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Schwarzschild Interior in Conformally Flat Form

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A unified conformally flat form of the static Schwarzschild interior space–times is provided. A new parameter that allows us to analyze the symmetry (spherical, plane or hyperbolic) of the three well known classes of metrics is introduced. In the spherically symmetric case, this parameter is related to the historical limit value of the mass to radius ratio found by Schwarzschild for a sphere of incompressible fluid.

KEY WORDS: Schwarzschild interior; conformal factor.

1. INTRODUCTION

As it has been well established, the Schwarzschild interior solution [1] is a conformally flat space–time because its Weyl tensor vanishes [2]. It has been mainly considered as a simplified stellar model filled with an ideal fluid whose energy density is a positive constant. Static conformally flat perfect fluid solutions having non positive energy density also exist [3], [4]. In fact, the Schwarzschild interior metrics constitute a static family of conformally flat perfect fluid space–times admitting a 4–dimensional group $G_4 = G_1 \times G_3$ of local isometries. Of course, the conformal flatness of these solutions suggests the possibility of using them as a curved background in Cosmology and in semi–classical or quantum gravity. For simplicity, in the following, we shall refer them as the Schwarzschild interior space–times (SIST). They have spherical, plane or hyperbolic symmetry depending on the (constant) curvature of the G_3 orbits [4, 5].

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Two invariant ways are known to distinguish geometrically these metrics according to the sign of the constant curvature of (i) the 3-spaces orthogonal to the privilegiated timelike Killing vector, or (ii) the spacelike 2-orbits of the isometry subgroup G_3. The former way is also dynamic because the corresponding curvature gives the energy density of the perfect fluid that is the source of the metric. The latest one is associated with a parameter that distinguishes among three different types of (radial) timelike Killing fields, as we will shown in this paper. Both characterizations are complementary.

The Schwarzschild interior solution and the Einstein universe were invariantly characterized by Shepley and Taub [6] as the only conformally flat static solutions whose source is a perfect fluid with rest particle density conserved. The uniqueness of the positive density conformally flat static perfect fluid solutions was also independently stablished by Barnes [4]. Other characterizations have been also discussed, [5, 7, 8], improving previous results by Collinson [9] about stationary axisymmetric space–times. In the case of spherical symmetry, the conformal uniqueness of the Schwarzschild interior and related metrics was presented in [10]. In this last reference, several conformal factors relating these metrics with Minkowski space-time and de Sitter and Einstein universes were also provided starting from the metric in isotropic coordinates. Then, an essential question follows: does it exist a procedure to obtain directly the whole SIST family in conformally flat form?

In this paper, we obtain a unified conformally flat form for the SIST (see expression (20)). The corresponding conformal factor is calculated by imposing only geometrical conditions: vanishing expansion of a timelike radial conformal field with orthogonal surfaces of constant curvature. Consequently, the energy content is a perfect fluid with constant energy density and whose velocity is the unit vector associated to the Killing field. Of course, we do not find a new family of solutions because of the results about uniqueness quoted previously. In the spherically symmetric case, expression (25) gives the reduced form associated with the Schwarzschild interior solution and related metrics.

It should be noted that the geometrical conditions we have considered can be also imposed on a spacelike radial conformal field. Then, an expression for the conformal factor can be obtained which applies for conformally flat space–times foliated by timelike hypersurfaces. However, we shall restrict here to static metrics because our main interest is about the Schwarzschild interior solution.

Next, we specify the terms we are using. A conformally flat space–time is one in which the metric has the local form $g = F^2\eta$, with F as a function of the coordinates (F^2 is the *conformal factor*) and η the flat metric. Then we shall start with a conformally flat metric in polar spherical coordinates:

$$g = F^2(t, r, \theta, \varphi)(-dt \otimes dt + dr \otimes dr + r^2 h) \tag{1}$$

where $h = d\theta \otimes d\theta + \sin^2\theta d\varphi \otimes d\varphi$ is the metric on the 2–sphere. A radial conformal field ξ of g has the form:

$$\xi = \alpha \frac{\partial}{\partial t} + \beta \frac{\partial}{\partial r} \tag{2}$$

where α and β are functions of the coordinates. In [11] we have shown that these functions are independent of the angular coordinates θ and φ and are given by:

