Complete intersections with isolated singularities (ICIS)

Algebraic methods and singularities

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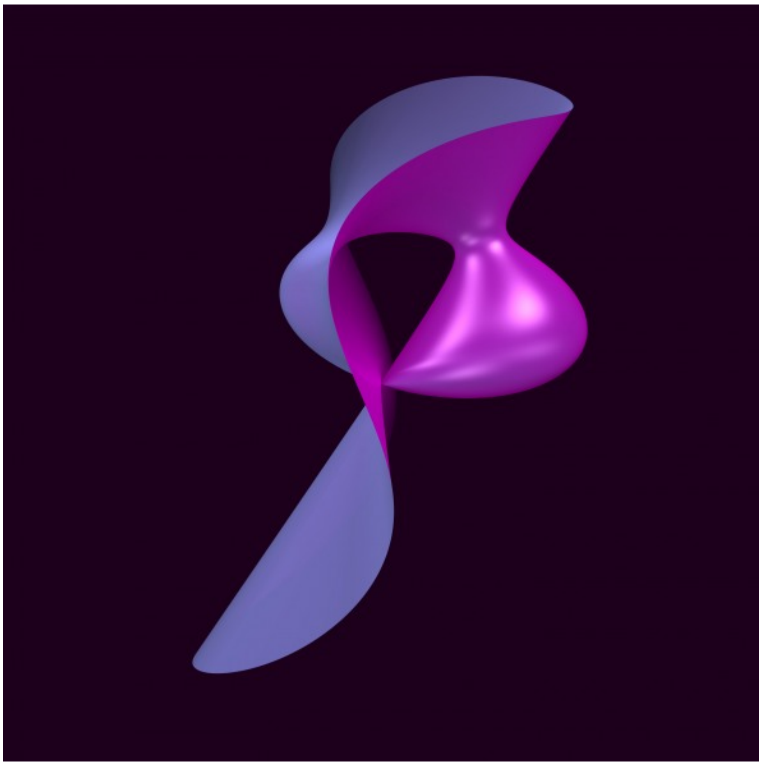

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Contents

Introduction

This is the second half of the course on ”Singularities and Algebraic Methods". The first half of the course was about hypersurfaces and was taught in 2021 in the Part 1 of the CIMPA Research School "Singularities and Applications" (online version). We will assume the reader is already familiar with the contents of that part.

In this second half we will focus on complete intersections with isolated singularities (ICIS). The complete intersections are a natural generalisation of hypersurfaces in the following sense: a hypersurface singularity (X, x) is set germ in $\mathbb{C}^{n+1}$ defined as the zero locus of a single non-constant holomorphic function $f\colon (\mathbb{C}^{n+1}, x) \to (\mathbb{C}, 0)$. This forces that (X, x) must have dimension n. If we consider now k holomorphic functions $f = (f_1, \dots, f_k)\colon (\mathbb{C}^{n+k}, x) \to (\mathbb{C}^k, 0)$ then in general it is not true that the zero locus (X, x) of f has also dimension n. But when this happens we say that (X, x) is a complete intersection.

In the hypersurface case, the algebraic methods to study the classification and the main invariants are provided by the $\mathscr{R}$-equivalence of smooth functions $f\colon (\mathbb{K}^{n+1}, 0) \to (\mathbb{K}, 0)$ where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\mathscr{R}$ is the Mather's group of right equivalences. We saw in the first half of the course how the infinitesimal methods can be applied to describe the tangent space to the $\mathscr{R}$-orbit and hence, to obtain algebraic criteria for finite determinacy, which is a crucial step in the classification process.

In the case of ICIS, we introduce the Mather's contact group $\mathscr{K}$ which acts on the space of smooth map germs $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$. When $\mathbb{K} = \mathbb{C}$ and $n \geq p$, we will see that there exists a one-to-one correspondence between the isomorphism classes of ICIS (X, x) and the $\mathscr{K}$-equivalence classes of map germs $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. Again, the infinitesimal machinery will enter into action to provide algebraic methods for the classification and the invariants of these singularities.

The course is organised into 4 lectures. In the first lecture, we will review some basic results about commutative algebra and local analytic geometry that will be needed for the course. We will give the precise definition of ICIS. We also introduce the contact group $\mathscr{K}$ and the notion of $\mathscr{K}$-equivalence of map germs $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$. The main result will be that two map germs f and g are $\mathscr{K}$-equivalent if and only if their local algebras $Q(f)$ and $Q(g)$ are isomorphic.

