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Emission coordinates in Minkowski space-time

J. A. Morales

Departament d'Astronomia i Astrofísica

Universitat de València

In collaboration with B.Coll and J. J Ferrando

Introduction

Navigation systems (GPS, Galileo,...) use as starting point Newtonian conceptions, the role of the Relativity theory being banished to monitor and correct, by trial and error, the running of the system.

Some time ago (ERE-2000, Valladolid), B. Coll proposed a fully relativistic approach to positioning systems.

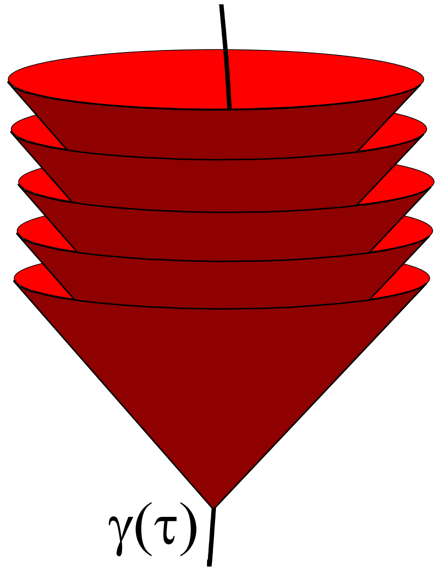
Progress in the theoretical understanding of the current navigation systems.

You should consider this talk to move forward in this proposal.

Relativistic positioning and emission coordinates

Clock (emitter) broadcasting its proper time τ by means of electromagnetic signals.

Foliation of future light cones $\tau = constant$ with vertices on the emitter world-line $\gamma(\tau)$



$$\gamma^A(\tau^A)$$

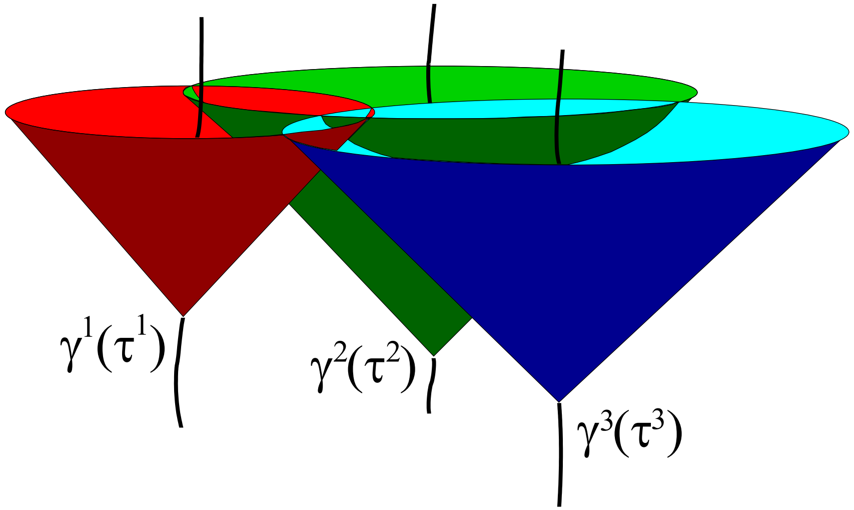
$$A = 1, 2, 3, 4$$

$$\{\tau^A\}$$

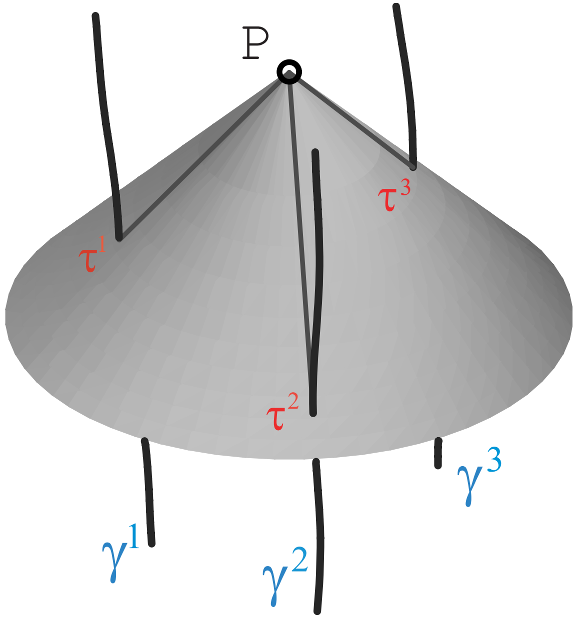
A relativistic positioning system is basically constituted by four such emitters.

