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NEWTONIAN AND RELATIVISTIC LOCATION SYSTEMS

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Abstract. The theory of location systems involves the geometric and physical description of the protocols allowing the realization of coordinate systems. In this communication, the incidence of the space-time causal structure (Newtonian or relativistic) on the construction of location systems is remarked. Specifically, we focus our attention: (i) on the construction of Newtonian emission coordinates that are contrasted with those associated with relativistic positioning systems, and (ii) on the role played by non-absolute synchronizations (like the one provided by the local Solar time) in the comprehension of Newtonian and relativistic location systems.

1 Introduction

A *location system* is a physical realization of a coordinate system (see Coll 2006 for motivations and details). Here, we will present some basic notions and results about location systems, both in Newtonian and relativistic physics. It must be pointed out that the content of this ERE-lecture is based in a more extended work presented elsewhere (see Coll *et al.* 2007) whose starting point is the causal classification of space-time frames and coordinate systems.

A complete geometric description of a space-time coordinate system may by given in different ways. For instance, from its associated four families of coordinate 3-surfaces whose mutual cuts give its six families of coordinate 2-surfaces and its four congruences of coordinate lines. Consequently, a location system must include the protocols for the physical construction of some of these geometric elements (lines, surfaces and hypersurfaces) of the coordinate system that it physically realizes. In this sense, timelike lines may be realized by means of clocks, null lines by lasers pulses, spacelike lines by synchronized inextensible threads, timelike surfaces by the history of threads or by lasers beams, null hypersurfaces by light-front signals and so on.

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The *causal signature* of a frame $\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3, \mathrm{v}_4\}$ is defined by a set of 14 causal orientations:

$$\{\mathrm{c}_1\mathrm{c}_2\mathrm{c}_3\mathrm{c}_4, \mathrm{C}_{12}\mathrm{C}_{13}\mathrm{C}_{14}\mathrm{C}_{23}\mathrm{C}_{24}\mathrm{C}_{34}, \mathit{c_1\,c_2\,c_3\,c_4}\}$$

where $\mathrm{c_i}$ is the causal orientation of the vector $\mathrm{v_i}$; $\mathrm{C_{ij}}$ $(\mathrm{i} < \mathrm{j})$ is the causal orientation of the 2-plane $\{\mathrm{v_i v_j}\}$, and c_i is the causal orientation of the covector θ^i of the dual coframe. The *causal class* of a frame is the set of all the frames that have the same causal signature. At a given point, the causal class of a coordinate system is the causal class of its associated natural frame at this point.

When the causal orientations of all the geometric elements of a coordinate system are uniform on a given space-time region we say that the region under consideration is a causal homogeneous region for the coordinate system in question. The point of interest here is that every protocol physically realizes coordinate lines, coordinate surfaces or coordinate hypersurfaces of specific causal orientations allowing to analyze the different causal homogeneous regions of the constructed coordinate system.

Let us denote by Roman letters $\mathrm{e}, \mathrm{t}, \mathrm{l}$ the causal orientations (spacelike, timelike, null) of vectors and coordinate lines; by capital letters $\mathrm{E}, \mathrm{T}, \mathrm{L}$ the causal orientations (spacelike, timelike, null) of the associated 2-planes and coordinate 2-surfaces; and by Italic letters e, t, l the causal orientation (spacelike, timelike, null) of covectors that also give the causal orientations (timelike, spacelike or null) of the coordinate 3-surfaces.

The causal structure of Newtonian space-time allows us to classify frames and coordinate systems in *four* causal classes, in accordance with the number of timelike vectors of the frame (Coll *et al.* 2007). The causal signatures of these classes are: $\{\mathrm{tttt}, \mathrm{TTTTTT}, \mathit{eeee}\}$, $\{\mathrm{ttte}, \mathrm{TTTTTT}, \mathit{eeee}\}$, $\{\mathrm{ttee}, \mathrm{TTTTTE}, \mathit{eeee}\}$ and $\{\mathrm{teee}, \mathrm{TTTEEE}, \mathit{teee}\}$. The last one is the standard causal class of Newtonian frames, i.e. the one based in the simultaneity synchronization, and it is constituted by frames with one timelike vector and three spacelike ones. Nevertheless, as we are going to see, the other three causal classes of Newtonian frames admit a simple physical description (by means of emission coordinates associated with Newtonian positioning systems) and an easy geometrical interpretation (by using non-standard (timelike) Newtonian synchronizations).