$$\alpha(t, r) = a(t^2 + r^2) + bt + c, \quad \beta(t, r) = r(2at + b) \tag{3}$$

with a, b and c as arbitrary constants. We will consider that ξ is timelike, then let $\mathbf{u}$ be the unit vector associated with it, $\mathbf{u} = \xi/\sqrt{-g(\xi, \xi)}$, where

$$-g(\xi, \xi) = F^2\{[a(t^2 - r^2) + bt + c]^2 - r^2\Delta\} > 0 \tag{4}$$

with $\Delta \equiv b^2 - 4ac$. This parameter Δ provides an algebraic classification of the radial conformal fields [12], and it will be used in order to simplify our expressions. The meaning of Δ is attached to the space–time considered. In [11] we showed that a Friedmann–Robertson–Walker space–time is an open, flat or closed universe if $\Delta > 0$, $\Delta = 0$ or $\Delta < 0$, respectively. For the SIST, Δ is related with the curvature of the G_3 orbits as it will be discussed in this paper.

This paper is organized as follows. In Section 2 we study the SIST, obtaining their conformally flat form and analyzing their isometries in conformally flat coordinates using the parameter Δ. Finally, in Section 3, we comment on the Schwarzschild interior metric relating Δ to the mass to radius ratio admissible for hydrostatic equilibrium.

2. SCHWARZSCHILD INTERIOR SPACE–TIMES

2.1. Conformally Flat Form of the SIST

We consider a timelike radial conformal field ξ in a conformally flat space–time with metric g, then its associated unit vector $\mathbf{u}$ is shear–free and vorticity–free. However, the expansion and the acceleration of $\mathbf{u}$ depend on the conformal factor and its first derivatives. This fact provides a kinematical method to obtain conformal factors. For example, the Robertson–Walker metrics are the conformally flat space–times admitting a timelike geodesic radial conformal field, and the form of their conformal factor is determined by these properties [11]. In order to obtain the SIST in conformally flat form, we impose that ξ is a Killing field of g. The nullity of the expansion leads to the following differential equation for the conformal factor:

$$\alpha \frac{\partial F}{\partial t} + \beta \frac{\partial F}{\partial r} + \frac{\beta}{r} F = 0. \tag{5}$$

In order to solve this equation we introduce the function

$$\omega(t,r) = \frac{a(t^2 - r^2) + bt + c}{r}, \tag{6}$$

which satisfies that $\omega^2 > \Delta$ due to (4). Then, the equation (5) reduces to $r\frac{\partial F}{\partial r} + F = 0$ and its general solution has the form:

$$F = \frac{1}{rf(\omega,\theta,\varphi)}, \tag{7}$$

$f(\omega,\theta,\varphi)$ being an arbitrary function.

The conformal factor becomes more determined if more conditions over ξ are added. For instance, we can impose that the orthogonal surfaces to the field have constant curvature. To do this we introduce the function s given by:

$$s(t,r) = \begin{cases} b(t^2 - r^2) + 2ct & \text{if} \quad a = 0 \\ \dfrac{a(t^2 - r^2) - c}{2at + b} & \text{if} \quad a \neq 0 \end{cases} \tag{8}$$

in such a way that s is a potential of the 1–form ξ_* associated with the field ξ by the metric g ($\xi_* \propto ds$, with d the exterior derivative). Then, the 3–surfaces $\Sigma_s = \{(t,r,\theta,\varphi) : s = constant\}$ are orthogonal to the field.

Note that if we use $\{s,\omega\}$ as coordinates, the metric (1) has the expression:

$$g = F^2 r^2 \left\{ -\frac{\omega^2 - \Delta}{4S^2} ds \otimes ds + \frac{1}{\omega^2 - \Delta} d\omega \otimes d\omega + h \right\}, \tag{9}$$

where S is a function of s given by the expression:

$$S(s) = \begin{cases} bs + c^2 & \text{if} \quad a = 0 \\ as^2 + bs + c & \text{if} \quad a \neq 0 \end{cases} \tag{10}$$

