius map on the sphere S^2_n. If $A \in Gl(2;\mathbb{C})$ and $\tilde{A}$ denotes the action of the corresponding Möbius map on S^2_n, then, if $z \in S^2_n$, we will have:

$$\tilde{A}(z) = \frac{2}{h(Az,Az)}\big(h(Az,n) - ih(Az,in)\big)Az = \frac{2H(Az,n)}{H(Az,Az)}Az.$$

A frequent chart of $P\mathbb{C}^1$ is given as follows. We consider the affine real plane B in $\mathbb{C}^2$ given by $B = \{z \in \mathbb{C}^2 : H(n,z) = 1\}$ and we choose a unit vector $m \in T$. Then, (m,n) is a H-orthonormal basis of $\mathbb{C}^2$. We have $B = \mathbb{C}m + n$. Now, for any $z \in \mathbb{C}^2$ we can write that $z = H(m,z)m + H(n,z)n$. Hence, if $\rho \in \mathbb{C}\backslash\{0\}$ and $\rho z \in B$ we will have $\rho H(n,z) = 1$. This is possible only if $H(n,z) \neq 0$, and in this case we get

$$\rho = \frac{1}{H(n,z)}, \qquad \rho z = \frac{H(m,z)}{H(n,z)}m + n.$$

Then, if we consider the following open dense subset of $P\mathbb{C}^1$, $U = \{\pi(z) \in P\mathbb{C}^1 : H(n,z) \neq 0\} \subset P\mathbb{C}^1$, we have that there is one element of B in the class $[z]$ if $\pi(z) \in U$. It is given by $wm + n$ for some uniquely defined $w \in \mathbb{C}$ that can be written as

$$w = \frac{H(m,z)}{H(n,z)}.$$

We put $f_n([z]) = w$. This establishes a bijection f_n between U and $\mathbb{C}$, and in the following, whenever we use this chart we will identify $w \in \mathbb{C}$ with $wm + n \in B$.

5. Circles

We keep here the notation of the preceding section. Our goal now is to prove the following well-known theorem, along whose proof we shall obtain effective formulas that may be used in computer calculations. Let us use the following notation

$$C_G = p_n(K(G)) \subset S^2_n,$$

where G is a Hermitian form in $\mathbb{C}^2$.

Theorem 5.1. *Let H and G be non-vanishing Hermitian forms in $\mathbb{C}^2$, the first one positive (or negative) definite; let $n \in \mathbb{C}^2$ be a H-unit vector and S^2_n be the 2-sphere of unit radius with center n in the real hyperplane $P_n = \{z \in \mathbb{C}^2 : h(z,in) = 0\}$. Let $p_n : P\mathbb{C}^1 \to S^2_n$ be the stereographic projection defined by n. Then:*

1. *If G is positive or negative definite then $C_G = \emptyset$. Otherwise C_G is a circle that may have zero radius. And conversely, every circle or point on S^2_n can be obtained in this manner.*
2. *Let $A \in Gl(2,\mathbb{C})$ and let $\tilde{G}$ be the pull-back of G by A, that is $\tilde{G}(x,y) = G(A^{-1}x, A^{-1}y)$. Then $C_{\tilde{G}} = \tilde{A}(C_G)$.*

Proof. Without loss of generality, suppose that H is positive definite. Let $g : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{R}$ be the real form associated to G. It is well known that then there is a basis (u,v) of $\mathbb{C}^2$ such that (u,iu,v,iv) is a real basis of $\mathbb{C}^2$ which is at the same time h and g-orthogonal. Moreover, if we put $G(u,u) = aH(u,u)$ and $G(v,v) = -bH(v,v)$, the numbers a and b do not depend on the choice or such basis. Those bases are not uniquely determined. In fact, if λ, μ are non-zero complex numbers, we may substitute λu and μv for u and v respectively without losing their properties.

Let $0 \neq z \in \mathbb{C}^2$. Then, $p_n(z) \in C_G$ iff $G(z,z) = 0$ and this is possible only if G is not definite (positive or negative). Otherwise we may suppose that $G(u,u) = aH(u,u),\ G(v,v) = -bH(v,v)$ with $a > 0,\ b \geq 0$. In the following we take this assumption.

It is possible that $G(z,z) = H(z,n) = 0$, so that $p_n([z]) = 0 \in C_G$. In this case the only complex multiple of z belonging to S^2_n is the zero vector 0. In the remaining cases, there is a nonvanishing complex multiple of z, namely $p_n(z)$ belonging both to S^2_n and $K(G)$. Thus, we shall compute the nonvanishing vectors in $S^2_n \cap K(G)$.

We start by defining the following vector: $\xi = p_n([u]) - n$ that is

$$\xi = \frac{2H(u,n)}{H(u,u)}u - n.$$

Note that ξ is a h-unit vector because $p_n([u]) \in S^2_n$. Any point c in S^2_n may be written as

$$c = n + \cos s\,\xi + \sin s(\cos t\,j + \sin t\,k),$$

where $\sin s \geq 0$ and j, k is any fixed h-orthonormal set of vectors of P_n, both h-orthogonal to ξ. We shall compute $g(c,c)$.

