

The momentum and the angular momentum of the Universe revisited. Some preliminary results

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Abstract. We consider the question of properly defining energy and momenta for non asymptotic Minkowskian spaces in general relativity. Only those of these spaces which have zero energy, zero linear 3-momentum, and zero intrinsic angular momentum would be candidates to creatable universes, that is, to universes which could have arisen from a vacuum quantum fluctuation. Given a universe, we completely characterize the family of coordinate systems in which it would make sense saying that this universe can be a creatable universe.

1. General considerations

Which is the most general universe which has null energy, null linear 3-momentum, and null intrinsic 3-angular momentum, and which could be the interest of such a question?

From the first seventies, people have speculated about a Universe which could have arisen from a quantum vacuum fluctuation [1] [2]. If this was the case, one could expect this Universe to have zero energy.

But, then, why we should consider only the energy? Why not expect that the linear 3-momentum and angular intrinsic 3-momentum of the Universe also be zero? And finally: why, in all, not to expect that, both, linear 4-momentum and angular intrinsic 4-momentum be zero?

So, in the present paper, we will consider both: linear 4-momentum, $P^\alpha = (P^0, P^i)$, and angular 4-momentum, $J^{\alpha\beta} = (J^{0i}, J^{ij})$. In all: it could be expected that only those universes with $P^\alpha = 0$, and $J^{\alpha\beta} = 0$ could have arisen from a quantum vacuum fluctuation. Then, we could say that only these ones would be 'creatable universes'.

Now, as it is well known (see, for example [3]), when dealing with an asymptotically flat space-time, one can define in a unique way its linear and angular 4-momentum, provided that one uses any coordinate system which goes fast enough to a Minkowskian coordinate system in the 3-space infinity.

Nevertheless, if, when dealing with the Universe as such, we only consider non asymptotically space-times, we cannot use such Minkowskian systems. Then, we will not know in advance which coordinate systems should we use, in order to properly define the linear and angular 4-momentum of the Universe. This is, of course a major problem, since, as we will see, and it is well known, P^α and $J^{\alpha\beta}$ are strongly coordinate dependent, and it is so whatever it be the energy-momentum complex we use (the one of Weinberg [3], or Landau [4], or any other one).

As we have just said, this strong coordinate dependence of P^α and $J^{\alpha\beta}$ is very well known, but in spite of this, in practice, is not always properly commented or even taken properly into

account. This can be seen by having a look on the different calculations of the energy of the Universe which have been appeared in the literature (see for example, among other references, [5][6]) from the pioneering paper by Rosen [7].

Even Minkowski space can have non null energy if we take non Minkowskian coordinate systems. This non null energy would reflect the energy of the *fictitious gravitational field* induced by such non Minkowskian coordinates, or in other words the energy tied to the family of the corresponding accelerated observers. So, in particular, to define the proper energy and momentum of a universe, we would have to use coordinate systems adapted, in some sense, to the symmetries of this universe, in order to get rid of this spurious energy supply. In the rest of the present paper we will address this question in some detail. We will look for the good family of coordinates systems in order to properly define the energy and momenta of the considered universe. Assuming some reasonable conditions, we will determine this family in the case in which we are interested, mainly when the universe has zero energy and momenta.

2. Which coordinate systems

We expect any well behaved universe to have well defined energy and momenta, i. e., P^α and $J^{\alpha\beta}$ would be finite and conserved in time. So, in order this conservation makes physical sense, we need to use a *physical* and *universal* time. Then we will use Gauss coordinates:

$$ds^2 = -dt^2 + dl^2, \quad dl^2 = g_{ij}dx^i dx^j, \quad i, j = 1, 2, 3. \quad (1)$$

In this way, the time coordinate is the proper time and so a *physical* time. Even more, it is an everywhere synchronized time (see for example [4]) and so an *universal* time.