In consequence, the induced metric over the surfaces Σ_s is given by $\widetilde{\gamma} = F^2\gamma, \gamma = r^2(\frac{d\omega \otimes d\omega}{\omega^2 - \Delta} + h)$ being a metric with constant curvature [11]. Then, the curvature form of the metric $\widetilde{\gamma}$ is:

$$\mathcal{R}(\widetilde{\gamma}) = F^2 \left[\mathcal{R}(\gamma) + \mathcal{H} \wedge \gamma\right] \tag{11}$$

where $\wedge$ is the exterior product of double 1–forms and

$$\mathcal{H} = F\nabla dF^{-1} - \frac{1}{2}\widetilde{\gamma}\,(d\ln|F|, d\ln|F|)\widetilde{\gamma} \tag{12}$$

with ∇ as the Levi–Civita connection of the metric γ. So, the condition of constant curvature over each hypersurface Σ_s is $\mathcal{H} \propto \widetilde{\gamma}$. Taking into account expression (7), this condition leads to the following system of differential equations:

$$f_{,\omega\theta} = f_{,\omega\varphi} = 0$$
$$f_{,\theta\varphi} = \cot\theta\ f_{,\varphi}$$

$$f_{,\varphi\varphi} = \sin\theta(\sin\theta\ f_{,\theta\theta} - \cos\theta\ f_{,\theta}) \tag{13}$$
$$f_{,\theta\theta} = [(\omega^2 - \Delta)\ f_{,\omega} - \omega\ f]_{,\omega}$$

Its general solution is:

$$f(\omega, \theta, \varphi) = \frac{\vec{\sigma} \cdot \vec{r}}{r} + \nu(\omega) \tag{14}$$

where

$$\vec{\sigma} \cdot \vec{r} = (\sigma_1 \cos\varphi + \sigma_2 \sin\varphi) r \sin\theta + \sigma_3 r \cos\theta \tag{15}$$

and

$$\nu(\omega) = \begin{cases} K_1\omega + K_2\sqrt{\omega^2 - \Delta} & \text{if} \quad \Delta \neq 0 \\ K_1\omega + \dfrac{K_2}{\omega} & \text{if} \quad \Delta = 0 \end{cases} \tag{16}$$

with $\vec{\sigma} \equiv (\sigma_1, \sigma_2, \sigma_3)$, K_1 and K_2 as arbitrary constants. We can rewrite the function $\nu(\omega)$ as:

$$\nu(\omega) = k_1(\omega + \sqrt{\omega^2 - \Delta}) + \frac{k_2}{\omega + \sqrt{\omega^2 - \Delta}}, \tag{17}$$

which includes both situations, $\Delta \neq 0$ and $\Delta = 0$. Here, k_1 and k_2 are two new arbitrary constants directly related to the arbitrary constants K_1 and K_2.

Consequently, the curvature of each surface Σ_s results independent of s and is the constant

$$K = 4k_1k_2 - \sigma^2 \quad \text{with} \quad \sigma^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 \tag{18}$$

and the energy content is a perfect fluid, $T = (\mu + p)\mathbf{u} \otimes \mathbf{u} + pg$, with constant energy density, $\mu = 3K$, and pressure given by

$$p(\omega, \theta, \varphi) = -\mu + \frac{2f}{\sqrt{\omega^2 - \Delta}}(k_2 - k_1\Delta). \tag{19}$$

We have taken the Einstein gravitational constant $\kappa = 1$ and to make clearer some geometrical aspects, no energy conditions have been imposed on T.

So, we have obtained the whole family of static conformally flat perfect fluid metrics with constant energy density, including its degenerations (Minkowski, de Sitter and Einstein universes). More precisely:

The conformally flat space–times admitting a timelike radial Killing field with orthogonal surfaces of constant curvature are locally isometric with the SIST. Their conformally flat form has the expression:

$$g = \frac{1}{(\vec{\sigma} \cdot \vec{r} + r\nu)^2}\eta \tag{20}$$

where $\vec{\sigma} \cdot \vec{r}$ and $\nu(\omega)$ are given in the form (15) and (17), respectively, and ω as in (6).

In fact, Killing's equations for this metric say that it admits a 4–dimensional isometry group $G_4 = G_1 \times G_3$, with G_1 generated by the timelike radial Killing field and with G_3 acting on spacelike 2–dimensional orbits with constant curvature as we will see in the following subsection.