By multiplying u and v, if it is necessary, by nonzero complex numbers, we may assume that u and v are h-unit vectors and belong to the real hyperplane P_n, that is $h(u,in) = h(v,in) = 0$. Hence in is a real linear combination of iu and iv, so that n is a real linear combination of u and v. We can write thus

$$n = u\cos\alpha + v\sin\alpha,$$

for some $\alpha \in \mathbb{R}$. Then, we will have

$$\xi = 2H(u,n)u - n = 2u\cos\alpha - u\cos\alpha - v\sin\alpha = u\cos\alpha - v\sin\alpha.$$

We choose for the vectors j and k the following values

$$k = -iu\sin\alpha + iv\cos\alpha, \qquad j = u\sin\alpha + v\cos\alpha.$$

One can easily verify that (ξ, j, k) is a h-orthonormal basis of P_n. For using them in the following calculations we record these values:

$$\begin{aligned}
g(n,n) &= a\cos^2\alpha - b\sin^2\alpha,\\
g(\xi,\xi) &= a\cos^2\alpha - b\sin^2\alpha,\\
g(k,k) &= a\sin^2\alpha - b\cos^2\alpha,\\
g(j,j) &= a\sin^2\alpha - b\cos^2\alpha,\\
g(\xi,j) &= (a+b)\sin\alpha\cos\alpha,\\
g(\xi,k) &= g(j,k) = g(n,k) = 0,\\
g(n,\xi) &= a\cos^2\alpha + b\sin^2\alpha,\\
g(n,j) &= (a-b)\sin\alpha\cos\alpha.
\end{aligned}$$

Thus

$$\begin{aligned}
g(c,c) = {} & g(n,n) + \cos^2 s\, g(\xi,\xi) + \sin^2 s \cos^2 t\, g(j,j) + \sin^2 s \sin^2 t\, g(k,k) \\
& + 2\cos s\, g(n,\xi) + 2\sin s \cos t\, g(n,j) + 2\sin s \sin t g(n,k) \\
& + 2\cos s \sin s \cos t\, g(\xi,j) + 2\cos s \sin s \sin t\, g(\xi,k) + 2\sin^2 s \sin t \cos t g(j,k) \\
= {} & (1+\cos^2 s) g(n,n) + \sin^2 s\, g(k,k) + 2\cos s\, g(n,\xi) \\
& + 2\sin s\ \cos t\big(g(n,j) + \cos s g(\xi,j)\big).
\end{aligned}$$

Now, we have

$$2\sin s\ \cos t\big(g(n,j) + \cos s g(\xi,j)\big) = \big(a - b + (a+b)\cos s\big)\sin s\ \cos t \sin 2\alpha$$

and

$$\begin{aligned}
& (1+\cos^2 s) g(n,n) + \sin^2 s\, g(k,k) + 2\cos s\, g(n,\xi) \\
= {} & (1+\cos^2 s) g(n,n) + (1-\cos^2 s) g(k,k) + 2\cos s\, g(n,\xi) \\
= {} & a - b + (a+b)\cos^2 s \cos 2\alpha + 2\cos s(a\cos^2\alpha + b\sin^2\alpha) \\
= {} & a - b + (a+b)\cos^2 s \cos 2\alpha + \cos s\big(a(1+\cos 2\alpha) + b(1-\cos 2\alpha)\big) \\
= {} & \big(a - b + (a+b)\cos s\big)(1 + \cos s \cos 2\alpha).
\end{aligned}$$

Therefore

$$g(c,c) = \big(a - b + (a+b)\cos s\big)(1 + \cos s\cos 2\alpha + \sin s\ \cos t \sin 2\alpha).$$

Assume that $c \neq 0$ and that the second factor vanishes. We can put $\cos s = r\cos\beta,\ \sin s \cos t = r \sin\beta$, for some $\beta \in [0, 2\pi)$, and $r = \sqrt{\cos^2 s + \cos^2 t \sin^2 s} \leq 1$. Then the second factor in the above expression for $g(c,c)$ is $0 = 1 + r\cos(\beta - 2\alpha)$. Therefore $\cos t = 1$ and then $\beta = s$ and $s = 2\alpha \pm \pi$, or $\cos t = -1$ and then $\beta = -s$ and $s = -2\alpha \pm \pi$. In all these cases we get $c = 0$, as one verifies by direct computation, but this is against our hypothesis. Therefore $g(c,c) = 0$ implies that the first factor vanishes, that is

$$\cos s = \frac{b-a}{a+b}, \qquad \sin s = \frac{2\sqrt{ab}}{a+b}.$$

The corresponding points of S_n^2 are exactly those of the circle

$$c(t) = n + \frac{b-a}{a+b}\xi + \frac{2\sqrt{ab}}{a+b}(\cos t\, j + \sin t\, k),$$

which has center

$$n + \frac{b-a}{a+b}\xi$$

and radius

$$\frac{2\sqrt{ab}}{a+b}.$$

If that circle passes by the origin, then by the compactness of C_G and the continuity of the map p_n we conclude that there is a vector $z \in K(G)$ such that $p_n(z) = 0$. Therefore, in all cases when G is not definite, C_G is a circle that may pass by the origin or have zero radius.