Obviously, we have so many Gauss coordinate systems in the considered universe (or in a part of it) as we have space-like 3-surfaces, Σ_3 . Then, P^α and $J^{\alpha\beta}$ will depend on Σ_3 (as the energy of a physical system in the Minkowski space-time does, which depends on the chosen Σ_3 , i.e., on the chosen Lorentz coordinates).

Now, in order to continue our preliminary inquire, we must choose one energy-momentum complex. Since besides linear momentum we will also consider angular momentum, we will need a symmetric energy-momentum complex. Then, we will take the Weinberg one [3]. This complex has the property that it allow us to write energy and momenta as some integrals over the boundary 2-surface, Σ_2 of Σ_3 . Then, any other symmetric complex with this property, as for example the one from Landau [4], will let us to obtain essentially the same results that the ones we will obtain in the present paper.

Then, taking the above Weinberg complex, one obtains, in Gauss coordinates, for the linear 4-momentum, $P^\alpha = (P^0, P^i)$, and the angular one, $J^{\alpha\beta} = (J^{0i}, J^{ij})$, the following expressions [3]:

$$P^0 = \frac{1}{16\pi G} \int (\partial_j g_{ij} - \partial_i g) d\Sigma_{2i}, \quad P^i = \frac{1}{16\pi G} \int (\dot{g}\delta_{ij} - \dot{g}_{ij}) d\Sigma_{2j} \quad (2)$$

$$J^{kj} = \frac{1}{16\pi G} \int (x_k \dot{g}_{ij} - x_j \dot{g}_{ki}) d\Sigma_{2i}, \quad (3)$$

$$J^{0i} = P^i t - \frac{1}{16\pi G} \int [(\partial_k g_{kj} - \partial_j g)x_i + g\delta_{ij} - g_{ij}] d\Sigma_{2j}, \quad (4)$$

where we have used the following notation: $g \equiv \delta^{ij} g_{ij}$, $\dot{g}_{ij} \equiv \partial_t g_{ij}$ and $d\Sigma_{2i}$ is the surface element of Σ_2 .

The area of Σ_2 could be zero, finite or infinite. In the first case the energy and momenta would be trivially zero.

3. More about the good coordinate systems

From what has been said in the above Section, one could erroneously concluded that, in order to calculate the energy and momenta of a universe, one needed to write the metric in all Σ_3 , in Gauss coordinates. Nevertheless, since, according to Eqs. (2), (3), (4), P^α and $J^{\alpha\beta}$ can be written as surface integrals on Σ_2 , all we need is this metric, in Gauss coordinates, on Σ_2 and its immediate neighborhood (notice that space derivatives of the metric appear in some of these integrals).

Furthermore, since P^α and $J^{\alpha\beta}$ are supposed to be conserved, we only need this metric in the neighborhood of a given time, say $t = t_0$, or in other words in the elementary vicinity of Σ_3 , whose equation, in the Gaussian coordinates we are using, is then $t = t_0$. Thus, we do not need our Gauss coordinate system to cover all the universe life. Nevertheless, in order to be consistent we will need to check that the conditions for this conservation are actually met (see next the end of Section 4 in relation to this question).

Now, the surface element $d\Sigma_{2i}$, which appears in the above expressions of P^α and $J^{\alpha\beta}$, is defined as if our space Gauss coordinates, (x^i) , were Cartesian coordinates. Thus, it has not any intrinsic meaning in front of a change of coordinates on the neighborhood of Σ_2 . So, which is the correct family of coordinate systems that we must use on this neighborhood, in order to properly define the energy and momentum of the universe?

