Lorentzian Comments on Stokes Parameters

Juan Antonio Morales and Emilio Navarro

Departament d'Astronomia i Astrofísica, Universitat de València
E-46100 Burjassot, (València), Spain

Abstract. The popular Stokes statements about polarized light are interpreted in a Minkowskian language using a Lorentzian representation for the Stokes parameters and the degree of polarization. The evolution equations for Stokes parameters on a curved space–time are obtained using the parallel transport of the polarization vector along a null geodesic. The interest of these equations in Astrophysics and Relativistic Cosmology is outlined.

1 Introduction

Stokes parameters [1] are a useful tool to describe polarized electromagnetic radiation. They contain exhaustive information about the degree of polarization (total, linear and circular), angle of polarization and ellipticity pattern. On a given space–time geometry, variations of such quantities are related with the way of transporting these parameters along light beams. Hence, the use of the Stokes parameters is of interest in Relativistic Astrophysics to study the transfer of polarized electromagnetic radiation, and also in Cosmology, dealing with the free propagation of polarized microwave background radiation on a perturbed Friedmann–Robertson–Walker universe. The hope of observing the associated cosmological polarization pattern is today an open prospect.

For practical purposes, these parameters are operationally defined for a quasi-monochromatic plane wave whose amplitude and phase are slowly varying functions at the scale of the coherence time [1], [2]. It is worth remembering that the Stokes parameters are both observer-dependent and basis-dependent quantities. Their definition involves the components of the electric field relative to a given observer and are referred to an orthonormal basis on the spacelike 2-plane perpendicular to the direction of propagation measured by the observer. According to conventional notation, they may be arranged in a four-element vector S, termed *Stokes vector*, in the following way

$$S = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} I \\ PI\cos 2\chi \cos 2\Psi \\ PI\cos 2\chi \sin 2\Psi \\ PI\sin 2\chi \end{pmatrix} \tag{1}$$

$I > 0$ being the light intensity and P the (total) degree of polarization,

$$0 \leq P = \frac{\sqrt{Q^2 + U^2 + V^2}}{I} \leq 1 \tag{2}$$

The *ellipticity* angle $\chi \in [-\pi/4, \pi/4]$ determines the eccentricity of the polarization ellipse, $\tan\chi = \pm b/a$, b and a denoting, respectively, the lengths of its minor and major semiaxes. The different helicity states of polarization, left-handed and right-handed, are associated respectively with the positive and negative values of χ. When $\chi = 0$ the polarization is said linear, and $\chi = \pm\pi/4$ correspond to circular polarization. The *polarization* angle $\Psi \in [0, \pi)$ measures the inclination of the major axis with respect to a given spacelike direction in the rest space of an observer.

The possibility of using a Lorentzian metric to describe light polarization was pointed out by Soleillet [3] but, to our knowledge, he did not develop this idea any further. Later, Perrin employed a Minkowskian four-dimensional space to analyze the algebraic structure of some scattering matrices [4] and, more recently, several authors have considered the Soleillet–Mueller matrices and some relevant generalizations from this point of view, [5] [6], [7], [8]. However, many other aspects of the Soleillet idea remain unexplored, and its development could provide a way of translating the Minkowskian language from relativistic physics to polarization phenomena. For instance, some properties about generalized Soleillet–Mueller matrices given in [7] can also be understood in terms of the algebraic classification of a symmetric 2–tensor in a Lorentzian four–dimensional metric and, in particular, in terms of the energy conditions on the matter tensor given by Plebański some time ago [9].

This contribution to the 24th edition of the Spanish relativistic meeting, E.R.E–2001, is organized as follows. Section 2 introduces the Stokes space which can be seen as a four-dimensional time-oriented Lorentzian vector space, using the familiar relativistic terminology. The elements of this space are called Stokes vectors. In this framework a distinguished timelike future-pointing vector is associated with ordinary unpolarized or natural light. Partially polarized light is represented by any other timelike vector of the same time orientation than natural light. Its degree of polarization is related with the hyperbolic angle between both timelike directions. In this picture, a future-pointing null direction represents totally polarized radiation. These null vectors generate the Stokes null cone. In Sect. 3 we analyze the (general form of the) transfer equations for Stokes distribution functions in a curved space–time. In the context of the geometrical optics approximation, the transport equations for freely propagating radiation proposed by Dautcourt and Rosen [10] and Bildhauer [11], [12] are recovered. Finally, in Sect. 4, we comment on the physical interest of the Lorentzian approach to polarization phenomena and transport equations.