Let us see now the converse. We can write the points $z(t)$ of any circle in S^2_n as $z(t) = n + \mu\xi + \rho(\cos t\, j + \sin t\, k)$, where $\mu^2 + \rho^2 = 1$, where the vectors $\xi, j, k \in P_n$ are an orthonormal basis and where $h(n,k) = 0$. We put

$$\cos 2\alpha = h(n,\xi), \qquad \sin 2\alpha = h(n,j),$$

from which we obtain α up to the addition of an integer multiple of π. Then we define

$$u = \xi\cos\alpha + j\sin\alpha, \qquad v = -\xi\sin\alpha + j\cos\alpha.$$

vectors that are defined up to their simultaneous multiplication by -1. Since $h(n,k) = 0$, we have $n = h(n,u)u + h(n,v)v$. Thus

$$\begin{aligned}
n =& \big(h(n,\xi)\cos\alpha + h(n,j)\sin\alpha\big)u + \big(-h(n,\xi)\sin\alpha + h(n,j)\cos\alpha\big)v\\
=&(\cos 2\alpha\cos\alpha + \sin 2\alpha\sin\alpha\big)u + (-\cos 2\alpha\sin\alpha + \sin 2\alpha\cos\alpha)v\\
=&u\cos\alpha + v\sin\alpha,\\
\xi =&u\cos\alpha - v\sin\alpha,\\
j =&u\sin\alpha + v\cos\alpha.
\end{aligned}$$

We consider the vectors u, v, iu, iv. We know already that

$$h(u,u) = h(v,v) = h(iu,iu) = h(iv,iv) = 1, \quad h(u,v) = h(iv,iv) = 0.$$

We need to prove that $h(u,iv) = 0$ in order to conclude that (u,v,iu,iv) is an h-orthonormal real basis of $\mathbb{C}^2$. We have

$$h(u,iv) = h(\xi\cos\alpha + j\sin\alpha, -i\xi\sin\alpha + ij\cos\alpha) = h(\xi, ij).$$

But

$$0 = h(\xi, in) = -h(i\xi, \cos 2\alpha\xi + \sin 2\alpha j) = h(\xi, ij)\sin 2\alpha,$$

and in the same manner we obtain $h(\xi,ij)\cos 2\alpha = 0$. Hence $h(\xi,ij) = h(u,iv) = 0$. Therefore, by using the same argument as before, we conclude that $k = -iu\sin\alpha + iv\cos\alpha$. Finally, we put

$$a = \frac{1-\mu}{2}, \qquad b = \frac{1+\mu}{2},$$

and then we will have $a+b = 1$, $\mu = (b-a)/(b+a)$, $\rho = 2\sqrt{ab}/(a+b)$. We complete thus our task by defining $g : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{R}$ by

$$g(w,z) = a\big(h(w,u)h(z,u) + h(w,iu)h(z,iu)\big) - b\big(h(w,v)h(z,v) + h(w,iv)h(z,iv)\big).$$

Then, it is evident that G, defined by $G(z,w) = g(z,w) - ig(z,iw)$ is an Hermitian form and that u, v, ξ, k, j are the vectors that define the circle determined by $K(G) \cap S^2_n$. Thus (1) is proved.

As for (2), if $0 \neq z \in \mathbb{C}^2$ we have

$$\tilde{G}(Az, Az) = G(z,z).$$

Therefore, $p_n(z) \in C_G$ iff $Az \in K(\tilde{G})$ and consequently iff $\tilde{A}(p_n(z)) = p_n(Az) \in C_{\tilde{G}}$. □

The image of $K(G)$ by the flat chart f_n of $P\mathbb{C}^1$ is also a circle or one of its limits, say a point or a straight line or the empty set. In fact, we have

Theorem 5.2. *Let H and G be non-vanishing Hermitian forms in $\mathbb{C}^2$, the first one positive (or negative) definite; let $(n,m) \in \mathbb{C}^2$ be a H-orthonormal basis of $\mathbb{C}^2$. We put $B = \mathbb{C}m + n = \{z \in \mathbb{C}^2 : H(n,z) = 1\}$ and $U = \pi(B)$. Let $f_n : U \to \mathbb{C}$ be the chart defined by $f_n([z]) = H(m,z)/H(n,z)$ and put $\hat{C}_G = f_n(K(G))$. Thus:*

If G is positive or negative definite then $\hat{C}_G = \emptyset$. Otherwise it consists of a circle, a straight line, a single point or the empty set. And conversely, every circle, straight line, single point or the empty set in $\mathbb{C}$ can be obtained in this manner.