At each event reached by the signals, the received four times define the emission coordinates of this event.

Emission coordinates (3-dimensional pictures)



Each event where three light cones intersect has emission coordinates $\{\tau^1, \tau^2, \tau^3\}$.



The past light cone of a given event P cuts the emitter world lines at $\gamma^A(\tau^A)$.

$\{\tau^A\}$ are the emission coordinates of the event P.

Emission coordinates may be defined in any space-time. The emitters needn't be synchronized.

B. Coll and J. M. Pozo (2006) Class. Quantum Grav. **23**, 7395
Relativistic positioning systems: the emission coordinates

- The coordinate covectors of emission coordinates, $d\tau^A$, are null and future-directed. They are subject to the coordinate inequality condition:

$$d\tau^1 \wedge d\tau^2 \wedge d\tau^3 \wedge d\tau^4 \neq 0$$

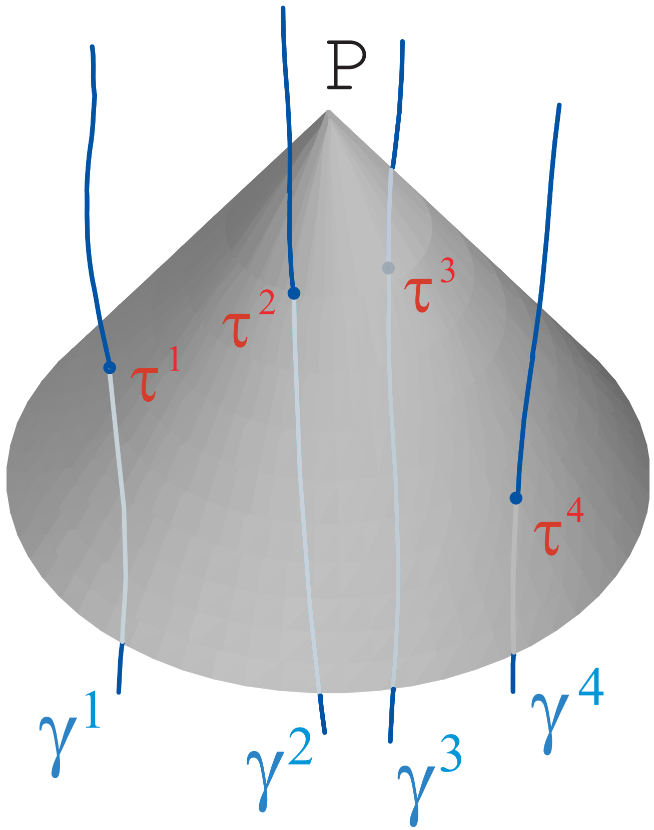
- Contravariant metric in emission coordinates [metric signature $(+ - - -)$].

$$(g^{AB}) = \begin{pmatrix} 0 & g^{12} & g^{13} & g^{14} \\ g^{12} & 0 & g^{23} & g^{24} \\ g^{13} & g^{23} & 0 & g^{34} \\ g^{14} & g^{24} & g^{34} & 0 \end{pmatrix} \quad , \qquad g^{AB} > 0 \quad \text{for} \quad A \neq B$$

- The coordinate vectors $\frac{\partial}{\partial \tau^A}$ of emission coordinates are space-like.
- The coordinate 2-planes spanned by each pair of coordinate vectors of emission coordinates are space-like.

The aim of this talk

Here we study emission coordinates in Minkowski space-time when the word-lines of the emitters, $\gamma^A(\tau^A)$, are known.



Goal: To find the inertial coordinates $\{x^\alpha\}$ of a given event P in terms of its emission coordinates $\{\tau^A\}$,

