Coordinates and frames from the causal point of view

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Abstract. Lorentzian frames may belong to one of the 199 causal classes. Of these numerous causal classes, people are essentially aware only of two of them. Nevertheless, other causal classes are present in some well-known solutions, or present a strong interest in the physical construction of coordinate systems. Here we show the unusual causal classes to which belong so familiar coordinate systems as those of Lemaître, those of Eddington-Finkelstein, or those of Bondi-Sachs. Also the causal classes associated to the Coll light coordinates (four congruences of real geodetic null lines) and to the Coll positioning systems (light signals broadcasted by four clocks) are analyzed. The role that these results play in the comprehension and classification of relativistic coordinate systems is emphasized.

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The main purpose of this short communication is to gain stimulus in the investigation of relativistic coordinate systems from the causal point of view. This issue might be relevant in several situations. For example, to investigate the coordinates which are appropriated to deal with evolution problems throughout horizons. Current $3+1$ numerical codes in relativistic hydrodynamics are being implemented using such a coordinates. Also, to consider admissible *cuts* of the space-time others than the very usual "space $\oplus$ time" decomposition. The causal classification of frames may help us to better understand other aspects of the space-time, for instance the "light $\oplus$ light $\oplus$ light $\oplus$ light" decomposition.

Firstly, we remember the causal classification of Lorentzian frames (also presented at the ERE-88 celebrated in Salamanca). Then, we give several examples of causal classes of relativistic coordinates: those associated with the coordinates introduced by Lemaître, by Bondi and Sachs, and by Coll.

In dimension $n=4$, the causal class of a frame $\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4\}$ is defined by a set of 14 characters:

$$\{\mathrm{c}_1\mathrm{c}_2\mathrm{c}_3\mathrm{c}_4, \mathrm{C}_{12}\mathrm{C}_{13}\mathrm{C}_{14}\mathrm{C}_{23}\mathrm{C}_{24}\mathrm{C}_{34}, c_1c_2c_3c_4\}$$

c_i being the causal character of the vector v_i, C_ij ($\mathrm{i} \neq \mathrm{j}$) being the causal character of the adjoint 2-plane $\{\mathrm{v}_\mathrm{i}\mathrm{v}_\mathrm{j}\}$, and c_i being the causal character of the covectors of the dual coframe $\{\theta^\mathrm{i}\}$, $\theta^\mathrm{i}(\mathrm{v}_\mathrm{j}) = \delta^\mathrm{i}_\mathrm{j}$. The covector θ^i is time-like (resp. space-like) iff the 3-plane generated by $\{\mathrm{v}_\mathrm{j}\}_{\mathrm{j}\neq\mathrm{i}}$ is space-like (resp. space-like). This applies for both the Newtonian and the Lorentzian causal structures. In addition, for the later, the covector θ^i is light-like iff the 3-plane generated by $\{\mathrm{v}_\mathrm{j}\}_{\mathrm{j}\neq\mathrm{i}}$ is light-like. Elsewhere (see [1] and [2]) we have presented the following result:

Theorem: *i) In the 4-dimensional Newtonian space-time there exist 4, and only 4, causal classes of frames, and ii) In the 4-dimensional relativistic space-time there exist 199, and only 199, causal classes of Lorentzian frames.*

This result provides the causal classification of coordinate systems in Newtonian and in Relativistic physics. A coordinate system $\{x^1,x^2,x^3,x^4\}$ belongs to the causal class $\{\mathrm{c_1c_2c_3c_4}, \mathrm{C_{12}C_{13}C_{14}C_{23}C_{24}C_{34}}, c_1c_2c_3c_4\}$ if the cobasis $\{dx^1,dx^2,dx^3,dx^4\}$ has causal type $(c_1c_2c_3c_4)$ and has associated four families of coordinate 3-surfaces whose mutual intersections give six families of coordinate 2-surfaces of causal characters $(\mathrm{C_{12}C_{13}C_{14}C_{23}C_{24}C_{34}})$ and four congruences of coordinate lines of causal characters $(\mathrm{c_1c_2c_3c_4})$.