Proof. Let $z \in B$. Then $H(n,z) = 1$. Therefore, if G is definite (positive or negative) we will have $G(z,z) \neq 0$, whence $z \notin K(G)$. Let us assume that G is not definite and $z = wm + n$ for some $w \in \mathbb{C}$. We will have $G(z,z) = |w|^2 G(m,m) + \bar{w}G(m,n) + w\overline{G(m,n)} + G(n,n)$. Let $a,b,c,d,x,y \in \mathbb{R}$ be such that $w = x+iy,\ G(m,m) = a,\ G(n,n) = b,\ G(m,n) = c+id$. Then $z \in K(G)$ iff

$$(x^2+y^2)a + 2(cx+dy) + b = 0.$$

Assume that $a = 0$. Then (x,y) must satisfy $2(cx+by)+b = 0$, and this is the equation of a straight line unless $c = d = 0$; if $c = d = 0$, then there is no solution for (x,y) because $b \neq 0$ since $G \neq 0$. If $a \neq 0$, then we can write the equation as

$$\left(x + \frac{c}{a}\right)^2 + \left(y + \frac{d}{a}\right)^2 = \frac{c^2+d^2-ab}{a^2}.$$

If the right hand side is negative this has no solutions. Otherwise it is the equation of a circle centered at $(-\frac{c}{a}, -\frac{d}{a})$ with radius $\frac{\sqrt{c^2+d^2-ab}}{|a|}$. Note that if the radius is zero the circle reduces to a single point. It is also clear now that the second part of the first claim is true. Note also that c^2+d^2-ab is equal to minus the determinant of the matrix of G in the basis (m,n). □

A proposition analogous to Theorem 5.1 is true, but some easy modifications are needed to account for the case when there are elements $z \in B$ such that $H(n,Az) = 0$.

6. Effective calculation of Circles

Assume that H is the canonical Hermitian form and that we know G. Let us put $g_{11} = G(e_1,e_1),\ g_{12} = G(e_1,e_2) = \overline{G(e_2,e_1)},\ g_{22} = G(e_2,e_2)$, where $g_{11}, g_{22} \in \mathbb{R}$ and $g_{12} \in \mathbb{C}$ and where (e_1,e_2) denotes the canonical basis of $\mathbb{C}^2$.

We look for a nonvanishing vector $z = z_1e_1 + z_2e_2 \in \mathbb{C}^2$ and a number $\lambda \in \mathbb{C}$ such that $G(z,w) - \lambda H(z,w) = 0$ for any $w \in \mathbb{C}^2$. This happens iff the same occurs for $w = e_1$ and for $w = e_2$. that is if

$$(g_{11}-\lambda)\overline{z}_1 + g_{21}\overline{z}_2 = 0, \quad g_{12}\overline{z}_1 + (g_{22}-\lambda)\overline{z}_2 = 0.$$

If $g_{12} = 0$, then we will have the solutions

$$\begin{aligned} \lambda &= a = g_{11}, & \lambda &= -b = g_{22} \\ z &= \tilde{u}_1 = e_1, & z &= \tilde{u}_2 = e_2. \end{aligned}$$

If $g_{12} \neq 0$, the determinant of that linear system must be zero, and this leads to the following equation

$$\lambda^2 - (g_{11}+g_{22})\lambda + g_{11}g_{22} - g_{12}\overline{g}_{12} = 0.$$

whose solutions are the following

$$a = \frac{1}{2}\left(g_{11} + g_{22} + \sqrt{(g_{11} - g_{22})^2 + 4g_{12}\overline{g}_{12}}, \quad\right),$$
$$-b = \frac{1}{2}\left(g_{11} + g_{22} - \sqrt{(g_{11} - g_{22})^2 + 4g_{12}\overline{g}_{12}}, \quad\right).$$

Then if $ab \geq 0$, we put

$$\mu = \frac{b-a}{b+a}, \qquad \rho = \frac{2\sqrt{ab}}{a+b},$$

and define the vector $\tilde{u} = g_{12}e_1 + (a - g_{11})e_2$. Then we have $(G - aH)(\tilde{u}, w) = 0$ for any $w \in \mathbb{C}^2$. In this manner, $\tilde{u}$ should be part of an h-orthogonal basis of $\mathbb{C}^2$ that is also g-orthogonal.

Then, we will put

$$\xi = \frac{2H(\tilde{u}, n)\tilde{u}}{H(\tilde{u}, \tilde{u})} - n.$$

The circle is thus given by $t \mapsto n + \mu\xi + \rho(\cos tj + \sin tk)$, where j, k is a h-orthonormal set of vectors of P_n, both h-orthogonal to ξ.