$$x^\alpha = x^\alpha(\tau^A)$$

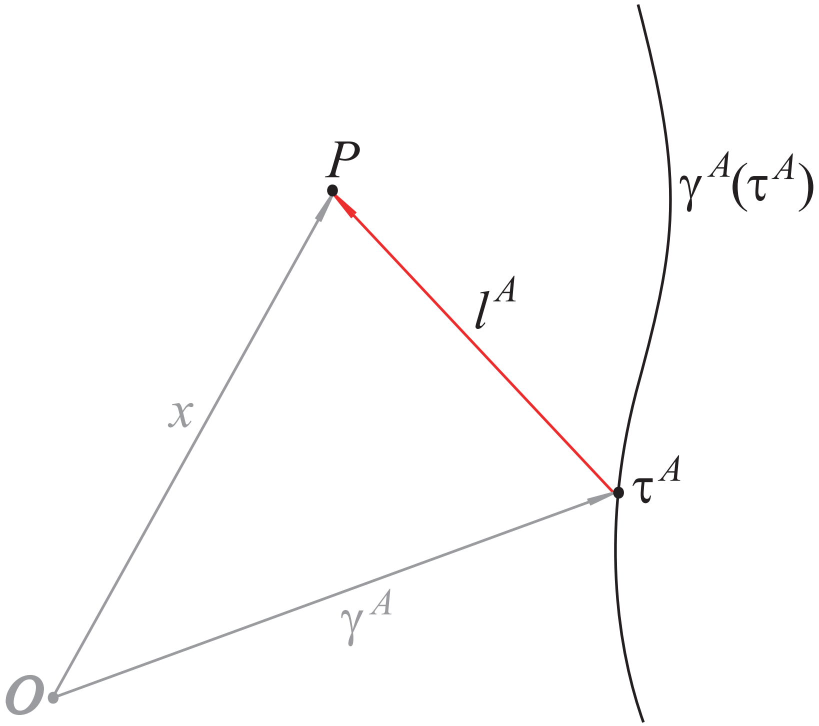
Talk plan

1. The main relations of a positioning system in flat space-time
2. The transformation Emission $\{\tau^A\} \longleftrightarrow$ Inertial $\{x^\alpha\}$
   (for any choice of the four emitters trajectories)

For special movements of the emitters see:

D. Bini, A. Geralico, M. L. Ruggiero, A. Tartaglia, arXiv:0809.0998.

The null propagation equations



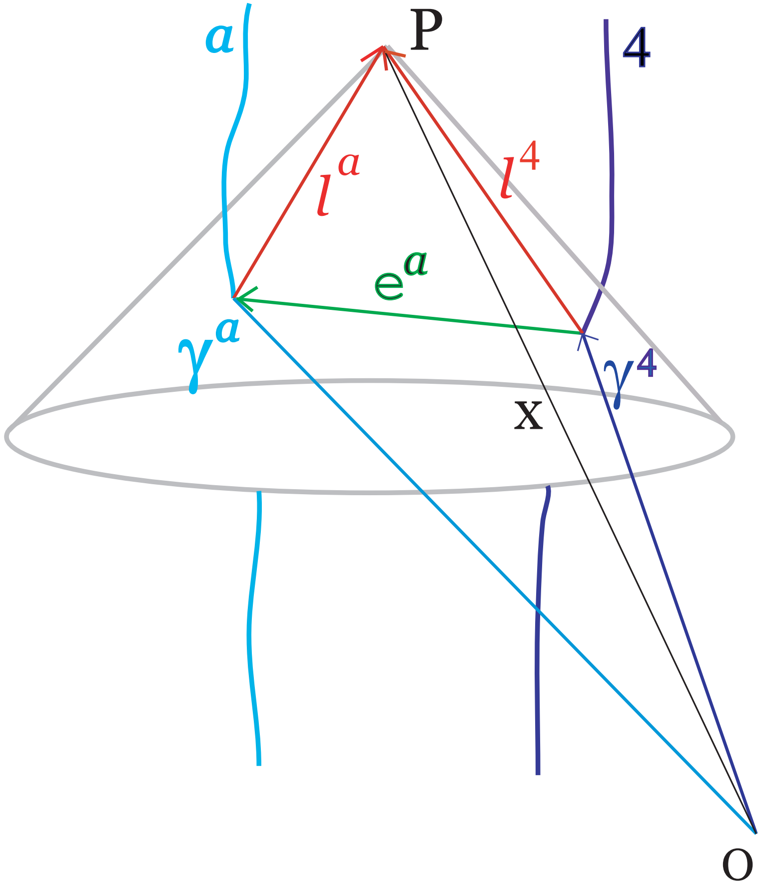
$$l^A = x - \gamma^A \quad A = 1, 2, 3, 4$$

null and future pointing

Null propagation equations

$$(x - \gamma^A) \cdot (x - \gamma^A) = 0\,, \quad \forall A$$

Splitting the null propagation equations in an inertial chart



$$\mathbf{x} = \gamma^4 + l^4 \qquad \mathbf{y} = l^4$$

$$e^a = \gamma^a - \gamma^4 = l^4 - l^a$$

space-like

$(a = 1, 2, 3)$

$$\Omega^a = \frac{1}{2}(e^a)^2 < 0$$

$$(x - \gamma^A) \cdot (x - \gamma^A) = 0 \quad \Longleftrightarrow \quad \left\{ \begin{array}{ll} y^2 = 0 \ , & A = 4 \\ e^a \cdot y = \Omega^a \ , & A = a \end{array} \right.$$