In relativity, according to a result by Coll and Morales (1992), there are other 198 causal classes of frames different from the standard one. Now, the standard causal class is constituted by those frames with one timelike vector and three spacelike ones generating a spacelike 3-space. Note that, concerning spacelike vectors, the main difference between Newtonian and relativistic causal structures comes from the essential property that in a Lorentzian metric two spacelike vectors generate a 2-plane that may be spacelike, null or timelike (depending on their mutual scalar product). Then, it can be easily proved that there are 13 non-standard causal classes of relativistic frames with a timelike vector and three spacelike ones.

Here we shall compare the incidences of the Newtonian and Lorentzian spacetime structures on the construction of location systems analyzing how some non-standard causal classes may be physically realized from two different, but complementary, protocols: by using emission coordinates and/or introducing non-standard timelike synchronizations. In particular, the study of the causal properties of a coordinate system has a significant incidence, for example, in post-Newtonian developments where it is convenient to choose coordinate systems such that their causal properties be the same with respect to the relativistic calculated metric structure as well as for the starting Newtonian one. This convenient choice of analogous causal properties is usually made by taking the starting Newtonian coordinate system to be the standard one, and considering weak gravitational fields that are unable to change, with the lower order perturbed relativistic values of the metric, these causal properties. However, new problems concerning strong gravitational fields, gravitational waves, binary systems, positioning systems or other relativistic situations, could induce to start from other Newtonian coordinate systems, best adapted to these problems.

2 Emission positioning systems

Now, we consider emission positioning systems in order to understand the main differences and analogies between the Newtonian and the relativistic situations. They are based in sound or light signals and they show that one can locate events in space-time domains *without* any use of the concept of synchronization.

Suppose an inertial (and non-dispersive) medium in which a class of signals (sound, light) propagates at constant velocity v. Consider the world-line of an emitter clock that uses such signals to continuously broadcast its time t. In the space-time, the front waves describe thus sound or light cones carrying the value $t = \textit{constant}$.

Four emitters $\kappa^A(t)$ $(A = 1, 2, 3, 4)$ fill a space-time domain with four one-parameter families of cones $t^A =$ constant that are the coordinate 3-surfaces of an emission coordinate system in this domain. An alternative point of view is to consider the past (sound, light) cone of every event that cuts the emitter world lines at $\kappa^A(t^A)$; then the *emission coordinates* of the event are $\{t^A\}$.

Here we will consider the simple case of four emitters at rest with respect to an inertial medium. In a standard coordinate system $\{t, x^i\} = \{t, \vec{r}\}$, the emitter world-lines are described by:

$$\kappa^A(t) = (t, \vec{c}^A)\,. \tag{2.1}$$

Then, the signal emitted by the clock κ^A at the instant t^A at velocity v describes in the space-time a cone of equation

$$v(t - t^A) = \left|\vec{r} - \vec{c}^A\right|\,, \tag{2.2}$$

so that the emission coordinates $\{t^A\}$ are related to the inertial ones $\{t, \vec{r}\}$ by

$$t^A = t - \frac{1}{v}\left|\vec{r} - \vec{c}^A\right|\,. \tag{2.3}$$

Newtonian emission coordinates.- In Newtonian space-time, the emission coordinate system generated by a positioning system is non (globally) causally homogeneous, but always presents three regions corresponding to the three non standard causal classes. In fact, at the events where the Jacobian (from inertial to emission coordinates) is not degenerate, the coordinate lines of the emission coordinates are generically of the type $\{\mathrm{t\,t\,t\,t}\}$; they are generically of the type $\{\mathrm{t\,t\,t\,e}\}$ on the events of the timelike 3-planes containing three emitters, and are of type $\{\mathrm{t\,t\,e\,e}\}$ on the events of the timelike strips generated by every pair of clocks. A detailed prove of this result may be found in the quoted work by Coll *et al.* 2007.