7. The principal bundle $P = (\mathbb{C}^2\backslash\{0\}, \pi, P\mathbb{C}^1, \mathbb{C}\backslash\{0\})$

As before let us denote by $\pi : \mathbb{C}^2\backslash\{0\} \to P\mathbb{C}^1$ the natural projection. This defines a principal bundle $P = (\mathbb{C}^2\backslash\{0\}, \pi, P\mathbb{C}^1, \mathbb{C}\backslash\{0\})$ whose right action is given by $R_\rho z = \rho z$ for $z \in \mathbb{C}^2\backslash\{0\}$ and $\rho \in \mathbb{C}\backslash\{0\}$.

The fundamental vector field μ^* generated on $\mathbb{C}^2\backslash\{0\}$ by the element $\mu \in \mathbb{C} \equiv T_1\mathbb{C}\backslash\{0\}$ of the Lie algebra of $\mathbb{C}\backslash\{0\}$ has $R_{\exp(\mu t)}$ as one-parameter subgroup. Therefore, its value at z is the tangent at $t = 0$ to the curve $R_{\exp(\mu t)}z = e^{\mu t}z$. That is $\mu^*_z = \mu z$. In particular we see that the vertical subspace V_z of $\mathbb{C}^2\backslash\{0\}$ at z is given by $V_z = \mathbb{C}\, z$ and also we see that $1^*_z = z$, that is 1^* is the "radius vector".

Let $H : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ be a positive (or negative) definite Hermitian form. Then, it is possible to define a *horizontal distribution* Q in $\mathbb{C}^2\backslash\{0\}$, invariant under right translations, by

$$Q_z = \{X \in T_z\mathbb{C}^2\backslash\{0\} \;:\; H(X, z) = 0\}.$$

In other words, a vector field $X \in \mathfrak{X}(\mathbb{C}^2\backslash\{0\})$ is horizontal iff $H(X, 1^*) = 0$. The invariance of Q is proved as follows. Let $X \in T_z(\mathbb{C}^2\backslash\{0\})$ be horizontal, that is $H(X, z) = 0$, and let $\rho \in \mathbb{C}\backslash\{0\}$. Then $H(dR_\rho X, \rho z) = H(\rho X, \rho z) = \rho\overline{\rho}H(X, z) = 0$, so that $dR_\rho(Q_z) \subset Q_{\rho z}$. Since dR_ρ is an isomorphism, we conclude that $dR_\rho(Q_z) = Q_{\rho z}$. Also, it is evident that being H definite, we have $Q_z \oplus V_z = T_z(\mathbb{C}^2\backslash\{0\})$.

This horizontal distribution defines a connection in the principal bundle P. Let us calculate its connection form. If $z \in \mathbb{C}^2\backslash\{0\}$, and $X \in T_z\mathbb{C}^2\backslash\{0\}$, we put $X = X^V + X^H$, where

$$X^V = \frac{H(z, X)z}{H(z, z)}, \quad X^H = X - \frac{H(z, X)z}{H(z, z)}.$$

Then, $H(X^H, z) = 0$, whence $H(X^H, iz) = 0$. In particular, we get that $h(X^H, z) = h(X^H, iz) = 0$. Therefore, X^H is the h-orthogonal projection of X into the h-orthogonal complement of the real plane generated by z and iz.

Now,

$$\frac{H(z,\mu_z^*)}{H(z,z)} = \frac{H(z,\mu z)}{H(z,z)} = \mu.$$

Therefore, the connection form ω of P with values in $\mathbb{C}$, determined by Q is given by $\omega_z(X) = H(z,X)/H(z,z)$, that is

$$\omega(X) = \frac{H(1^*,X)}{H(1^*,1^*)}.$$

Let us compute its curvature. Let $X, Y \in \mathfrak{X}(\mathbb{C}^2\backslash\{0\})$ be two constant fields and denote by X^H, Y^H their respective horizontal components, that obviously are not constant. Then having in mind that 1^* is the radius vector and that X, Y are constant we have $[X, 1^*] = X$ and $[Y, 1^*] = Y$. Also, since $\omega(X)$ is homogeneous of degree -1, we will have $1^*(\omega(X)) = -\omega(X)$. Also, by the properties of the Lie derivative we have

$$\begin{aligned}
X(\omega(Y)) &= X\left(\frac{H(1^*,Y)}{H(1^*,1^*)}\right) = \frac{H(X,Y)}{H(1^*,1^*)} - \frac{H(1^*,Y)(H(X,1^*)+H(1^*,X))}{H(1^*,1^*)^2} \\
&= \frac{H(X,Y) - \omega(Y)H(X,1^*)}{H(1^*,1^*)} - \omega(Y)\omega(X) = \frac{H(X,Y^H)}{H(1^*,1^*)} - \omega(X)\omega(Y) \\
&= \frac{H(X^H,Y^H)}{H(1^*,1^*)} - \omega(X)\omega(Y).
\end{aligned}$$