The main relations of a positioning system

Null propagation equations:

$$(x-\gamma^A)\cdot(x-\gamma^A)=0 \quad \Longleftrightarrow \quad \left\{\begin{array}{ll} y^2=0\,, & A=4 \\ e^a\cdot y=\Omega^a\,, & A=a \end{array}\right.$$

Emission inequalities:

$$\forall A \quad \text{and for an arbitrary observer } u, \quad (x-\gamma^A)\cdot u>0$$

Coordinate inequality:

$$d\tau^1\wedge d\tau^2\wedge d\tau^3\wedge d\tau^4\neq 0 \Longleftrightarrow l^1\wedge l^2\wedge l^3\wedge l^4\neq 0 \Longleftrightarrow e^1\wedge e^2\wedge e^3\wedge l^4\neq 0$$

$$\Longrightarrow \quad \chi\equiv *(e^1\wedge e^2\wedge e^3)\neq 0$$

The general solution of the linear system

Proposition In the domain of emission coordinates, the general solution to the linear system

$$e^a \cdot y = \Omega^a \equiv \frac{1}{2}(e^a)^2$$

is of the form

$$y = y_* + \lambda\chi \ ,$$

where y_* is a particular solution, λ is a real parameter and $\chi \neq 0$

$$\chi \equiv *(e^1 \wedge e^2 \wedge e^3)$$

$*$ being the Hodge dual operator.

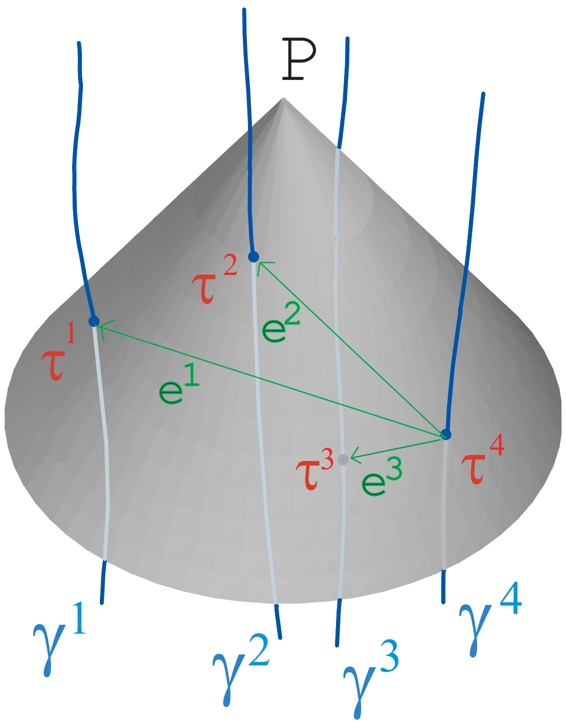
Note $\quad y^2 = 0 \iff \chi^2\lambda^2 + 2(\chi \cdot y_*)\,\lambda + y_*^2 = 0 \ .$

Internal configuration of the emitters for a given event

When $\{\tau^1, \tau^2, \tau^3, \tau^4\}$ are the emission coordinates of the event, the four events

$$\{\gamma^1(\tau^1), \gamma^2(\tau^2), \gamma^3(\tau^3), \gamma^4(\tau^4)\}$$

defines the internal configuration of the emitters for this event.



Take γ^4 as reference emitter,

$$e^a = \gamma^a - \gamma^4 \quad (a = 1, 2, 3)$$

$$e^1 \wedge e^2, \quad e^1 \wedge e^3, \quad e^2 \wedge e^3$$

$$e^1 \wedge e^2 \wedge e^3, \quad \chi \equiv *(e^1 \wedge e^2 \wedge e^3)$$

$$\chi^2 = [*(e^1 \wedge e^2 \wedge e^3)]^2$$

Regions of the emission coordinate domain

$$\Upsilon = \Upsilon_s \cup \Upsilon_l \cup \Upsilon_t$$

$$\chi \equiv *(e^1 \wedge e^2 \wedge e^3)$$

$$\Upsilon_s \equiv \{(\tau^A) \,|\, \chi^2 > 0\}$$

$$\Upsilon_l \equiv \{(\tau^A) \,|\, \chi^2 = 0\}$$

$$\Upsilon_t \equiv \{(\tau^A) \,|\, \chi^2 < 0\}$$

- In order to obtain a sole expression relating emission and inertial coordinates in all the domain of emission coordinates, an external element ξ is also required,

Any ξ such that $\xi \cdot \chi \neq 0$

(One could take $\xi = \chi$ when $\chi^2 \neq 0$).