Note that, from expression (19), de Sitter space–times are recovered taking $k_2 = k_1\Delta$. To recover the Einstein universes we need to look at the expression of the field acceleration. From (4) and (20), we get that $g(\xi, \xi) = -(\omega^2 - \Delta)/f^2$, which is a potential of the acceleration of $\mathbf{u}$:

$$\mathbf{a} = d \ln \sqrt{-g(\xi, \xi)} = \frac{1}{rf} \left\{ \frac{A}{\omega^2 - \Delta} d\omega - (\vec{\sigma} \cdot \vec{r})_{,\theta} d\theta - (\vec{\sigma} \cdot \vec{r})_{,\varphi} d\varphi \right\} \tag{21}$$

with

$$A = \omega\, \vec{\sigma} \cdot \vec{r} + r(k_1\Delta + k_2).$$

Then, the condition of null acceleration necessarily implies $\vec{\sigma} = 0$ and the relation $k_1\Delta + k_2 = 0$, which corresponds to the Minkowski space-time and the Einstein universes ($\mu = -3p$) as it follows from expressions (14), (17), (18) and (19).

2.2. Isometries of the SIST

As we have mentioned in Section 1, the isometry group of the SIST has been widely analyzed [4, 5] considering a coordinate system adapted to the timelike Killing field. In this subsection we solve the Killing's equations $\mathcal{L}_\xi g = 0$ to determine the groups of motions of a conformally flat space–time given by (20), following closely the Levine's method [13, 14] and using conformally flat coordinates. It is convenient to consider that a t-independent rotation over the surfaces $t = constant$ defines new conformally flat coordinates $\{t, x', y'z'\}$ for which $\vec{\sigma} = (0, 0, \sigma)$, then we can always take $\sigma_1 = \sigma_2 = 0$ without loss of generality. Under these considerations, we arrive to the following expressions for a basis of the Lie algebra:

$$\xi_0 = \xi = [a(t^2 + r^2) + bt + c]\frac{\partial}{\partial t} + [r(2at + b)]\frac{\partial}{\partial r} = 2S(s)\frac{\partial}{\partial s}$$

$$\xi_1 = \sigma(\omega^2 - \Delta)\sin\theta \sin\varphi \frac{\partial}{\partial \omega} - (\lambda + \sigma\omega\cos\theta)\sin\varphi\frac{\partial}{\partial\theta} - (\sigma\omega + \lambda\cos\theta)\frac{\cos\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\xi_2 = \sigma(\omega^2 - \Delta)\sin\theta\cos\varphi\frac{\partial}{\partial\omega} - (\lambda + \sigma\omega\cos\theta)\cos\varphi\frac{\partial}{\partial\theta}$$

$$+ (\sigma\omega + \lambda\cos\theta)\frac{\sin\varphi}{\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\xi_3 = \frac{\partial}{\partial\varphi}$$

where $S(s)$ is given by expression (10) and the parameter λ is defined as:

$$\lambda \equiv k_1\Delta + k_2. \tag{22}$$

Note that this parameter λ also appears in the acceleration of the Killing field ξ. Then, these expressions for the Killing vectors stand when $\sigma \neq 0$ or $\lambda \neq 0$ because the case $\sigma = \lambda = 0$ corresponds to the Minkowski space–time or the Einstein universes, as we have mentioned before.

So, we are going to analyze the generic cases. Clearly $[\xi, \xi_1] = [\xi, \xi_2] = [\xi, \xi_3] = 0$, and the G_3 Lie subalgebra is given by

$$[\xi_1, \xi_2] = (\lambda^2 - \sigma^2\Delta)\xi_3, \quad [\xi_1, \xi_3] = -\xi_2, \quad [\xi_2, \xi_3] = \xi_1. \tag{23}$$

The structure constant $\lambda^2 - \sigma^2\Delta$ is related to the curvature of the G_3 orbits, giving spherical, plane or hyperbolic symmetry according to $\lambda^2 > \sigma^2\Delta$, $\lambda^2 = \sigma^2\Delta$ or $\lambda^2 < \sigma^2\Delta$, respectively. The corresponding Bianchi types are *IX*, *VII*$_0$ and *VIII*.

Next, we are going to study the relation between the symmetry type and the parameters Δ and K associated with ξ. Clearly, if $\sigma = 0$ we only have spherical symmetry. In the case $\sigma \neq 0$, taking into account (22) and (23), we have to analyze the sign of the polynomial function:

$$p(\Delta) = k_1^2\Delta^2 + (2k_1k_2 - \sigma^2)\Delta + k_2^2, \tag{24}$$

whose discriminant is $-\sigma^2 K$. The result of this analysis is summarized in Table I.

