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Flat synchronizations in spherically symmetric space-times

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Abstract. It is well known that the Schwarzschild space-time admits a spacelike slicing by flat instants and that the metric is regular at the horizon in the associated adapted coordinates (Painlevé-Gullstrand metric form). We consider this type of flat slicings in an arbitrary spherically symmetric space-time. The condition ensuring its existence is analyzed, and then, we prove that, for any spherically symmetric flat slicing, the densities of the Weinberg momenta vanish. Finally, we deduce the Schwarzschild solution in the extended Painlevé-Gullstrand-Lemaître metric form by considering the coordinate decomposition of the vacuum Einstein equations with respect to a flat spacelike slicing.

1. Introduction

The Schwarzschild solution was extended inside its horizon by Painlevé [1], Gullstrand [2] and Lemaître [3], who used a non-orthogonal curvature coordinate system $\{t, r, \theta, \varphi\}$ in which the metric takes the form

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + 2\sqrt{\frac{2m}{r}}\, dt\, dr + dr^2 + r^2 d\Omega^2 \tag{1}$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$. In these Painlevé-Gullstrand (PG) coordinates, the metric is non diagonal, asymptotically flat, and regular at the horizon. So, the space-time appears foliated by a synchronization of flat instants.

This kind of foliations is being studied with renewed interest nowadays due to its applications in both, relativistic and non-relativistic, situations (see, for example, [4, 5, 6]).

In general, for a spherically symmetric space-time (SSST), the existence of flat synchronizations is usually taken for granted but, recently, an obstruction to it has been pointed out [7]. However, as far as the authors are aware, a definitive interpretation of this obstruction as well as the analysis of the domains where a flat synchronization exists have not been done up to now. The question is: does every SSST admit a region of physical interest where a synchronization by flat instants exists?

Here, in Section 2, we consider flat synchronizations in SSST analyzing their existence. In Section 3 we show that the energy and momenta densities of a flat slice vanish for any SSST. Next, Section 4 is devoted to recovering the Painlevé-Gullstrand-Lemaître (PGL) extension of the Schwarzschild solution by solving the vacuum Einstein equations for a SSST in PG coordinates.

2. Existence of flat synchronizations in SSST

Let us consider a SSST, whose metric can be written in the form [8, 9]

$$ds^2 = A(T,R)dT^2 + B(T,R)dR^2 + 2C(T,R)dTdR + D(T,R)d\Omega^2$$

being $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$, $D(T,R) \neq 0$ an arbitrary function, and $A(T,R)$, $B(T,R)$ and $C(T,R)$ arbitrary functions satisfying the condition $\delta \equiv AB - C^2 < 0$ in order to guarantee the Lorentzian character of the metric.

In general, these coordinates are not PG coordinates. In fact, if we consider the induced metric γ on the 3-surfaces $T = constant$,

$$\gamma = B(T,R)dR \otimes dR + D(T,R)\sigma$$

($\sigma \equiv d\theta \otimes d\theta + \sin^2\theta\, d\varphi \otimes d\varphi$ being the metric on the unit 2-sphere) then we can calculate its Ricci tensor,

$$\mathcal{R}ic(\gamma) = \left(\frac{B}{2}\mathcal{R} - \frac{B}{D}F\right) dR \otimes dR + \left(\frac{D}{4}\mathcal{R} + \frac{F}{2}\right)\sigma$$

and its scalar curvature,

$$\mathcal{R} \equiv \mathcal{R}(\gamma) \quad = \quad \begin{cases} \dfrac{2F}{D} + \dfrac{4\partial_R F}{\partial_R D} & \qquad \partial_R D \neq 0 \\ \dfrac{2}{D} & \qquad \partial_R D = 0 \end{cases}$$

where

$$F = 1 - \frac{(\partial_R D)^2}{4BD}.$$

So, we have that γ is a flat metric ($\mathcal{R}ic(\gamma) = 0$) if and only if $F = 0$, that is, if and only if $(\partial_R D)^2 = 4BD$.

Consequently, in the general case, we can look for a coordinate transformation of the form $T(t,r)$ and $R(t,r)$, which leads to write the metric of the SSST in PG coordinates. This transformation will allow to put the metric in the form

$$ds^2 = \xi^2\, dt^2 + \chi^2 dr^2 + 2\xi \cdot \chi\, dt\, dr + \mathcal{D}(t,r)d\Omega^2 \tag{2}$$

with $\mathcal{D}(t,r) \equiv D(T(t,r), R(t,r))$, and the vector fields ξ and χ defined as

$$\xi \equiv \dot{T}\frac{\partial}{\partial T} + \dot{R}\frac{\partial}{\partial R}, \qquad \chi \equiv T'\frac{\partial}{\partial T} + R'\frac{\partial}{\partial R}$$

where $J \equiv \dot{T}R' - T'\dot{R} \neq 0$, in order to assure coordinate regularity (over-dot and prime stand for partial derivative with respect to t and r, respectively).

Given that the induced metric on the 3-surfaces $t = constant$ is flat if and only if

$$F = 1 - \frac{\mathcal{D}'^2}{4\xi^2\mathcal{D}} = 0 \;\Leftrightarrow\; 4\mathcal{D}\xi^2 = \mathcal{D}'^2 \;\Leftrightarrow\; (d\sqrt{\mathcal{D}})^2 \leq 1,$$

and r is a coordinate of curvature for the spherically symmetric form (2) if $\mathcal{D}(t,r) = r^2$, then we have that

$$\chi^2 = 1, \quad \mathcal{A} \equiv \xi^2 = J^2\delta(dr)^2, \quad \xi \cdot \chi = \pm J\sqrt{\delta[(dr)^2 - 1]}$$

So, the real function $\mathcal{B}(t,r) \equiv \xi \cdot \chi$ exists if $(dr)^2 \leq 1$ and in this case the metric results

$$ds^2 = \mathcal{A}(t,r)dt^2 + 2\mathcal{B}(t,r)dt\ dr + dr^2 + r^2 d\Omega^2$$

Then, we have proved the following result.