2 The Stokes space

From (2), one has $I^2 \geq Q^2 + U^2 + V^2$, and the equality takes place for totally polarized light, $P = 1$. The above relation may be interpreted in a *Lorentzian* terminology considering the *Stokes space*, that is, the set of points

$$\mathbf{S} = \{(I, Q, U, V),\ I > 0\} \subset \mathbb{R}^4$$

endowed with a Lorentzian metric. Let $S = (I, Q, U, V)$ and $S' = (I', Q', U', V')$ be two *Stokes vectors*, that is $S, S' \in \mathbf{S}$, then their scalar product is given by

$$(S, S') = II' - QQ' - UU' - VV'$$

Completely polarized lights are represented by null vectors S, $(S, S) = 0$, and they generate the *Stokes cone*. A completely unpolarized or *natural* light of intensity I is represented by a privileged positive vector $S_n = (I, 0, 0, 0)$ having $P = 0$. Any other positive Stokes vector $S \neq S_n$, $(S, S) > 0$ (i.e. pointing into the Stokes cone) represents a *partially* polarized light with $0 < P < 1$.

Next, we consider the Lorentzian interpretation of the degree of polarization P. Let u be the unitary positive Stokes vector representing natural light of unit intensity, $(u, u) = 1$. Then, the intensity of a light represented by the Stokes vector S is defined as the scalar product of u and S,

$$I = (u, S) \tag{3}$$

We have the following decomposition

$$S = (I - I_p)u + l \tag{4}$$

where l is a totally polarized light (null vector) whose intensity is given by $I_p = (u, l)$. From (4), the scalar product gives

$$(S, S) = (I - I_p)^2 + 2(I - I_p)I_p = I^2 - I_p^2$$

and from (2), the degree of polarization is given by the ratio of the intensity of the partially polarized component I_p to the total intensity I,

$$P = \frac{I_p}{I} = \sqrt{1 - \frac{(S, S)}{(u, S)^2}} \tag{5}$$

Using the familiar relativistic notation,

$$s = \frac{S}{\sqrt{(S, S)}} = \gamma(1, \boldsymbol{\beta}), \qquad \gamma = (u, s), \qquad \boldsymbol{\beta} = (\mathrm{q}, \mathrm{u}, \mathrm{v}) \tag{6}$$

where $\mathrm{q} \equiv Q/I$, $\mathrm{u} \equiv U/I$ and $\mathrm{v} \equiv V/I$ are the *normalized* Stokes parameters, the expression (5) is written as

$$P = \sqrt{1 - \frac{1}{\gamma^2}} = \beta \equiv |\boldsymbol{\beta}| \tag{7}$$

which provides a new interpretation of the degree of polarization.

Proposition 1: *In the Lorentzian representation for polarized radiation, the degree of polarization P is kinematically interpreted as a "relative velocity" between the (unitary) Stokes vectors, u and s, respectively associated with natural and partially polarized lights.*

In particular, the same interpretation can be made for the linear and circular degrees of polarization which are defined respectively as

$$\beta_l = \sqrt{\mathrm{q}^2 + \mathrm{u}^2} = \beta \cos 2\chi, \qquad \beta_c = \mathrm{v} = \beta \sin 2\chi \tag{8}$$

β being the total degree of polarization previously considered, $\beta = \sqrt{\beta_l^2 + \beta_c^2}$.

Also, from (4) and writing $u = (l + k)/(2I_p)$, every positive vector can be decomposed according to the following expression

$$S = \frac{1}{2\beta}\big[(1+\beta)\, l + (1-\beta)\, k\big] \tag{9}$$

where both k and l are null vectors with opposite projections in the 3-space orthogonal to u. The physical meaning of (9) is clear because it reflects the well known equivalence between a light beam having intensity I and polarization degree β, and two incoherent streams of elliptically polarized light having intensities $I(1+\beta)/2$ and $I(1-\beta)/2$ in the states of opposite polarizations (χ, Ψ) and $(-\chi, \Psi + (\pi/2))$. In particular, it is also suitable for the decomposition of natural light, of intensity I, in two incoherent oppositely polarized waves with the same intensity $I/2$. These waves can be linearly polarized and mutually perpendicular, or circularly polarized with opposite helicities, one right-handed and the other left-handed (cf. [1], [2]).