As a matter of notation, romanic letters $(\mathrm{e,t,l})$ represent the causal character of vectors (space-like, time-like, light-like , respectively), and capital $(\mathrm{E,T,L})$ and italic letters (e,t,l) denote the causal character of 2-planes and covectors, respectively. Accordingly, the four Newtonian causal classes are denoted as:

$$\{\mathrm{teee}, \mathrm{TTTEEE}, teee\}, \{\mathrm{ttee}, \mathrm{TTTTTE}, eeee\}, \{\mathrm{ttte}, \mathrm{TTTTTT}, eeee\}, \{\mathrm{tttt}, \mathrm{TTTTTT}, eeee\}.$$

Among the 199 Lorentzian classes, four of them have the same set of causal characters as the Newtonian ones [1]. Next, we consider another examples of relativistic classes.

1.- The three causal classes of Lemaître coordinates. The familiar metric form of the Schwarzschild solution in coordinates $\{t,r,\theta,\phi\}$ is written as

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \frac{1}{1-\dfrac{2m}{r}}\,dr^2 + r^2 d\Omega^2$$

where $r > 2m$ and $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. The coordinate basis $\{\partial_t, \partial_r, \partial_\theta, \partial_\phi\}$ belong to the causal class $\{\mathrm{teee}, \mathrm{TTTEEE}, teee\}$. Lemaître [3] extended the Schwarzschild solution at the region $0 < r \leq 2m$, obtaining a metric form

$$ds^2 = -\left(1-\frac{2m}{r}\right)dT^2 + 2\varepsilon\sqrt{\frac{2m}{r}}\,dT\,dr + dr^2 + r^2 d\Omega^2 \qquad (\varepsilon = \pm 1)$$

that is regular at $r = 2m$. The causal character of the coordinate lines are given by the sign of the four diagonal elements $g_{\alpha\alpha}$ of the metric $g_{\alpha\beta}$ in this basis. The causal character of the coordinate 2-surfaces is given by the sign of the principal second order minors $g_{\alpha\alpha}g_{\beta\beta} - (g_{\alpha\beta})^2$. And the causal character of the coordinate 3-surfaces is related to the sign of the diagonal elements $g^{\alpha\alpha}$ of the contravariant metric expression $g^{\alpha\beta}$. Consequently, the Lemaître coordinate basis $\{\partial_T, \partial_r, \partial_\theta, \partial_\phi\}$ belong to the causal class $\{\mathrm{teee}, \mathrm{TTTEEE}, teee\}$ if $r > 2m$, $\{\mathrm{leee}, \mathrm{TLLEEE}, tlee\}$ if $r = 2m$, or $\{\mathrm{eeee}, \mathrm{TEEEEE}, ttee\}$ if $r < 2m$.

2.- The thirteen causal classes of Bondi-Sachs coordinates. Any space-time metric may be expressed in the form [4]:

$$g_{\alpha\beta} \equiv g\left(\frac{\partial}{\partial x^\alpha}, \frac{\partial}{\partial x^\beta}\right) = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{01} & 0 & 0 & 0 \\ g_{02} & 0 & g_{22} & g_{23} \\ g_{03} & 0 & g_{23} & g_{33} \end{pmatrix}$$

For the associated contravariant metric one has $g^{00} = g^{02} = g^{03} = 0$. Therefore, $\{x^0 = \lambda\}$ (with λ a real parameter) are null hypersurfaces. When $g_{22}g_{33} - g_{23}^2 > 0$, the coordinates are called (generalized) Bondi-Sachs coordinates [4] [5], and are usually denoted by $x^0 \equiv u$, $x^1 \equiv r$, $x^2 \equiv \theta$, $x^3 \equiv \phi$. We have that *the Bondi-Sachs coordinate systems are classified in 13 causal classes:*

$$\{\mathrm{tlee}, \mathrm{TTTLLE}, leee\}$$

$$\begin{array}{ll}
\{\mathrm{llee}, \mathrm{TTTLLE}, leee\} & \{\mathrm{elee}, \mathrm{TTTLLE}, leee\} \\
\{\mathrm{llee}, \mathrm{TTLLLE}, leee\} & \{\mathrm{elee}, \mathrm{TTLLLE}, leee\} \\
\{\mathrm{llee}, \mathrm{TLLLLE}, llee\} & \{\mathrm{elee}, \mathrm{TLLLLE}, leee\}
\end{array}$$