Relativistic emission coordinates.- Next, we are going to discuss the causal properties of positioning systems in Minkowski space-time. Now, every emitter κ^A is supposed to continuously broadcast their proper time τ^A by means of sound or light signals that propagate in the medium at constant velocity $v \leq 1$. For simplicity, the four emitters will be considered at rest with respect to the medium referred to a standard coordinate system $\{t, x^i\} = \{t, \vec{r}\}$. Then, the inertial time t is also the proper time of the four emitters and their world-lines take the expression (2.1); the equation of the cones that describe the signals is like (2.2) and the emission coordinates $\{t^A\}$ are related to the inertial ones by (2.3).

In the light case ($v = 1$) we have $(dt^A)^2 = 0$, so that, the coframe of the relativistic emission coordinate system is of causal type $\{l\,l\,l\,l\}$. Consequently, the relativistic positioning systems with light signals define in their whole domains a sole causal class, of causal signature $\{\mathrm{eeee}, \mathrm{EEEEEE}, llll\}$. This result, obtained for an inertial homogeneous medium and four static clocks, may be shown true also for arbitrary clocks in general space-times (see Coll and Pozo 2006).

In the sound case ($v < 1$) the coframe of the relativistic emission coordinate system is of the causal type $\{e\,e\,e\,e\}$ and it can be proved that, depending on the different configurations of the stationary emitters, and/or of the different values of the velocity $v < 1$, the emission coordinate systems may present space-time regions of 102 different causal classes (see Coll *et al.* 2007).

In both, the Newtonian and the the relativistic situations, the coordinate lines of emission coordinates are hyperbolas. Nevertheless, their causal types are different. In the Newtonian case every hyperbola is everywhere timelike up to at its base point, where it is spacelike. In the relativistic case: when $v = 1$ the hyperbolas are spacelike everywhere, but when $v < 1$ each hyperbola is spacelike over an arc (including the base point) which is bounded by two points where it is lightlike, the rest of the branches being timelike. This is at the basis of the richness of the sound-based relativistic positioning systems previously considered.

3 The role played by the synchronizations

First of all, in order to deal with causal properties of coordinate systems, we need to precise the current terminology. There are two natural variations associated with a given coordinate x^α: its variation ∂_α along the coordinate lines with variable x^α, and its gradient, dx^α, associated with the coordinate hypersurfaces having constant x^α. Such variations have, in general, different causal orientations. Consequently, when the causal orientation of ∂_α (resp. dx^α) is c_α (resp. c_α) we say that the coordinate x^α is a c_α *coordinate parameter* (resp. a c_α *gradient coordinate*).

If a coordinate t is a timelike coordinate parameter and a timelike gradient coordinate we say that it defines a *spacelike synchronization*. Obviously, the absolute Newtonian time t, whose gradient dt is the Newtonian time current (the sole timelike codirection that the Newtonian causal structure admits) defines a spacelike synchronization.

If a coordinate t is a timelike coordinate parameter and a spacelike gradient coordinate we say that it defines a *timelike synchronization*. The Solar time of the different places on the Earth surface provides a timelike Newtonian synchronization. Let us take a simplified model for a spherical rotating Earth (with uniform angular velocity ω and taking in consideration neither its translational motion nor the ecliptic inclination) in which the local Solar time T of a given place is related to its azimuthal angle ϕ (from a given fixed direction to the sky) by $T = \phi/\omega$. The longitude Φ of this place is obtained (up to an additive constant) as $\Phi = \phi - \omega t$ (with t the absolute Newtonian time). Then $T = t + (\Phi/\omega)$, and T is a spacelike gradient coordinate (dT is a spacelike 1-form because it is not proportional to the Newtonian time currrent dt).

Of course, in relativity, we say that a coordinate t defines a lightlike synchronization if its gradient is lightlike. In any case, the locus of synchronous events of the coordinate lines $t =$ variable are the coordinate hypersurfaces $t =$ constant.