Also, the usual matrix representation of optical devices as polarizers and retarders (Soleillet–Mueller matrices) has a Lorentzian meaning. Up to an overall factor, they can be seen as elements of the proper orthochronous subgroup of the Lorentz group acting on the Stokes space. Matrices representing polarizers are homothetic to ordinary boosts, and retarders are represented as Euclidean rotations, cf. [7], [13], [14], [15], [16].

3 Propagation of polarized radiation

Next, let us consider a one-parameter family of Stokes vectors $S(\lambda)$ that, from (1) and (8), can be written as

$$S(\lambda) = \begin{pmatrix} I \\ I\beta_l \cos 2\Psi \\ I\beta_l \sin 2\Psi \\ I\beta_c \end{pmatrix} \tag{10}$$

where every quantity in this expression depends on the real parameter λ, that is, $I(\lambda)$, $\beta_l(\lambda)$, $\beta_c(\lambda)$ and $\Psi(\lambda)$. From (8), denoting with a prime the derivation with respect to λ, we obtain

$$\beta_l' = (\ln \beta)'\, \beta_l - 2\chi' \beta_c \tag{11}$$

$$\beta_c' = 2\chi' \beta_l + (\ln \beta)'\, \beta_c \tag{12}$$

Moreover, from (1) the polarization angle is given by

$$\tan 2\Psi = \frac{U}{Q} = \frac{\mathrm{u}}{\mathrm{q}} \tag{13}$$

and its derivative has the expression,

$$\Psi^{'} = \frac{\mathrm{u}'\mathrm{q} - \mathrm{q}'\mathrm{u}}{2(\mathrm{u}^2 + \mathrm{q}^2)} \tag{14}$$

So, we obtain the following relations that give the variations of the *normalized* Stokes parameters with respect the parameter λ

$$\mathrm{q}' = -2\Psi^{'}\mathrm{u} + \beta_l^{'} \cos 2\Psi \tag{15}$$

$$\mathrm{u}' = 2\Psi^{'}\mathrm{q} + \beta_l^{'} \sin 2\Psi \tag{16}$$

$$\mathrm{v}' = \beta_c^{'} \tag{17}$$

Note that $(\mathrm{q}(\lambda), \mathrm{u}(\lambda), \mathrm{v}(\lambda))$ is a parametrized curve in the domain bounded by the *Poincaré sphere*, which is also an extended and well known representation for light polarization [1]. So, (15), (16) and (17) give the velocity of a motion across this domain which is represented by a smooth path or sequence of polarized states. This equations provide the transport of Stokes parameters along a space–time curve $x(\lambda)$ under the fair hypothesis that these parameters vary smoothly along the curve, $S(x(\lambda))$. In physical applications only causal curves (light rays and observers) will be relevant.

In the kinetic theory of a relativistic gas on a given space–time geometry [17], the relevant quantity is the particle distribution function f. For an unpolarized photon gas, the specific intensity I (at a given light frequency ν) and the photon distribution function f are related by $I = \nu^3 f$. In the polarized case, the Stokes parameters $S_A (A = I, Q, U, V)$ are defined for quasi-monochromatic light and can also be seen as specific intensities for the given frequency. As a conventional extension for polarized radiation, the distribution function f_A associated with S_A, can be defined as $f_A = S_A/\nu^3$ (see, for instance, [18], [19]). Units are taken so that $h = c = 1$. When unpolarized radiation is freely propagating, the photon distribution function f is constant along each null geodesic, that is, it must satisfy the collisionless Liouville equation $\mathcal{L}f = 0$. The effect of the space–time geometry is involved in the *Liouville operator* $\mathcal{L}$ that represents a *total* derivative along the light trajectory $x(\lambda)$. This operator acts on the photon distribution function f as defined on the phase space for massless particles, i.e. $f\big(x(\lambda), k(\lambda)\big)$, where $k(\lambda) = dx(\lambda)/d\lambda$ is the photon 4-momentum, and $k^2 = 0$ in the given space–time metric, due to the photon masslessness. When radiation is partially polarized the distribution functions $f_A\big(x(\lambda), k(\lambda)\big)$ associated with the Stokes parameters contain exhaustive information about polarization. In the unfreely propagating case, a source polarization term $\mathcal{J}$ enters in the transfer equation for the total intensity, as expressed by the corresponding Boltzmann-type equation $\mathcal{L}f_I = \mathcal{J}$. The particular form of this term depends on the physical process under consideration (Thomson scattering, synchrotron radiation, bremsstrauhlung, etc.).