A particular solution of the linear system

Proposition In all the domain of emission coordinates, a particular solution to the linear system

$$e^a \cdot y = \Omega^a \equiv \frac{1}{2}(e^a)^2 \ , \quad a = 1, 2, 3$$

is given by

$$y_* \equiv \frac{1}{\xi \cdot \chi} \, i(\xi) H \ ,$$

where the bivector H is, like the vector $\chi \equiv *(e^1 \wedge e^2 \wedge e^3)$, a function of the internal configuration of the emitters,

$$H \equiv *(\Omega^1 \, e^2 \wedge e^3 + \Omega^2 \, e^3 \wedge e^1 + \Omega^3 \, e^1 \wedge e^2)$$

and where ξ is any vector transversal to χ, $\xi \cdot \chi \neq 0$.

(This particular solution y_* is the one that is orthogonal to the chosen ξ, $\xi \cdot y_* = 0$)

The solution taking γ^4 as reference emitter

Proposition The transformation from emission coordinates to inertial ones is given by

$$y = y_* + \lambda \chi$$

where y_*, χ and λ are given in terms of the emitter configuration as

$$y_* = \frac{1}{\xi \cdot \chi} i(\xi) H, \quad H \equiv *(\Omega^1 \, e^2 \wedge e^3 + \Omega^2 \, e^3 \wedge e^1 + \Omega^3 \, e^1 \wedge e^2)$$

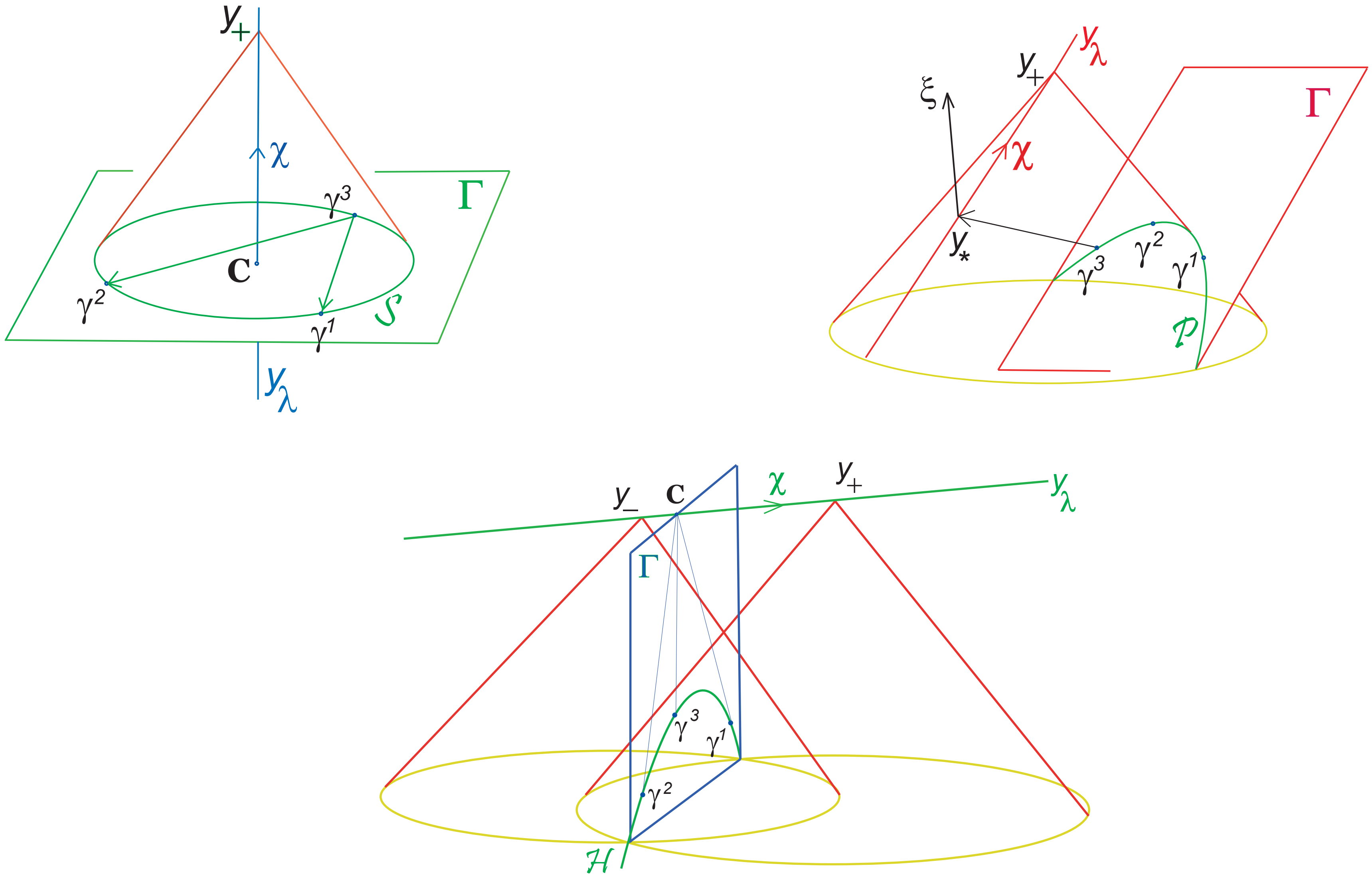
$$\Omega^a = \frac{1}{2}(e^a)^2 , \qquad e^a = \gamma^a - \gamma^4 , \qquad \chi \equiv *(e^1 \wedge e^2 \wedge e^3)$$