The above example of the solar synchronization suggests us that we will be able to generate all the Newtonian causal classes using the linear synchronization group,

$$X^0 = x^0 + a_i x^i \ , \quad X^i = x^i \, . \tag{3.1}$$

The natural frame and coframe of the new system $\{X^\alpha\}$ are given by

$$\partial_{X^0} = \partial_{x^0} \ , \qquad \partial_{X^i} = -a_i \partial_{x^0} + \partial_{x^i} \ , \tag{3.2}$$

$$dX^0 = dx^0 + a_i dx^i \ , \qquad dX^i = dx^i \, . \tag{3.3}$$

Non-standard synchronizations in Newtonian space-time.- Starting from a standard coordinate system $\{x^0, x^i\}$ of causal type $\{\mathrm{t\,e\,e\,e}\}$, the linear synchronization transformations (3.1) define a coordinate system $\{X^\alpha\}$ whose causal type is: $\{\mathrm{t\,t\,e\,e}\}$ if there is a sole i such that $a_i \neq 0$; $\{\mathrm{t\,t\,t\,e}\}$ if there is a sole i such that $a_i = 0$, and $\{\mathrm{t\,t\,t\,t}\}$ if for all i, $a_i \neq 0$. In this way, the different causal classes have been obtained by simple changes of synchronization of the given system of inertial observers.

Non-standard synchronizations in Minkowski space-time.- It follows, by direct scalar products of the above expressions (3.2) and (3.3) that all the causal classes obtained by a linear synchronization transformation have a causal signature of the form $\{\mathrm{tc_1c_2c_3}, \mathrm{TTTC_{12}C_{13}C_{23}}, c_0eee\}$. The non-fixed causal orientations, $\mathrm{c_1, c_2, c_3, C_{12}, C_{13}, C_{23}}, c_0$ depend on the a_i parameters and then, it results that the number of different causal classes that may be generated by a linear synchronization transformation is 29, in contrast with the only four Newtonian ones (see Coll *et al.* 2007).

Non-standard synchronizations in Schwarzschild space-time.- Painlevé (1921) and Gullstrand (1922) expressed the Schwarzschild solution in the form

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dT^2 + 2\sqrt{\frac{2m}{r}}\, dT\, dr + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2).$$

There is no divergence at the horizon $r = 2m$ (see also Lemaître 1933). It is easy to prove (see Morales 2006) that the coordinate system $\{T, r, \theta, \phi\}$ has causal signature: $\{\mathrm{teee}, \mathrm{TTTEEE}, \mathrm{teee}\}$ if $r > 2m$, $\{\mathrm{leee}, \mathrm{TLLEEE}, \mathrm{tlee}\}$ if $r = 2m$, and $\{\mathrm{eeee}, \mathrm{TEEEEE}, \mathrm{ttee}\}$ if $r < 2m$. Note that T is a timelike gradient coordinate, but it is a timelike, null or spacelike coordinate parameter outside, over or inside the horizon, respectively. T is the proper time of a freely falling observer whose initial velocity at $r = \infty$ is zero with respect to a static observer. The relation between Schwarzschild time t and the T-coordinate is obtained as a non-linear synchronization transformation over the congruence of the static observers. It is given by $T = t + 2mf(r)$ where $f(r) = \sqrt{2r/m} + \ln(\sqrt{r} - \sqrt{2m}) - \ln(\sqrt{r} + \sqrt{2m})$.

To conclude, we would like to add a final comment. We have showed the interest of the causal classification of frames in the theory of classical and relativistic location systems. Among the 198 admissible *cuts* of the space-time others than the very usual space $\oplus$ time decomposition, a lot of them admit simple physical realizations (from synchronization transformations and/or emission coordinates). Nevertheless, in order to better understand the role that location systems play in the analysis of experimental observations, a lot of basic work still needs to be developed.

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Disapproval.- We want to emphasize here our blame for the biased and unscientific arguments of the board member's report of Class. and Quantum Gravity in rejecting our work on this subject.

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