Now, we suppose that the Stokes parameters may vary smoothly along the photons paths $x(\lambda)$, so that we have a one-parameter family of Stokes vectors $S(x(\lambda))$ on each path. Then (15), (16) and (17) are also suitable for this situation, whether the total derivative of the normalized Stokes parameters with respect to λ is replaced by the Liouville derivation of the associated specific distribution functions. Therefore, the actual form of these equations is

$$\mathcal{L}\Big(\frac{f_Q}{f_I}\Big) = -2\Psi' \frac{f_U}{f_I} + \beta_l' \cos 2\Psi \tag{18}$$

$$\mathcal{L}\Big(\frac{f_U}{f_I}\Big) = 2\Psi' \frac{f_Q}{f_I} + \beta_l' \sin 2\Psi \tag{19}$$

$$\mathcal{L}\Big(\frac{f_V}{f_I}\Big) = \beta_c' \tag{20}$$

Hence, we have the following *transfer equations* for polarized radiation

$$\mathcal{L} f_I = \mathcal{J} \tag{21}$$

$$\mathcal{L} f_Q = \frac{f_Q}{f_I}\mathcal{J} - 2\Psi' f_U + \beta_l' f_I \cos 2\Psi \tag{22}$$

$$\mathcal{L} f_U = \frac{f_U}{f_I}\mathcal{J} + 2\Psi' f_Q + \beta_l' f_I \sin 2\Psi \tag{23}$$

$$\mathcal{L} f_V = \frac{f_V}{f_I}\mathcal{J} + \beta_c' f_I . \tag{24}$$

For freely propagating photons the linear and circular polarization degrees are assumed constant along each null geodesic. Specifically, from the above transfer equations we arrive at the following result.

Proposition 2: *In the free propagation of electromagnetic radiation, the necessary and sufficient condition for the degrees of polarization to be constant along each light ray,* $\beta_l' = \beta_c' = 0$, *is that the Stokes parameters distribution functions satisfy the transport equations*

$$\mathcal{L} f_I = 0 \tag{25}$$

$$\mathcal{L} f_Q = -2\Psi' f_U \tag{26}$$

$$\mathcal{L} f_U = 2\Psi' f_Q \tag{27}$$

$$\mathcal{L} f_V = 0 \tag{28}$$

where the prime denotes total derivation along the light ray.

Transport equations describing the free propagation of polarized electromagnetic radiation in any space–time and in the geometrical optics approximation were anticipated by Dautcourt and Rosen [10] and deduced by Bildhauer [11],

[12] using Wigner distribution theory. From the above proposition, we can recover the equations proposed by these authors when the transport of the polarization angle is conveniently referred to an orthonormal tetrad $\{e_A\}_{A=0}^3$ as will be seen at once.

Let us consider a null geodesic with tangent vector k. Let us take $e_o = u$ an arbitrary space–time observer and $e_3 = n$ the instantaneous spacelike unitary vector along the light trajectory as measured by u. So, $n = k/\nu - u$, where $\nu = u \cdot k$ is the observed light frequency. The space–time signature is taken $(+ - --)$. The unitary vector e *(polarization vector)* along the major axis of the polarization ellipse always stays on the 2-plane orthogonal to u and k; it is determined up to a multiple of the null vector k and may be chosen

$$e = \cos\Psi e_1 + \sin\Psi e_2$$

Now the covariant derivative of e along k is given by

$$\nabla_k e = (\nabla_k e_1 + \Psi^{'} e_2)\cos\Psi + (\nabla_k e_2 - \Psi^{'} e_1)\sin\Psi$$