We observe that for $\Delta < 0$ we only have spherical symmetry, as well as for $K > 0$ and also, in the case $\Delta = K = 0$.

As for the case $\Delta = 0$ and $K < 0$, the constant k_2 appearing in expression (17) allows us to distinguish the type of symmetry, giving spherical or plane symmetry according to $k_2 \neq 0$ or $k_2 = 0$, respectively.

Table I. (S)pherical, (P)lane and (H)yperbolic Symmetry of the SIST Depending on the Δ and K Parameters for $\sigma \neq 0$

	$K < 0$	$K = 0$	$K > 0$
$\Delta < 0$	S	S	S
$\Delta = 0$	S, P	S	S
$\Delta > 0$	S, P, H	S, P	S

The case $\Delta > 0$ and $K = 0$ only needs the ratio k_2/k_1 to draw the distinction, having spherical or plane symmetry according to $\Delta \neq k_2/k_1$ or $\Delta = k_2/k_1$, respectively.

Finally, in the case $\Delta > 0$ and $K < 0$ the three possible algebras are also easily distinguished by the constants k_1 and k_2 in (17):

(i) If $k_1 = 0$ we have spherical, plane or hyperbolic symmetry when Δ is minor, equal or major than $(k_2/\sigma)^2$, respectively.
(ii) If $k_1 \neq 0$, let $\Delta_\pm$ be the two real roots of (24). Then, the SIST have spherical symmetry when $\Delta < \Delta_-$ or $\Delta > \Delta_+$, plane symmetry when $\Delta = \Delta_-$ or $\Delta = \Delta_+$, and hyperbolic symmetry when $\Delta_- < \Delta < \Delta_+$.

3. COMMENTS ON THE SCHWARZSCHILD INTERIOR SOLUTION

In this section we deal mainly with the Schwarzschild interior solution, which is a special case of spherically symmetric SIST. For these metrics a reduced conformally flat form can be directly obtained from expression (20) taking $\vec{\sigma} = 0$. So, one has

$$f = \nu(\omega), \quad g(\xi, \xi) = -\frac{\omega^2 - \Delta}{\nu^2} \quad \text{and} \quad \mathbf{a} = \frac{k_1\Delta + k_2}{(\omega^2 - \Delta)\nu} d\omega.$$

Then, it results:

A reduced conformally flat form for the spherically symmetric SIST is:

$$g = \frac{\eta}{r^2\,\nu^2}, \tag{25}$$

$\nu(\omega)$ being the function (17) with ω defined by (6). The function ω is a potential of the acceleration of the timelike radial Killing field ξ, whose expression is given by (2) and (3).

To interpret the parameter Δ for the case of Schwarzschild interior solution we are going to recover this metric in curvature coordinates. So, we shall consider a transformation from the conformally flat coordinates $\{t, r\}$ to new coordinates $\{\tau, \rho\}$. We start with the metric (25) written in $\{s, \omega\}$ coordinates as in expression (9). Then, the functions:

$$\rho = \frac{1}{\nu(\omega)} \quad \text{and} \quad \tau = \int \frac{ds}{S(s)},$$

with $S(s)$ given by (10), allow to write the metric in curvature coordinates as:

$$g = -\frac{\omega^2 - \Delta}{4\nu^2(\omega)} d\tau \otimes d\tau + \frac{\nu^2(\omega)}{(\omega^2 - \Delta)\nu'^2(\omega)} d\rho \otimes d\rho + \rho^2 h.$$

In the case $K = 4k_1k_2 \neq 0$ this metric results in the Schwarzschild interior form:

$$g = -(A + B\sqrt{1 - K\rho^2})^2 d\tau \otimes d\tau + \frac{1}{1 - K\rho^2} d\rho \otimes d\rho + \rho^2 h, \tag{26}$$

A and B being constants related to the constants k_1 and k_2 by the following expressions:

$$A^2 = \frac{1}{4K^2}(k_1\Delta - k_2)^2 \quad \text{and} \quad B^2 = \frac{1}{4K^2}(k_1\Delta + k_2)^2.$$