$$\begin{array}{ll}
\{\mathrm{leee}, \mathrm{TLLLEE}, llee\} & \{\mathrm{elee}, \mathrm{TTELLE}, leee\} \\
\{\mathrm{leee}, \mathrm{TLLEEE}, llee\} & \{\mathrm{elee}, \mathrm{TLELLE}, leee\} \\
\{\mathrm{leee}, \mathrm{TLLEEE}, tlee\} & \{\mathrm{elee}, \mathrm{TEELLE}, leee\}
\end{array}$$

For example, a coordinate system of the causal class $\{\mathrm{elee}, \mathrm{TEELLE}, leee\}$ has associated a family of null coordinate 3-surfaces and three families of time-like 3-surfaces. Their mutual cuts give one family of time-like surfaces, two families of null 2-surfaces and three families of space-like 2-surfaces. The intersections of these surfaces give a congruence of null lines and three congruences of space-like lines.

Note that the Bondi-Sachs coordinates are a generalization of the familiar Eddington-Finkelstein coordinates used in the Schwarzschild space-time. These coordinates belong to the class $\{\mathrm{tlee}, \mathrm{TTTLLE}, leee\}$ outside the horizon, to the class $\{\mathrm{llee}, \mathrm{TLLLLE}, llee\}$ at the horizon $r = 2m$, and to the class $\{\mathrm{elee}, \mathrm{TEELLE}, ltee\}$ inside the horizon. The later is the class $\{\mathrm{leee}, \mathrm{TLLEEE}, tlee\}$ (as it results when the first and the second vectors are changed).

3.- The two causal classes of Coll coordinates. Let us consider the classes

$$\{\mathrm{eeee}, \mathrm{EEEEEE}, llll\} \quad \text{and} \quad \{\mathrm{llll}, \mathrm{TTTTTT}, eeee\}$$

A coordinate systems of the first class has associated four families of null 3-surfaces whose mutual cuts give six families of space-like 2-surfaces and four congruences of space-like lines. This class includes the *emission coordinates* of the *Coll positioning systems* [6] [7]. A coordinate system of the second class has associated four families of time-like 3-surfaces whose mutual cuts give six families of time-like 2-surfaces and four congruences of null lines. This class includes the *Coll light coordinates* built from the intersection of four beams of light [8]. We call them the *Coll causal classes*.

The 199 causal classes of relativistic coordinates may be wholly visualized in a table that we call the 199-Table (for details, see [1] and [2]). Here, we reproduce the 199-Table showing the location of the Coll causal classes.