In order to answer this question, we will first prove that, on Σ_2 , in any given time instant t_0 , $dl_0^2 \equiv dl^2(t = t_0)$ has a conformally flat form, that is, there is a coordinate systems such that

$$dl_0^2|_{\Sigma_2} = f \delta_{ij} dx^i dx^j, \quad i, j = 1, 2, 3, \quad (5)$$

where f is a function defined on Σ_2 . The different coordinate systems, in which $dl_0^2|_{\Sigma_2}$ exhibits explicitly its conformal form, are connected among themselves by the conformal group in three dimensions. Then, one or some of these different conformal coordinate systems are to be taken as the good coordinate systems to properly define the energy and momenta of the considered universe. This is a natural assumption since the conformal coordinate systems allow us to write, in an explicit way, the space metric on Σ_2 in the most approximate form to the explicit Euclidean space metric. But, which ones of all the conformal coordinates should be used? We will not treat to answer here this question, since our final goal in the present paper is to consider universes with zero energy and momenta. Then, we will see next that, in the particular case where the energy and momenta of the universe are zero in one of the above conformal coordinate systems, the energy and momenta are zero in any other conformal coordinate system.

So, according to what we have just announced, we must prove that $dl_0^2|_{\Sigma_2}$ has a conformal form. In order to do this, let us use Gaussian coordinates, (y^i) in Σ_3 , based in Σ_2 . Then, we will have

$$dl_0^2 = (dy^3)^2 + g_{ab}(y^3, y^c) dy^a dy^b, \quad a, b, c = 1, 2 \quad (6)$$

In the new (y^i) coordinates the equation of Σ_2 is then $y^3 = L$, where L is a constant.

Then, we can always find a new coordinate system (x^a) on Σ_2 , such that we can write dl_0^2 on Σ_2 , that is to say, $dl_0^2|_{\Sigma_2}$, as:

$$dl_0^2|_{\Sigma_2} = (dy^3)^2 + f(L, x^a) \delta_{ab} dx^a dx^b, \quad i, j = 1, 2, 3, \quad (7)$$

Finally, we make the coordinate transformation

$$x^3 = \frac{y^3 - L}{f^{\frac{1}{2}}} + C, \quad (8)$$

with C an arbitrary constant, which can be seen to allow us to write $dl_0^2|_{\Sigma_2}$ in the form (5), as we wanted to prove.

Furthermore, if $r \equiv \delta_{ij}x^i x^j$, and we assume that the equation of Σ_2 in spherical coordinates is $r = R(\theta, \phi)$, in the vicinity of Σ_2 , we will have:

$$dl^2 \approx [f\delta_{ij} + (r - R) {}^1g'_{ij} + (t - t_0) {}^1\dot{g}_{ij}]dx^i dx^j. \quad (9)$$

where ${}^1g'_{ij}$ and ${}^1\dot{g}_{ij}$ are functions which do not depend neither on r nor on t . Let us note that if the equation of the boundary, Σ_2 , is $r = \infty$, then we should consider an expansion in $1/r$ of g_{ij} instead of the above expansion in $r - R$.

According to Eqs. (2), (3), (4), notice that the ${}^1g'_{ij}$ functions will appear in P^0 , whereas the ${}^1\dot{g}_{ij}$ functions will appear in P^i and J^{ij} . In the components J^{0i} will appear all functions which are present in Eq. (9).

The f , ${}^1g'_{ij}$ and ${}^1\dot{g}_{ij}$ functions will change when we do a conformal change of coordinates. But, this is the only change these functions can undergo. To see this, let us see first which coordinate transformation are allowed, besides the conformal transformations, if the the explicit conformal character of $dl_0^2|_{\Sigma_2}$ is to be preserved. In an evident notation, these transformations will have the form

$$x^i = x'^i + y^i(x^j)(t - t_0), \quad (10)$$

in the vicinity of Σ_3 . But it is easy to see that here the functions $y^i(x^j)$ must all three be zero, if the Gaussian character of the coordinates has to be preserved. That is, the only coordinate transformations which we can do on the vicinity of Σ_2 metric are the coordinate transformations of the conformal group in three dimensions.