$$\lambda = -\frac{y_*^2}{y_* \cdot \chi \pm \sqrt{\Delta}}, \qquad \Delta \equiv (y_* \cdot \chi)^2 - \chi^2 y_*^2$$

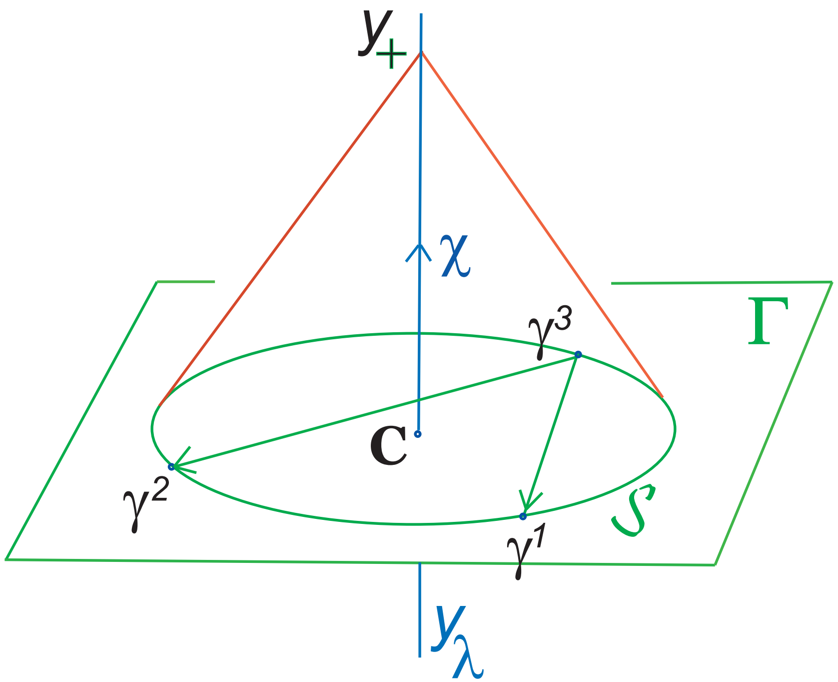
and where ξ is any vector transversal to χ, $\xi \cdot \chi \neq 0$.

<u>Consistence</u> $\dfrac{\partial y}{\partial \xi^\mu} \wedge \chi = 0 \,$, the change in ξ may be absorbed by λ.