Contracting the above expression with e_1 and e_2 it results

$$e_1 \cdot \nabla_k e = (e_1 \cdot \nabla_k e_2 + \Psi^{'})\sin\Psi$$

$$e_2 \cdot \nabla_k e = (e_2 \cdot \nabla_k e_1 - \Psi^{'})\cos\Psi$$

Then, according to [20], the polarization vector e is *quasi-parallel* transported along the light beam, that is, the field $\nabla_k e$ belongs to the (distribution of timelike) 2-planes expanded by u and k if, and only if, the polarization angle varies as

$$\Psi^{'} = -e_1 \cdot \nabla_k e_2 = e_2 \cdot \nabla_k e_1 \tag{29}$$

where the second equality also follows from the relation $e_1 \cdot e_2 = 0$. This condition is satisfied in the geometrical optics approximation where, in particular, the parallel transport of the polarization vector e along any light ray occurs, that is $\nabla_k e = 0$. On the other hand, the covariant derivatives of the tetrad fields are given by

$$\nabla_{e_A} e_B = \Gamma^C_{BA}\, e_C$$

Γ^C_{BA} being the connection coefficients. Now, from (29) taking into account that $k = \nu(u + n) = \nu(e_0 + e_3)$, the variation of the polarization angle may be expressed as

$$\Psi' = \nu(\Gamma^1_{20} + \Gamma^1_{23}) = \nu(\Gamma^2_{10} + \Gamma^2_{30}). \tag{30}$$

The transport equations are expressed in the form obtained by Bildhauer [11] when (30) is replaced in Proposition 2. Note the very simplicity of the arguments we have used to obtain the general form for the transfer equations (21)–(24). Starting from (25)–(28) and assuming the relation $f_A = S_A/\nu^3$, the transfer equations immediately follows.

4 Comments and discussion

We have considered the unusual Lorentzian interpretation of the Stokes vectors and the degree of polarization (see (5) and Proposition 1). In this framework the popular Stokes statement about lights [1], [2]: *"Any partially polarized light may be regarded as the incoherent mixture of an unpolarized light and a completely polarized one"*, has a direct geometric meaning (see (4)) . It comes as a consequence of the fact that any vector inside the positive shell of the null cone of a Lorentzian structure (positive oriented vector) may be decomposed as the sum of another positive oriented vector and a null vector with the same orientation (positive intensity). In this sense, (9) refers to the incoherent decomposition of a partially polarized light as two totally polarized lights with opposite polarizations. Moreover, the sum of positive and null vectors of the same orientation always is a positive vector with the given orientation. This property reflects the fact that any incoherent mixture of polarized beams of light may be represented by a sole positive oriented Stokes vector.

On the other hand, the general form of transfer equations for polarized radiation in curved space–times has been obtained considering the variation of the Stokes parameters along a null curve, (21)–(24). For a given physical situation (i.e. a specification of how the source term $\mathcal{J}$ depends on the distribution functions f_A) each of the solutions of these equations provide an initial condition for the general problem of free-propagating radiation in a curved background. This propagation is governed by (25)–(28) when the linear and circular degrees of polarization are constant along each ray. When the variation of the polarization angle is referred to an orthonormal tetrad, the transport equations are put in Bildhauer's form. In the geometrical optics approximation, the polarization vector is parallely propagated along each ray. However, the rotation of the polarization vector with respect to a screen expanded by the tetrad fields e_1 and e_2 must be taken into account whenever these fields are not parallel transported along the ray. These provide a sort of frame-dependent effect (which would be called *kinematic Faraday rotation*) that needs to be controlled.

The above comment may be important in Relativistic Cosmology, in relation to the challenge of measuring the polarization of the cosmic microwave background radiation (CMB). At the last scattering surface, the solutions of the Boltzmann equation (with a Thomson collisional term) provide the initial conditions for free propagating radiation in a perturbed Friedmann universe. The election of an appropriate background geometry, since decoupling until now, and the study of the transport equations in this geometry, are essential questions to be analyzed in connection with the theoretical prediction of a polarized CMB.

Also, in astrophysical scenarios where a general relativistic treatment would be necessary (propagation of polarized radiation through a magnetized plasma [20], polarization from accretion disks near compact objects [21], etc.) the detailed analysis of the transport equations describing the change of polarization along a null geodesic in a given space–time ought to have been considered. In a pioneering work, Plebański [22] investigated the rotation of the plane of polarization in the gravitational field of an isolated system, considering the linearized approximation. The gravitational Faraday rotation has been investigated in the Kerr geometry from the parallel transport of the polarization vector along its null geodesics [21], [23] and also, considering the expression of the terms of the Bildhauer's transport equations for this metric [12]. Because these equations follow from the transport equations of Proposition 2, when the parallel transport of the polarization vector is taken into account, both points of view seem to be consistent.

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Added note.[1]

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