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On the local existence of maximal slicings in spherically symmetric spacetimes

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Abstract. In this talk we show that any spherically symmetric spacetime admits locally a maximal spacelike slicing. The above condition is reduced to solve a decoupled system of first order quasi-linear partial differential equations. The solution may be accomplished analytical or numerically. We provide a general procedure to construct such maximal slicings.

1. Introduction

A *maximal* hypersurface is one that has vanishing mean extrinsic curvature, $K = 0$, K being the trace of the extrinsic curvature of the hypersurface. The name comes from the fact that the induced volume functional reaches a local maximum with respect the variations that keep fixed a given boundary. Maximal hypersurfaces were considered by Lichnerowicz [1] to solve Einstein's constraint equations, giving motivation for subsequent studies on the subject (see, for example [2, 3, 4, 5]). In fact, the existence of maximal hypersurfaces is extensively used in Mathematical Relativity. This property is a very simple geometric assumption to establish general results for broad classes of spacetimes, for instance, local or asymptotically stationary or conformally flat spacetimes.

We will use the term *maximal slicing* when referring to a (non intersecting) family of spacelike maximal hypersurfaces which locally foliates a certain domain of spacetime. This type of slicing has very nice properties as, for example: i) the well-known singularity avoidance capability [6], ii) it is well adapted to the propagation of gravitational waves [7, 8], and, iii) it gives the natural Newtonian analogous when, in addition, conformal flatness is imposed on each slice [9]. Maximal slicing condition has been recently used in the Fully Constrained Formulation of Einstein equations derived by the Meudon group [10, 11].

In spite of their extended use, the existence of maximal slicings in spherically symmetric spacetimes (SSSTs) has been only established for vacuum and for some particular energy contents (see [9, 12, 13, 14, 15, 16, 17]). There is, as far as we know, no theorem stating that always it is possible to build a maximal slicings in a SSST.

In this work, we aim to prove the local existence of maximal slicings in any SSST. We will follow a purely geometrical approach, independent of Einstein equations, according to [14], complementary to the standard time evolution strategy [9, 12, 13, 15, 16, 17].

Although our study is independent of the field equations, one by-product of our approach, which could be of interest in the field of Numerical Relativity, is that it provides a means to

assess complex and sophisticated 3D numerical codes built to solve Einstein equations.

2. Local existence of maximal slicings

In this section we establish the following result:

Theorem. *Any spherically symmetric spacetime can be locally sliced by a family of maximal spacelike hypersurfaces.*

In order to prove the previous theorem, we derive a decoupled system of three first order partial differential equations that proves the local existence of a maximal slicings, and provides a general procedure allowing its construction.

Let us start with the canonical form of the metric of a SSST,

$$ds^2 = A dt^2 + 2C dt dr + B dr^2 + D d\Omega^2, \quad (1)$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the metric of the 2-sphere, A, B, C, D are smooth functions of t and r , and $AB - C^2 < 0$ to ensure the Lorentzian character of the metric. In addition we choose the signature $(-, +, +, +)$, and accordingly $D > 0$. Partial derivatives with respect to r will be denoted as $\frac{\partial f}{\partial r} = f'$, and with respect to t as $\frac{\partial f}{\partial t} = \dot{f}$. The spatial metric γ_{ij} induced on the hypersurfaces Σ_t , defined by $t = \text{constant}$, is $\gamma_{ij} = \text{diag}(B, D, D \sin^2\theta)$, where $B > 0$ since we are considering spacelike hypersurfaces. Let n be the future pointing timelike unit normal to the hypersurfaces Σ_t ,

$$n = \frac{1}{\alpha} \left(\frac{\partial}{\partial t} - \frac{C}{B} \frac{\partial}{\partial r} \right), \quad \alpha = \sqrt{\frac{C^2}{B} - A}. \quad (2)$$

The mean extrinsic curvature K of Σ_t is related with the expansion of n , $K = -\nabla \cdot n$, where ∇ is the covariant derivative with respect to the spacetime metric. In the given metric in Eq. (1), this relation is

$$K = \frac{1}{2\alpha B} \left(-\dot{B} - 2B \frac{\dot{D}}{D} + 2C' - C \frac{B'}{B} + 2C \frac{D'}{D} \right). \quad (3)$$

In the following, we assume that A, B, C and D are known functions. We look for a change of coordinates $\{\tilde{t} = \tilde{t}(t, r), \tilde{r} = \tilde{r}(t, r), \theta, \varphi\}$ such that the hypersurfaces $\tilde{t} = \text{constant}$ are maximal. We introduce two fields, X and Y , satisfying the commutation relation $[X, Y] = 0$. This condition assures the existence of two coordinate parameters, namely $\tilde{t} = \tilde{t}(t, r)$ and $\tilde{r} = \tilde{r}(t, r)$, such that

$$X = \frac{\partial}{\partial \tilde{t}}, \quad Y = \frac{\partial}{\partial \tilde{r}}. \quad (4)$$

Then, we decompose these fields as $Y = \lambda \bar{Y}$, $X = a\bar{Y} + b\bar{Y}^\perp$, with $\bar{Y}^2 = 1$, $\bar{Y} \cdot \bar{Y}^\perp = 0$, $b \neq 0$, and $\lambda > 0$. The condition $\bar{Y}^2 = 1$ is equivalent to

$$\bar{Y} = f \frac{\partial}{\partial t} + P \frac{\partial}{\partial r}, \quad P = B^{-1} \left(-fC + \epsilon \sqrt{f^2 l^2 + B} \right), \quad (5)$$

being f an unknown function to be determined, $\epsilon = \pm 1$ and $l^2 = -AB + C^2 > 0$. Fixing the coefficient of $\frac{\partial}{\partial t}$ in the decomposition of \bar{Y}^\perp , $\bar{Y} \cdot \bar{Y}^\perp = 0$ leads to

$$\bar{Y}^\perp = \frac{\partial}{\partial t} + Q \frac{\partial}{\partial r}, \quad Q = B^{-1} \left(-C + \frac{\epsilon f l^2}{\sqrt{f^2 l^2 + B}} \right). \quad (6)$$