Three configurations (3-dimensional pictures)



Space-like configuration (3-dimensional picture)



y_λ: straight line of equidistant points from $\gamma^A(\tau^A)$

Γ: 3-plane defined by $\gamma^A(\tau^A)$

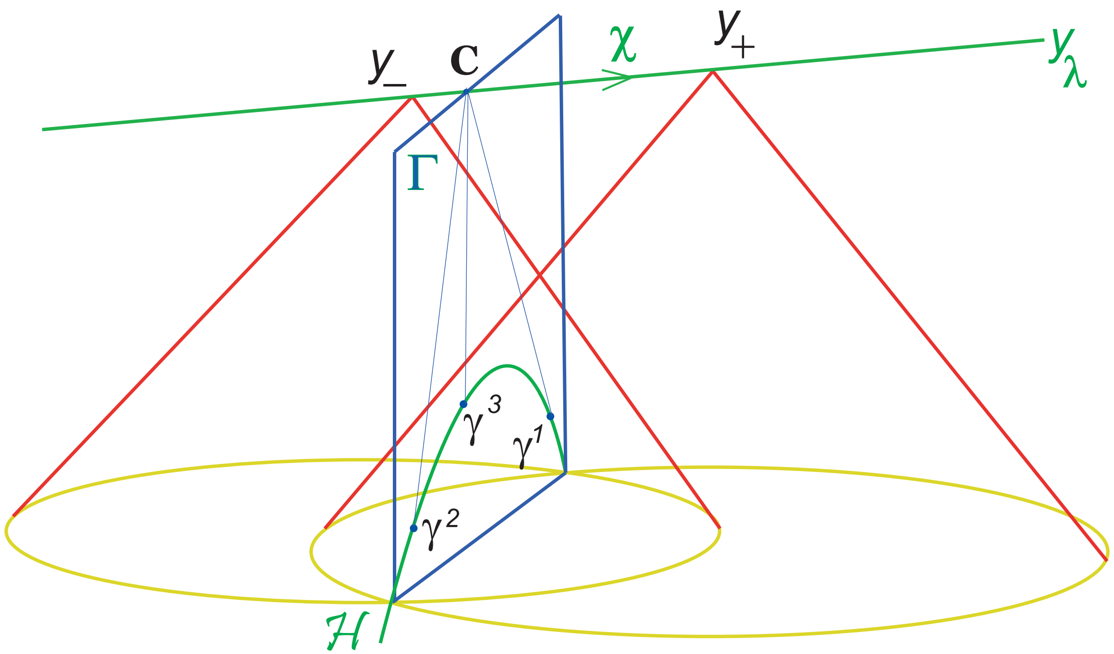
$$y_\pm = \mathbf{C} \mp \frac{\mathbf{C^2}}{\sqrt{-\chi^2\, \mathbf{C^2}}}\,\chi$$

Take y_+ when χ is future-oriented
Take y_- when χ is past-oriented

$$y_* \equiv \frac{1}{\xi \cdot \chi}\; i(\xi) H \;, \qquad y_* = \mathbf{C} \quad \text{when} \quad \xi \equiv \chi$$

$$H \equiv *(\Omega^1\, e^2 \wedge e^3 + \Omega^2\, e^3 \wedge e^1 + \Omega^3\, e^1 \wedge e^2)$$

Time-like configuration (3-dimensional picture)