Thus, given Σ_3 , that is, given the 3-surface which serves us to build our Gauss coordinates, we have defined uniquely P^α and $J^{\alpha\beta}$, modulus some transformation which is conformal on Σ_2 . So, the question is: how do P^α and $J^{\alpha\beta}$ change under such a conformal transformation? As we have said above, we are not going to try to answer this question here. Instead of this, since we are mainly concerned with 'creatable universes', we will explore under what reasonable assumptions the energy and momentum of a universe are zero for all the above conformal systems.

4. Zero energy and momentum irrespective of the conformal system

The first thing to be noticed in relation to the question is that P^α and $J^{\alpha\beta}$ are invariant in front of the groups of dilatations, translations, and rotations, respectively, in three dimensions. These all three groups are subgroups of the conformal group in three dimensions. Then, we are left with the subgroup of elements that have been sometimes called the *essential* conformal transformations. But it is well known [8] that these transformations are equivalent to apply first an inversion, that is, r going to $1/r$, then a translation, and finally another inversion. So, in order to see how P^α and $J^{\alpha\beta}$ change when we do a conformal transformation, one has only to see how they change when we apply an inversion, that is, r going to r' , such that

$$r' = \frac{1}{r}, \quad r^2 \equiv \delta_{ij}x^i x^j. \quad (11)$$

Assume as a first case that the equation of the boundary Σ_2 is $r = \infty$. In this case, the 2-surface element, $d\Sigma_{2i}$, which appears in the Eqs. (2), (3), (4), can be written as $d\Sigma_{2i} = r^2 n_i d\Omega$, where $n_i \equiv x^i/r$, and $d\Omega$ is the elementary solid angle.

Now, let us consider first the energy, P^0 . How does it change when we apply an inversion? This lead us to see how its integrand, $I \equiv r^2(\partial_j g_{ij} - \partial_i g)n_i d\Omega = r^2(n_i \partial_j g_{ij} - \partial_r g)d\Omega$, changes. After some calculation, one sees that the new value, I' , for I is

$$I' = r^3(r\partial_r g - r n_i \partial_j g_{ij} + 2n_i n_j g_{ij} + 2g)d\Omega, \quad (12)$$

which, according to Eq. (9), can still be written for $t = t_0$ as

$$I' = r^3(r\partial_r g - rn_i\partial_j g_{ij} + 8f)d\Omega. \quad (13)$$

Now, in this expression of I' there is an r^3 common factor. Thus, if we want $P^{0'}$ to be zero, it suffices that f goes at least as r^{-4} when r goes to ∞ . Then, according to what we have commented in the last Section, the functions $g_{ij} - f\delta_{ij}$, which must go to zero faster than f , will go at least as r^{-5} . Of course, this asymptotical behavior of f and $g_{ij} - f\delta_{ij}$ makes the original P^0 equal zero too. Thus, on the assumption that the equation of Σ_2 is $r = \infty$, we have proved that this behavior is a sufficient condition in order that $P^0 = 0$ be independent of the used conformal system.

This sufficient condition is not a necessary one, since it could happen that P^0 were zero because the angular dependence of I . An angular dependence which makes zero the integral of I on the boundary 2-surface, Σ_2 , independently of the behavior of I when r goes to ∞ . But, in this case, from the above expressions of I' and the expression of I , one sees that the sufficient and necessary condition to have $P^{0'}$ equal zero is that the integral of f on Σ_2 be zero because of the special angular dependence of the function f .

In all, under the assumption that the equation of Σ_2 is $r = \infty$, we have given the necessary and sufficient conditions in order to have $P^0 = 0$ independently of the used conformal system. Also, one can easily see that the same is true for $P^i = 0$ and $J^{\alpha\beta} = 0$, provided that the above asymptotic behavior for f and $g_{ij} - f\delta_{ij}$ can be extended to ${}^1\dot{g}_{ij}$.

We will see next that all this can be applied to the closed and flat Friedmann-Robertson-Walker (FRW), whose energy and momenta become then zero.

Let us continue with the question of the nullity of energy and momenta, leaving now the case where the equation of Σ_2 is $r = \infty$ and considering the case where this equation is $r = R(\theta, \phi)$.