Thus

$$\begin{aligned}
\Omega(X,Y) &= d\omega(X^H,Y^H) = d\omega(X-\omega(X)1^*, Y-\omega(Y)1^*) \\
&= d\omega(X,Y) - \omega(Y)d\omega(X,1^*) - \omega(X)d\omega(1^*,Y) \\
&= \frac{H(X^H,Y^H) - H(Y^H,X^H)}{H(1^*,1^*)} + \omega(Y)d\omega(1^*,X) - \omega(X)d\omega(1^*,Y).
\end{aligned}$$

But

$$d\omega(1^*,X) = 1^*(\omega(X)) - X(\omega(1^*)) - \omega([1^*,X]) = -\omega(X) + \omega(X) = 0.$$

Therefore

$$\Omega(X,Y) = \frac{H(X^H,Y^H) - H(Y^H,X^H)}{H(1^*,1^*)} = \frac{-2i\,h(X^H,iY^H)}{h(1^*,1^*)}.$$

8. The Fubini-Study metric in $P\mathbb{C}^1$

The Hermitian form H defines a $\mathbb{C}\backslash\{0\}$-invariant Hermitian form field $\mathcal{H}$ in $\mathbb{C}^2\backslash\{0\}$ by $\mathcal{H} = H/H(1^*,1^*)$, that is, if $z \in \mathbb{C}^2\backslash\{0\}$ and $X, Y \in T_z\mathbb{C}^2\backslash\{0\}$ we have

$$\mathcal{H}_z(X,Y) = \frac{H(X,Y)}{H(z,z)}.$$

In fact, if $\rho \in \mathbb{C}\backslash\{0\}$ we have

$$(R_\rho^*\mathcal{H})_z(X,Y) = \mathcal{H}_{\rho z}(dR_\rho X, dR_\rho Y) = \frac{H(\rho X,\rho Y)}{H(\rho z,\rho z)} = \mathcal{H}_z(X,Y).$$

Thus, $R_\rho^*\mathcal{H} = \mathcal{H}$. Note that the Riemannian metric defined by $\mathrm{Re}(\mathcal{H})$ is also $\mathbb{C}\backslash\{0\}$-invariant. This allows for the construction of a Riemannian metric in $P\mathbb{C}^1$, the

Fubini-Study metric γ, that makes of the bundle P a Riemannian bundle. It is enough to put, for $\tilde{X}, \tilde{Y} \in T_{\pi(z)} P\mathbb{C}^1$:

$$\gamma_{\pi(z)}(\tilde{X}, \tilde{Y}) = \mathrm{Re}(\mathcal{H})_z(X_z^H, Y_z^H),$$

where X_z^H and Y_z^H are the horizontal lift of $\tilde{X}$ and $\tilde{Y}$, respectively, to z. The proof that this definition is consistent is immediate. We will have that if $X, Y \in T_z\mathbb{C}^2 \backslash \{0\}$ are such that $d\pi(X) = \tilde{X}$ and $d\pi(Y) = \tilde{Y}$, then

$$X_z^H = X - \frac{H(z,X)z}{h(z,z)}, \quad Y_z^H = Y - \frac{H(z,X)z}{h(z,z)}.$$

Thus

$$\begin{aligned} H(X_z^H, Y_z^H) &= \frac{1}{h(z,z)} H\Big(X - \frac{H(z,X)z}{h(z,z)}, Y - \frac{H(z,Y)z}{h(z,z)}\Big) \\ =&\frac{1}{h(z,z)} H\Big(X, Y - \frac{H(z,Y)z}{h(z,z)}\Big) = \frac{H(X,Y)h(z,z) - H(X,z)H(z,Y)}{h(z,z)^2} \end{aligned}$$

Now,

$$\begin{aligned} \mathrm{Re}\big(H(X,z)H(z,Y)\big) &= \mathrm{Re}\Big(\big(h(X,z) - ih(X,iz)\big)\big(h(z,Y) - ih(z,iY)\big)\Big) \\ &= h(X,z)h(Y,z) - h(X,iz)h(iY,z) = h(X,z)h(Y,z) + h(X,iz)h(Y,iz). \end{aligned}$$

Therefore

$$\gamma_{\pi(z)}(\tilde{X}, \tilde{Y}) = \frac{h(X,Y)h(z,z) - h(X,z)h(Y,z) - h(X,iz)h(Y,iz)}{h(z,z)^2}.$$

Let us see how is the Fubini-Study metric in the stereographic image of $P\mathbb{C}^1$ described in section 4. We take a h-unit vector m such that $H(m,n) = 0$. In particular we see that $H(m,in) = H(im,in) = 0$, and as a consequence that $h(m,n) = h(m,in) = h(im,in) = h(im,n) = 0$. Thus, (m, im, n) is a h-orthonormal basis of P_n. The sphere S_n^2 can thus be parameterized in usual spherical coordinates as

$$\psi(\theta,\phi) = n(1+\cos\theta) + \sin\theta(m\cos\phi + i\,m\sin\phi).$$

Therefore

$$\psi_\theta = \cos\theta(m\cos\phi + i\,m\sin\phi) - n\sin\theta, \quad \psi_\phi = \sin\theta(-m\sin\phi + i\,m\cos\phi).$$