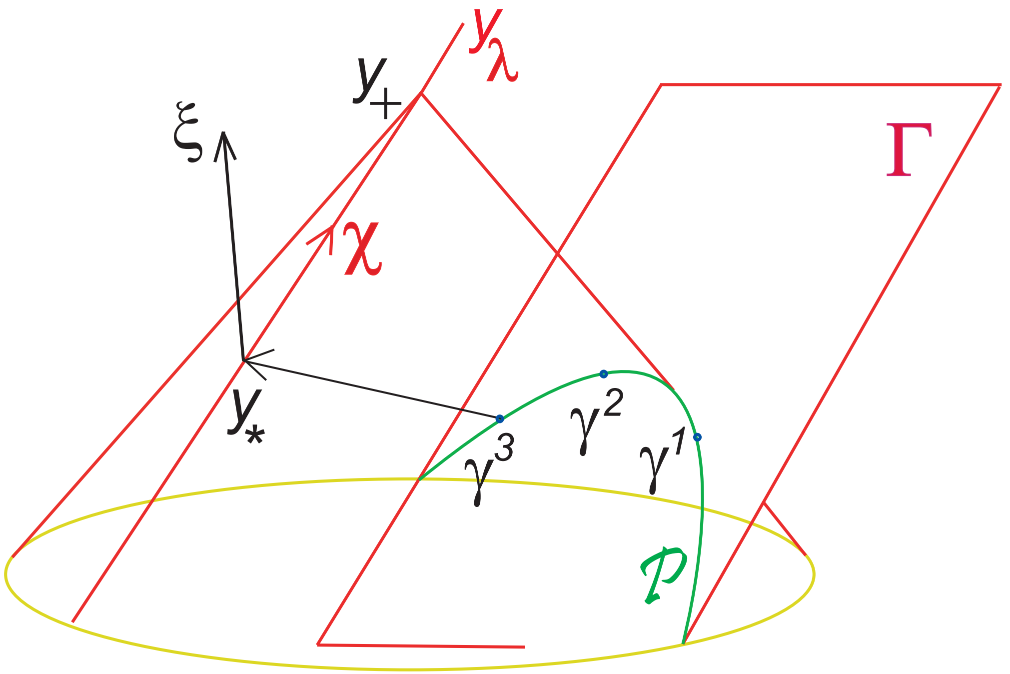


$$y_\pm = \mathbf{C} \mp \frac{\mathbf{C^2}}{\sqrt{-\chi^2\,\mathbf{C^2}}}\,\chi$$

$$y_* \equiv \frac{1}{\xi \cdot \chi}\, i(\xi) H \;, \qquad y_* = \mathbf{C} \quad \text{when} \quad \xi \equiv \chi$$

$$H \equiv *(\Omega^1\, e^2 \wedge e^3 + \Omega^2\, e^3 \wedge e^1 + \Omega^3\, e^1 \wedge e^2)$$

Light-like configuration (3-dimensional picture)



$$\xi \cdot \chi \neq 0, \qquad y_* \cdot \xi = 0$$

$$y_+ = y_* - \frac{(y_*)^2}{2(\chi \cdot y_*)}\,\chi$$

$$y_* \equiv \frac{1}{\xi \cdot \chi}\; i(\xi)H \;,$$

$$H \equiv *(\Omega^1\, e^2 \wedge e^3 + \Omega^2\, e^3 \wedge e^1 + \Omega^3\, e^1 \wedge e^2)$$

A general expression for the solution

Proposition The solution of the main positioning system in Minkowski space-time may be written as

$$x = \gamma + y_* + \lambda\chi$$

where γ, χ, y_* and λ are given in terms of the emitter configuration as

$$\gamma \equiv \frac{1}{4}\epsilon_A \gamma^A \;\; \text{(barycenter)}, \quad \chi \equiv -\frac{1}{4!}\epsilon^A \epsilon_{ABCD} * (\gamma^B \wedge \gamma^C \wedge \gamma^D)\,,$$

$$y_* = \frac{1}{\xi \cdot \chi} i(\xi) H, \quad H \equiv \frac{1}{4}(\epsilon^A \Omega^{BA}) E_B, \quad E_B \equiv \frac{1}{2}\epsilon_{BACD} * (e_A^C \wedge e_A^D)\,,$$

$$\Omega^{AB} = \frac{1}{2}(e_B^A)^2, \quad e_B^A = \gamma^A - \gamma^B\,,$$

$$\lambda = -\frac{y_*^2 + \Omega}{y_* \cdot \chi \pm \sqrt{\Delta}}, \quad \Omega \equiv \frac{1}{8}\sum_{A<B}\Omega^{AB}, \quad \Delta \equiv (y_* \cdot \chi)^2 - \chi^2 (y_*^2 + \Omega)$$

and where ξ is any vector transversal to χ, $\xi \cdot \chi \neq 0$.

Last comments

1. Emission coordinates have a simple definition and an easy realization.
2. The coordinate transformation from emission to Cartesian coordinates we have presented could deserve some attention in practical applications.

   a) First of all, it is possible to decompose all the above expressions relatively to an arbitrary observer, applying standard $3+1$ methods.

   b) From the received times, the users of a relativistic positioning system are able to locate themselves with respect an inertial frame.

   c) One can use the above transformation to express the flat metric in emission coordinates,

$$g_{AB} = \frac{\partial x^\mu}{\partial \tau^A} \frac{\partial x^\nu}{\partial \tau^B} \eta_{\mu\nu} \,, \qquad x^\alpha(\gamma^A(\tau^A)) \,.$$

3. The control sector of the current navigation systems is based on the Earth surface. But, in a fully relativistic scheme, this is a secondary sector which might be referred to the satellite constellation, according with the Coll proposal. Our result is very preliminary, but open the door for more investigation, for example, by taking into account the Earth gravitational field. Bini and collaborations have make some progress in this direction (see the aforementioned arXiv-paper). But more people should be involved in relativistic positioning system in order to gain more and more theoretical and practical advances in this area.