Let us consider first the special case where R is a finite constant. Then, according to the above expressions of I' and I , one sees that in this case the necessary and sufficient condition to have $P^0 = 0$ irrespective of the used conformal system is that the integral of f on Σ_2 be zero. This could be so, either because f is zero for $r = R$, or because the special angular dependence of f .

In the general case, when it is $r = R(\theta, \phi)$, a natural sufficient condition to have energy zero, irrespective of the used conformal system, is that $g_{ij} = 0$ at first order in $r - R(\theta, \phi)$, in the time $t = t_0$, so that, in Eq. (9), it would be $f = 0$ and ${}^1\dot{g}'_{ij} = 0$. This is the same sufficient condition which was present, in a natural way, in the above cases, i.e., when it was $r = \infty$, and when it was $r = R \equiv \text{constant}$, respectively.

It can be easily seen that extending this condition to ${}^1\dot{g}_{ij} = 0$ (see again Eq. (9)) entails that not only the energy of the considered universe, but also the linear 3-momenta and the angular 4-momentum will be zero irrespective of the used conformal system.

Finally, in order to make sure that P^α and $J^{\alpha\beta}$ are actually conserved, we need the nullity of the second time-time and time-space derivatives of the original space metric on Σ_2 . This is the answer to the consistency question raised at the end of the second paragraph, at the beginning of Section 3.

5. The example of FRW universes

As it is well known, in these universes one can use Gauss coordinates such that the 3-space exhibits explicitly its conformal flat character:

$$dl^2 = \frac{a^2(t)}{\left[1 + \frac{k}{4}r^2\right]^2} \delta_{ij} dx^i dx^j, \quad r^2 \equiv \delta_{ij} x^i x^j, \quad (14)$$

where $a(t)$ is the expansion factor and $k = 0, k = \pm 1$ are the index of the 3-space curvature.

Then, this conformal flat character will be valid, *a fortiori* on any vicinity of Σ_3 and of Σ_2 . Therefore, according to Section 3, we can apply our definitions to the metric (14). We will have then:

$$P^0 = -\frac{1}{8\pi G} \int r^2 \partial_r f d\Omega, \quad P^i = \frac{1}{8\pi G} \int r^2 \dot{f} n_i d\Omega, \quad (15)$$

$$J^{kj} = \frac{1}{16\pi G} \int r^2 \dot{f} (x_{0k} n_j - x_{0j} n_k) d\Omega, \quad (16)$$

$$J^{0i} = P^i t - \frac{1}{8\pi G} \int r^2 (f n_i - x_{0i} \partial_r f) d\Omega \quad (17)$$

with $d\Omega = \sin \theta d\theta d\phi$, $n_i \equiv x^i/r$, where x_{0i} is the origin of angular momentum, and where we have put

$$f \equiv \frac{a^2(t)}{\left[1 + \frac{k}{4} r^2\right]^2}. \quad (18)$$

Then, one can easily obtain the following results, in according with most literature on the subject:

$$\begin{aligned} k = 0, +1 : & \quad P^\alpha = 0, \quad J^{\alpha\beta} = 0, \\ k = -1 : & \quad P^0 = -\infty. \end{aligned} \quad (19)$$

Thus, the flat and closed FLRW universes are ‘creatable universes’, but the open one is not.

6. Work in progress

Before we finish the paper, we must point out two open questions which we are addressing at this moment.

The first one is, if in the case that $P^\alpha = 0$ and $J^{\alpha\beta} = 0$ for a chose 3-surface, Σ_3 , this nullity will remain preserved when we move to a new Σ_3 , as it is suggested by what happens in the Minkowski space.

The second one would be to study, next to the considered LFRW universes, another ones, as for instance the Bianchi universes, in order to see which of them, if any, are ‘creatable universes’.

The detailed calculations and proofs of the present paper, jointly with the possible results obtained on the above questions, will be considered elsewhere.

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