▷ *Let r be the radius of curvature of the orbits (2-spheres) of the isometry group of a spherically symmetric space-time with metric g. In the region defined by the condition*

$$(dr)^2 \equiv g^{\mu\nu}\partial_\mu r \partial_\nu r \leq 1 \tag{3}$$

a curvature coordinate system $\{t, r, \theta, \varphi\}$ exists in which the metric line element may be written as

$$ds^2 = \mathcal{A}(t,r)dt^2 + 2\mathcal{B}(t,r)dt\ dr + dr^2 + r^2 d\Omega^2 \tag{4}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$.

The above condition $(dr)^2 \leq 1$ is equivalent to say that the Misner-Sharp gravitational energy of a sphere of radius r is nonnegative [10].

3. Energy and momenta densities of a flat slice in SSST

In this section we consider a SSST in PG coordinates and write the metric (4) in the form $g = \eta + h$, with η the Minkowski metric and $h_{00} = 1 + \mathcal{A}$, $h_{0i} = \mathcal{B}x_i/r$, and $h_{ij} = 0$.

Moreover, we have that the Weinberg pseudotensor is defined by [11]

$$2Q^{i0\lambda} = \frac{\partial h^\mu_\mu}{\partial x_0}\eta^{i\lambda} - \frac{\partial h^\mu_\mu}{\partial x_i}\eta^{0\lambda} - \frac{\partial h^{\mu 0}}{\partial x^\mu}\eta^{i\lambda} + \frac{\partial h^{\mu i}}{\partial x^\mu}\eta^{0\lambda} + \frac{\partial h^{0\lambda}}{\partial x_i} - \frac{\partial h^{i\lambda}}{\partial x_0},$$

where Latin and Greek indexes go from 1 to 3 and from 0 to 3, respectively, and all indexes are raised and lowered with the flat metric η.

So, the Weinberg pseudotensor for a SSST in PG coordinates results

$$Q^{i00} = 0, \qquad 2Q^{i0j} = \left(\frac{\mathcal{B}}{r} + \mathcal{B}'\right)\delta_{ij} + \left(\frac{\mathcal{B}}{r} - \mathcal{B}'\right)\frac{x_i}{r}\frac{x_j}{r}.$$

By differentiating and contracting indexes in this expression we directly obtain

$$\tau^{0\lambda} \equiv -\frac{1}{8\pi G}\frac{\partial Q^{i0\lambda}}{\partial x^i} = 0$$

and then, the angular momentum densities, $\mathrm{j}^{i\lambda} = x^i\tau^{0\lambda} - x^\lambda\tau^{0i}$ also vanish.

So, we have proved the following result.

▷ *In any spherically symmetric space-time, the energy and momenta densities vanish for every Painlevé-Gullstrand synchronization.*

4. Painlevé-Gullstrand-Lemaître extension of the Schwarzschild solution

In this section we recover the Schwarzschild solution, including the horizon and its interior region. For that, we consider a SSST in PG coordinates whose metric expression is given by (4) and write the vacuum Einstein equations for this synchronization. According to the standard 3+1 splitting of the Einstein equations (see for example [12]), in this case the constraint equations are

$$\Phi(\Phi + 2\Psi) = 0, \qquad r\Phi' + \Phi - \Psi = 0 \tag{5}$$

and the evolution equations result

$$\dot{\Psi} = \Psi(2\Phi - \Psi) + \frac{1}{\mathcal{B}}(\mathcal{B}^2\Psi)', \qquad \dot{\Phi} = \Psi\Phi + \frac{\mathcal{B}}{r^2}(r^2\Phi)' \tag{6}$$

where

$$\Phi = \frac{1}{\alpha}\frac{\mathcal{B}}{r}, \qquad \Psi = \frac{\mathcal{B}'}{\alpha} \tag{7}$$

being $\alpha^2 = \mathcal{B}^2 - \mathcal{A}$ the lapse function.

The first constraint equation in (5) implies that, for $\Phi \neq 0$, $\Phi = -2\Psi$ and then, from the expressions in (7), we have that

$$\mathcal{B} = f(t) r^{-1/2}$$

with $f(t)$ and arbitrary function. Substituting this expression in the other constraint equation (5), it results that the α is a function of the sole variable t. So, by re-scaling the t coordinate parameter, we can take $\alpha = 1$. Then, we have that

$$\Phi = f(t) r^{-3/2} = -2\Psi$$

and taking into account the evolution equations (6) one has that $f(t)$ must be a constant function. So, if we take $f(t) = \sqrt{2m}$ we have that the metric can be written as in (1). Then, we have recovered the Schwarzschild metric and now the r coordinate can take any positive value, $0 < r < \infty$. In fact, from (1), we have $(dr)^2 = g^{rr} = 1 - \frac{2m}{r} < 1$, and the domain of the PG coordinates extends for every value of $r \neq 0$.

In the case $\Phi = 0$ the resolution of the equations (5) and (6) lead to the Minkowski space-time.

5. Conclusions

Summarizing, in this work we have interpreted the condition allowing the existence of a flat synchronization in any SSST in terms of the curvature radius of the 2-spheres, see Eq. (3). We have also shown that, for these flat slices, the energy and momenta densities vanish. And, finally, we have recovered the Painlevé-Gullstrand-Lemaître extension of the Schwarzschild solution by solving the vacuum Einstein equations for a SSST in a coordinate system adapted to a flat synchronization.

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