Thus

$$\begin{aligned} &h(\psi(\theta,\phi), \psi(\theta,\phi)) = 2(1+\cos\theta), \\ &h(\psi_\theta, \psi_\theta) = 1, \\ &h(\psi(\theta,\phi), \psi_\theta) = -\sin\theta, \\ &h(\psi(\theta,\phi), i\psi_\theta) = h(\psi(\theta,\phi), \cos\theta(im\cos\phi - m\sin\phi) - in\sin\theta) = 0, \\ &\gamma(\psi_\theta, \psi_\theta) = \frac{2(1+\cos\theta) - \sin^2\theta}{4(1+\cos\theta)^2} = \frac{1}{4}. \end{aligned}$$

In the same manner we have

$$\begin{aligned}
&h(\psi_\phi, \psi_\phi) = \sin^2\theta,\\
&h(\psi(\theta,\phi), \psi_\phi) = 0,\\
&h(\psi(\theta,\phi), i\psi_\phi) = -\sin^2\theta,\\
&\gamma(\psi_\phi, \psi_\phi) = \frac{2(1+\cos\theta)\sin^2\theta - \sin^4\theta}{4(1+\cos\theta)^2} = \frac{1}{4}\sin^2\theta,
\end{aligned}$$

and

$$\begin{aligned}
h(\psi_\theta, \psi_\phi) = 0,\\
\gamma(\psi_\theta, \psi_\phi) = 0.
\end{aligned}$$

Thus, $\gamma = \frac{1}{4}(d\theta \otimes d\theta + \sin^2\theta d\phi \otimes d\phi)$, which is $\frac{1}{4}$ times the usual metric of S^2, or also the metric of a sphere of radius $1/2$. Its curvature is constant and equal to 4. We conclude that the conformal structure defined by γ in $P\mathbb{C}^1$ is the same structure preserved for $P\mathbb{C}^1$ under the action of elements of $Gl(2;\mathbb{C})$.

Now, we describe the same Fubini-Study metric when we use the "flat" presentation of $P\mathbb{C}^1$. By putting $x(w) = \mathrm{Re}(w), y(w) = \mathrm{Im}(w)$, we have the coordinate functions (x,y) that we shall use for B. Then if ∂_x, ∂_y are the coordinate vector fields corresponding to x and y, the inclusion B into $P\mathbb{C}^1$ gives the following parameterization B :

$$\Phi(x,y) = xm + y\, im + n,$$

After an easy calculation, this gives the following metric that can be identified with the metric of U in the chart Φ :

$$\begin{aligned}
&\gamma(\Phi_x, \Phi_x) = \frac{1}{(1+x^2+y^2)^2},\\
&\gamma(\Phi_x, \Phi_y) = 0,\\
&\gamma(\Phi_y, \Phi_y) = \frac{1}{(1+x^2+y^2)^2},
\end{aligned}$$

Hence, we have

$$\gamma = \frac{dx^2 + dy^2}{(1+x^2+y^2)^2},$$

Theorem 8.1. *Let $U(H)$ be the unitary group defined by H, that is $U(H) = \{A \in Gl(2;\mathbb{C}) : H(Ax, Ay) = H(x,y), \forall x, y \in \mathbb{C}^2\}$, and let $A \in U(H)$. Then γ is invariant by $\tilde{A}$.*

Proof. Let $0 \neq z \in \mathbb{C}^2$ and $\tilde{X}, \tilde{Y} \in T_{\pi(z)}P\mathbb{C}^1$. We want to compute the pull-back of γ by $\tilde{A} : P\mathbb{C}^1 \to P\mathbb{C}^1$. We have

$$(\tilde{A}^*\gamma)_{\pi(z)}(\tilde{X}, \tilde{Y}) = \gamma_{\tilde{A}(\pi(z))}\big(d\tilde{A}(\tilde{X}), d\tilde{A}(\tilde{Y})\big).$$

Let $X, Y \in T_z(\mathbb{C}^2)$ be such that $d\pi(X) = \tilde{X},\ d\pi(Y) = \tilde{Y}$. Then $d\pi\big(dA(X)\big) = d(\pi \circ A)(X) = d\tilde{A}(d\pi(X)) = d\tilde{A}(\tilde{X})$ and $dA(X) = AX$. Therefore we have

$$
\begin{aligned}
&(\tilde{A}^*\gamma)_{\pi(z)}(\tilde{X}, \tilde{Y}) \\
&\quad = \frac{h(AX, AY)h(Az, Az) - h(AX, Az)h(AY, Az) - h(AX, iAz)h(AY, iAz)}{h(Az, Az)^2} \\
&\quad = \frac{h(X, Y)h(z, z) - h(X, z)h(Y, z) - h(X, iz)h(Y, iz)}{h(z, z)^2} = \gamma_{\pi(z)}(\tilde{X}, \tilde{Y}).
\end{aligned}
$$