	eeee	leee	elee	teee	llee	tlee	ttee	llle	tlle	ttle	ttte	llll	tlll	ttll	tttl	tttt
eeee	EEEEEE LEEEEE TEEEEE LLEEEE TLEEEE TTEEEE LLLEEE TLLEEE TTLEEE TTTEEE LLLLEE TLLLEE TTLLEE TTTLEE TTTTEE LLLLLE TLLLLE TTLLLE TTTLLE TTTTLE TTTTTE LLLLLL TLLLLL TTLLLL TTTLLL TTTTLL TTTTTL TTTTTT	TTLEEE TTTEEE TTLLEE TTLELE TTTLEE TTLTEE TTLETL TTTTEE TTLLLE TTLELL TTTLLE TTLTLE TTLLTE TTLETL TTTTLE TTLTTE TTLETT TTTTTE TTLLLL TTTLLL TTLTLL TTLLTL TTTTLL TTLTTL TTLLTT TTTTTL TTLTTT TTTTTT		TTTEEE TTTLEE TTTTEE TTTLLE TTTTLE TTTTTE TTTLLL TTTTLL TTTTTL TTTTTT	TTLTLE TTLLTE TTTTLE TTTTTE TTLTLL TTLLTL TTTTLL TTLLTT TTLTLT TTTTTL TTTTLT TTTTTT	TTTTLE TTTTTE TTTTLL TTTTTL TTTTLT TTTTTT	TTTTTE TTTTTL TTTTTT	TTLTLL TTTTLL TTTTTL TTTTTT	TTTTLL TTTTTL TTTTTT	TTTTTL TTTTTT	TTTTTT	TTTTTT	TTTTTT	TTTTTT	TTTTTT	TTTTTT
leee	EEEEEE LEEEEE EEELEE TEEEEE LELEEE LEELEE EELLEE TLEEEE TEELEE EETLEE TTEEEE LLLEEE LLELEE LELLEE TLLEEE TLELEE TELLEE LETLEE TTLEEE TTELEE TETLEE TTTEEE LLLLEE TLLLEE LLTLEE TTLLEE TLTLEE TTTLEE	TTLEEE TTTEEE TTLLEE TTLELE TTTLEE	TEELLE TLELLE TTELLE TLLLLE TTLLLE TTTLLE	TTTEEE TTTLEE	TTLLLE TTTLLE	TTTLLE										
teee	EEEEEE LEEEEE TEEEEE LLEEEE TLEEEE TTEEEE LLLEEE TLLEEE TTLEEE TTTEEE	TTLEEE TTTEEE		TTTEEE												
llee	EEEEEE LEEEEE ELEEEE TEEEEE LLEEEE EELLEE ELELEE TLEEEE LLELEE LELLEE TLELEE TELLEE	TLLEEE TLLLEE			TLLLLE											
tlee	EEEEEE LEEEEE ELEEEE TEEEEE LLEEEE TLEEEE	TLLEEE														
ttee	EEEEEE LEEEEE TEEEEE															
llle	EEEEEE LEEEEE LLEEEE LLELEE															
tlle	EEEEEE LEEEEE LLEEEE															
ttle	EEEEEE LEEEEE															
ttte	EEEEEE															
llll	EEEEEE															
tlll	EEEEEE															
ttll	EEEEEE															
tttl	EEEEEE															
tttt	EEEEEE															

Coll causal classes

The Coll causal classes {llll, TTTTTT *eeee*} and {eeee, EEEEEE, *llll*}. They are dual each other (see references [1] and [2]).

Each causal class has univocally associated its *dual* causal class [2]. For example the Coll causal classes are dual each other. The same occurs for the causal classes of the Eddington-Finkelstein coordinates at $r > 2m$ and $r < 2m$. When a causal class and its dual are equal, the class is said *self-dual* (as the class of the Eddington-Finkelstein coordinates at $r = 2m$). Orthonormal frames and real null frames belong, respectively, to the self-dual classes $\{\text{teee}, \text{TTTEEE}, eeee\}$ and $\{\text{llee}, \text{TLLLLE}, llee\}$. There exist eleven self-dual causal classes which are easily located looking at the diagonal region of the 199-Table. They are:

$$
\begin{array}{ll}
\{\text{eeee}, \text{TTTEEE}, eeee\} & \{\text{leee}, \text{TTTEEE}, leee\} \\
\{\text{eeee}, \text{TTLLEE}, eeee\} & \{\text{leee}, \text{TTLLEE}, leee\} \\
\{\text{eeee}, \text{TLLLLE}, eeee\} & \{\text{leee}, \text{TTLELE}, leee\} \\
\{\text{eeee}, \text{LLLLLL}, eeee\} & \\
\\
\{\text{elee}, \text{TTELLE}, leee\} & \{\text{teee}, \text{TTTEEE}, teee\} \\
\{\text{elee}, \text{TLLLLE}, leee\} & \{\text{llee}, \text{TLLLLE}, llee\}
\end{array}
$$

To gain more comprehension of the role that coordinates play in Relativity, the 199 causal classification would be investigated through and through. The causal classes may be associated to admissible *cuts* of the space-time others than the well-known three-space $\oplus$ one-time usual in the at present evolution conception of physics. Other cuts (among the other 198 possible ones) may help us to better understand other aspects of the space-time, and even to wake up our interest for other variations of physical fields than the time-like ones associated to the evolution formalism. A lot of work still remains to be done in this direction.

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