Consequently, the resulting fields are

$$X = (af + b)\frac{\partial}{\partial t} + (aP + bQ)\frac{\partial}{\partial r}, \quad Y = \lambda \left(f\frac{\partial}{\partial t} + P\frac{\partial}{\partial r} \right) = \lambda \left(f\alpha n + \frac{\epsilon}{B}\sqrt{f^2 l^2 + B}\frac{\partial}{\partial r} \right), \quad (7)$$

where we have taken into account Eqs. (2) and (5). The condition $[X, Y] = 0$ is then equivalent to

$$\left[\frac{P}{bp} \right]' = - \left[\frac{f}{bp} \right]', \quad - \left[\frac{aP + bQ}{bp\lambda} \right]' = \left[\frac{af + b}{bp\lambda} \right]', \quad (8)$$

where $p = P - fQ = \frac{\epsilon}{\sqrt{f^2 l^2 + B}} \neq 0$.

Now, we denote with \tilde{K} the trace of the extrinsic curvature of the new hypersurfaces $\tilde{t} = \text{constant}$. The condition $\tilde{K} = 0$ and the commutation relation provide 3 equations for 4 unknown functions, a, b, f, λ . Taking into account that D is a scalar under the above change of coordinates, we can add, without loss of generality, the following coordinate condition

$$\tilde{r}^2 Y^2 = D, \quad (9)$$

saying that the metric on the hypersurfaces $\tilde{t} = \text{constant}$ is written in isotropic conformally flat form. From Eqs. (3) and (9), the condition $\tilde{K} = 0$ is equivalent to

$$2Y(X \cdot Y) - 3X(Y^2) + \left[\frac{4}{\tilde{r}} + \frac{Y(Y^2)}{Y^2} \right] X \cdot Y = 0. \quad (10)$$

From the decompositions (7), Eqs. (9) and (10) are expressed as $\lambda = \sqrt{D}/\tilde{r}$, that can be viewed as a definition of \tilde{r} in terms of λ , and as

$$f\dot{a} + Pa' - a \left(\frac{f\dot{\lambda} + P\lambda'}{\lambda} - \frac{2}{\sqrt{D}} \right) = 3b \frac{\dot{\lambda} + Q\lambda'}{\lambda}. \quad (11)$$

After some algebraic calculations, the previous definition of λ and Eqs. (8) and (11) are equivalent to

$$a = b\sqrt{D} \left(\frac{\dot{\lambda} + Q\lambda'}{\lambda} - \frac{\dot{D} + QD'}{2D} \right), \quad (12)$$

$$f\frac{\dot{\lambda}}{\lambda} + P\frac{\lambda'}{\lambda} = f\frac{\dot{D}}{2D} + P\frac{D'}{2D} - \frac{1}{\sqrt{D}}, \quad (13)$$

$$f\frac{\dot{b}}{b} + P\frac{b'}{b} = P' - P\frac{p'}{p} + \dot{f} - f\frac{\dot{p}}{p}, \quad (14)$$

and

$$\frac{\dot{p}}{p} - \frac{\dot{D}}{D} - Q' + Q \left[\frac{p'}{p} - \frac{D'}{D} \right] = 0. \quad (15)$$

Notice that Eq. (15) involves only f when p and Q are written explicitly in terms of f .

First, Eq. (15) can be solved for f . Second, Eqs. (13) and (14) can be solved for λ and b . Finally, a can be obtained from Eq. (12). Assuming that A, B, C, D are continuously differentiable functions, the initial value problem with respect to this set of equations has always local (both in space and time) solution [18] (which is also continuously differentiable). Therefore, we have proved the announced theorem.

Notice that in order to solve this set of equations, it can be useful to distinguish two different cases, $f = 0$ and $f \neq 0$. In the case of $f = 0$, Eq. (15) is reduced to $K = 0$, and the rest of equations can be integrated easily. In the case of $f \neq 0$, it can be defined the variable

$F = \frac{\epsilon f}{\sqrt{f^2 l^2 + B}} \Leftrightarrow f = \epsilon F \sqrt{\frac{B}{1 - l^2 F^2}}$, and Eqs. (13), (14) and (15) can be rewritten as a hyperbolic system of equations for F , λ and b .

3. Conclusions

Two basic results have been displayed: i) A theorem ensuring the existence of maximal slicings in any SSST. ii) A geometrical method to build up such slices by solving three decoupled first order quasi-linear partial differential equations (13), (14) and (15). The first result aims to fill a theoretical gap in the scientific literature. The second one tries to achieve an algorithmic procedure to obtain maximal slicings. An interesting by-product for Numerical Relativity of the approach presented in this paper has to do with the assessment of 3D codes written, as customary, in Cartesian coordinates. Let us consider two codes NC1 and NC2 such that only NC1 uses a gauge which is maximal. Hence, the evolution with code NC2 of any initial data admitting a spherically symmetric limit could be compared to the evolution produced by code NC1, by simply using our procedure to generate a SSST satisfying the maximal slicing condition.

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