□

9. The intrinsic isometric circle

Let $A \in Gl(2; \mathbb{C})$. We have seen in section 4 that it induces a transformation $\tilde{A}$ in $P\mathbb{C}^1$ and, consequently in S^2_n. We study now the subset of $P\mathbb{C}^1$ where that transformation preserves the inner product γ. Let $Am = am + cn$ and $An = bm + dn$. If $z \in \mathbb{C}^2$, we have $z = H(m, z)m + H(n, z)n$. Hence

$$
\begin{aligned}
Az =&H(m, z)Am + H(n, z)An = H(m, z)(am + cn) + H(n, z)(bm + dn) \\
=&(aH(m, z) + bH(n, z))m + (cH(m, z) + dH(n, z))n.
\end{aligned}
$$

for some well defined complex numbers a, b, c, d such that $ad - bc \neq 0$. Then, the image of Az in the “flat” presentation will be

$$\tilde{A}(p(z)) := \frac{aH(m, z) + bH(n, z)}{cH(m, z) + dH(n, z)} = \frac{aw + b}{cw + d},$$

where $w = H(m, z)/H(n, z)$ is the image of z in the same presentation. Therefore, the map $\tilde{A}$ in $\tilde{A}^{-1}(U)$ is seen in the flat presentation as the complex function

$$f(w) = \frac{aw + b}{cw + d}.$$

Let us denote by w' a vector tangent to B at w. Then

$$df(w') = \frac{acw + ad - caw - cb}{(cw + d)^2} w' = \frac{ad - bc}{(cw + d)^2} w'.$$

let us put $\Delta = ad - bc,\ q = cw + d,\ r = aw + b$. Then $df(w') = \Delta w'/q^2$. Also

$$1 + |f(w)|^2 = 1 + \frac{|r|^2}{|q|^2}.$$

The squared length of $df(w')$ will be thus

$$\gamma_{z(w)}(d\tilde{A}(w'), d\tilde{A}(w')) = \frac{|\Delta|^2 |w'|^2}{|q|^4 \big(1 + \frac{|r|^2}{|q|^2}\big)^2} = \frac{|\Delta|^2 |w'|^2}{(|q|^2 + |r|^2)^2}.$$

Since

$$\gamma_w(w', w') = \frac{|w'|^2}{(1 + |w|^2)^2},$$

we see that the points w where the map $\tilde{A}$ preserves the inner product of tangent vectors are those that satisfy the equation:

$$|q|^2 + |r|^2 - |\Delta|(1 + |w|^2) = 0.$$

After calculation we get the equation:

$$(|a|^2 + |c|^2 - |\Delta|)|w|^2 + (c\overline{d} + a\overline{b})w + (\overline{c}d + \overline{a}b)\overline{w} + |b|^2 + |d|^2 - |\Delta| = 0,$$

which is the equation of a circle in B. I shall call it the *intrinsic isometric circle* because it is defined through the natural Fubini-Study metric of $P\mathbb{C}^1$.

One sees by an easy calculation that if we put $G(u,v) = H(Au,Av) - |\Delta| H(u,v)$, then

$$G(wm+n, wm+n) = |q|^2 + |r|^2 - |\Delta|(1+|w|^2).$$

Therefore, we have proved the following result (cfr. section 5):

Theorem 9.1. *The intrinsic isometric circle in B (resp. S_n^2) defined by the Hermitian form H and the automorphism $A \in GL(2;\mathbb{C})$ is the intersection of B (resp. S_n^2) with $K(G)$, where G is the Hermitian form in $\mathbb{C}^2$ given by $G(u,v) = H(Au,Av) - |\det A| H(u,v)$.*

Note that when $A \in U(H)$ we obtain $G(u,v) = 0$ because then $\Delta^2 = 1$. Then $K(G) = \mathbb{C}^2 \backslash \{0\}$, so that the isometric circle degenerates to all of $P\mathbb{C}^1$.

Usually by isometric circle is understood the locus of points of B where the map induced by A is an isometry with respect to the Euclidean metric of B. We compute now the Hermitian form associated with the usual isometric circle. If instead of the metric γ for B as isometric to $P\mathbb{C}^1$ with the Fubini-Study metric, we use the metric induced in B by h, then it is clear that a point w is in the isometric circle iff

$$\left| \frac{ad-bc}{(cw+d)^2} \right| = 1,$$

that is iff $|cw+d|^2 - |\Delta| = |c|^2|w|^2 + |d|^2 + c\overline{d}w + \overline{c}d\overline{w} - |\Delta| = 0$. Then, that Hermitian metric G is given by the following matrix coefficients:

$$G(m,m) = |c|^2, \quad G(n,n) = |d|^2 - |\Delta|, \quad G(m,n) = \overline{c}d, \quad G(n,m) = c\overline{d}.$$

The ugliness of those formulas, that is their lack of symmetry, reflects their hybrid character.

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