We observe that $\Delta = 4K(B^2 - A^2)$. The case $A = 0$ gives $k_1\Delta - k_2 = 0$, that is, $p = -\mu$ (de Sitter metrics), and $B = 0$ is the geodesic case which corresponds to the Einstein universes ($\Delta \neq 0$). To recover the Schwarzschild interior solution we have to take $K > 0$, $AB \neq 0$ in metric (26) and match it with the Schwarzschild exterior metric g_e across a timelike 3–surface Σ. The metric g_e writes in curvature coordinates as:

$$g_e = -\left(1 - \frac{2m}{\rho}\right) d\tau \otimes d\tau + \frac{1}{1 - \frac{2m}{\rho}} d\rho \otimes d\rho + \rho^2 h. \tag{27}$$

and Σ is taken as $\rho = \rho_0 > 2m$. The continuity of the metric potential $g_{\rho\rho}$ across Σ is equivalent to take $K = 2m/\rho_0^3$. The continuity of the timelike component $g_{\tau\tau}$ and its first derivative $\partial_\rho g_{\tau\tau}$ across Σ fixes (up to a sign $\epsilon = \pm 1$) the integration constants: $A = (3\epsilon/2)\sqrt{1 - K\rho_0^2}$ and $B = -\epsilon/2$. Under the above requirements, the constant Δ is expressed as:

$$\Delta = \frac{4m}{\rho_0^3}\left(\frac{9m}{\rho_0} - 4\right), \tag{28}$$

which allows us to give an interpretation of this parameter:

For the Schwarzschild interior solution, the condition $\Delta < 0$ *is equivalent to* $m/\rho_0 < 4/9$, *which gives the mass/radius limit ratio for the stability of a spherically symmetric static configuration of uniform density.*

As it is well known, the derivative $\partial_\rho g_{\rho\rho}$ presents a discontinuity across Σ and the coordinate ρ is not an admissible coordinate in the sense of Darmois and Lichnerowicz. Several types of admissible coordinate systems for this and other matching problems has been analyzed by different authors (see, for instance, [1, 15–18]). From these studies, when an expression $r(\rho)$ as a radial admissible coordinate is provided, it could be used to express our relation (28) in terms of $r_0 = r(\rho_0)$.

Note that the particular situation with $A^2 = B^2$ corresponds to $\Delta = 0$; the case $A = B$ has been interpreted in [19] as a model of universe and, in [20] an interior solution with $A = -B$ has been matched with the exterior Schwarzschild solution in an harmonic coordinate system.

On the other hand, we can also comment on solutions with $K = 0$ for which the metric is expressed as:

$$g = -(A\rho^2 + B)^2 d\tau \otimes d\tau + d\rho \otimes d\rho + \rho^2 h, \tag{29}$$

A and B being two constants given by:

1) If $k_1 = 0,\ k_2 \neq 0$ then $A = \frac{k_2}{4}$ and $B = \frac{-\Delta}{4k_2}$.
2) If $k_2 = 0,\ k_1 \neq 0$ then $A = \frac{-\Delta k_1}{4}$ and $B = \frac{1}{4k_1}$.

Here, if $\Delta = 0$, the second case reduces to Minkowski space–time while the first case does not. Moreover, the pressure of a spherically symmetric SIST can be decomposed using these metrics. In fact, expression (19) (with $f = \nu$) is written as $p(\omega) = -2K + p_1(\omega) + p_2(\omega)$, where $p_1(\omega)$ is the pressure corresponding to the case $k_1 = 0,\ k_2 \neq 0$ and $p_2(\omega)$ the corresponding to the case $k_1 \neq 0,\ k_2 = 0$.

Finally, it should be remarked that the simplest Schwarzschild interior generalizations are the so called *generalized Schwarzschild interiors* which are the conformally flat space–times whose energy content is a non–expanding perfect fluid [21]. In addition, the fluid is shear–free and vorticity–free due to the Bianchi identities and the nullity of the Weyl tensor. The nullity of the expansion implies that the energy density is a constant that also gives the curvature of the spacelike sections orthogonal to the fluid flow. In general, this flow is non geodesic, according to the pressure inhomogeneities. This family of solutions is obtained and presented in a non–conformally flat form, using adapted coordinates to the fluid flow, althought the Weyl tensor nullity plays an essential role in its determination [21]. In this paper we have presented a way to obtain the conformal factor of the (static) SIST. Its possible extension to cover the *generalized Schwarzschild interiors* is an essential question that remains open.

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