The second lecture is dedicated to finite determinacy and versality of unfoldings of maps for the contact group $\mathscr{K}$ and for map germs $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$. We will present first the infinitesimal criterion of finite $\mathscr{K}$-determinacy, which says that f is finitely $\mathscr{K}$-determined if and only if its $\mathscr{K}_e$-codimension is finite. In the complex case, finite $\mathscr{K}$-determinacy is equivalent to that f is finite-to-one on its critical locus, which is known as the geometric criterion of Mather-Gaffney. Then, we will introduce the complete transversal method, which is very useful to obtain classifications under $\mathscr{K}$-equivalence. The last part of this lecture is about $\mathscr{K}$-versality of unfoldings of maps. The main result is that the r-parameter unfolding F is versal if and only if it is transversal to the $\mathscr{K}$-orbit, that is, the residue classes of the partial derivatives of F with respect to the parameters generate the quotient $\theta(f)/T\mathscr{K}_e f$ over $\mathbb{K}$. Then, some consequences are obtained about the relationship between $\mathscr{K}$-versality and $\mathscr{A}$-stability.

In the third lecture we will study versal deformations of ICIS and also the topology of the generic fibres. Here we restrict ourselves to the complex case $\mathbb{K} = \mathbb{C}$ and consider an ICIS (X, x) given as the fibre of a $\mathscr{K}$-finite map germ $f\colon (\mathbb{C}^{n+k}, x) \to (\mathbb{C}^k, 0)$. We will see how the notion of versal deformation of the ICIS (X, x) is strongly related to the $\mathscr{K}$-versal unfoldings of f. Then, we will deduce some interesting properties of the discriminant of a versal unfolding. We will introduce the link of an ICIS (X, x) and also the Milnor fibration. As in the case of an hypersurface with isolated singularity, the Milnor fibre has the homotopy type of a wedge of n-spheres and the number of such spheres is called the Milnor number, denoted by $\mu(X, x)$. We will prove a theorem of Gaffney and Hauser that gives a criterion to prove that two ICIS are isomorphic in terms of the modules of infinitesimal deformations.

Finally, in the fourth lecture we will present an introduction to equisingularity, with applications to the case of ICIS. We will present Whitney's conditions (a) and (b) as well as Verdier's W condition. Next, we will give basic definitions and properties about integral closure of ideals and modules and their connection with equisingularity. In the last section, we will give an application to characterise the Whitney equisingularity of families of ICIS in terms of multiplicities and also the constancy of the μ^*-sequence.

São Carlos, July 2022.

Part I

Lecture 1

Chapter 1

Preliminary results

1.1 Necessary algebraic results

We will assume that all the rings have identity and thta if $\varphi : R \to S$ is a homomorphism of rings then $\varphi(1_R) = 1_S$.

Definition 1.1.1. A ring R is called Noetherian if every ideal in R is finitely generated.

Lemma 1.1.2. *Let R be a ring. Then the following conditions are equivalent:*

1. *R is Noetherian.*

2. *Any chain of ideals in R*
$$I_1 \subset I_2 \subset \cdots \subset I_k \subset \cdots$$
becomes stationary, that is, there exists an $n > 0$ such that $I_k = I_n$ for all $k \geq n$.

3. *Every nonempty set of ideals in R has a maximal element with respect to inclusion.*

Proof. See, for instance, [6, Lemma 1.1.2]. □

Definition 1.1.3. A ring R is called *local* if it has an unique maximal ideal, $\mathfrak{m}$. One often says that $(R, \mathfrak{m})$ is a local ring to indicate that $\mathfrak{m}$ is its unique maximal ideal.

Lemma 1.1.4. *Let R be a ring and $\mathfrak{m} \subset R$ be an ideal. Then R is local, with maximal ideal $\mathfrak{m}$, if and only if $R\backslash\mathfrak{m}$ is the set of the units of R.*

Proof. Assume that R is local with maximal ideal $\mathfrak{m}$. Take an element $a \in R$. If $(a) \neq R$, then a belongs to a maximal ideal, therefore $a \in \mathfrak{m}$. That is, if $a \notin \mathfrak{m}$ then a is a unit. Moreover, if $a \in \mathfrak{m}$ than$a$ is not a unit, as otherwise $(a) = R$.

Suppose, on the other hand, that $R\backslash\mathfrak{m}$ is the set of units. Take an ideal $I \neq R$. Then the ideal I does not contain units, and therefore must be contained in $\mathfrak{m}$. This shows that $\mathfrak{m}$ is the unique maximal ideal of R. □

Theorem 1.1.5. *[6, Theorem 1.3.4] (Nakayama's Lemma). Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module with $\mathfrak{m} \cdot M = M$. Then $M = 0$.*

Proof. We assume that $M \neq 0$. Let t be the minimal number of generators of M and $m_1, m_2, \ldots, m_t$ be a set of generators of M. Since $\mathfrak{m}M = M$, there exist $a_1, \ldots, a_t \in \mathfrak{m}$ such that

$$m_t = a_1 m_1 + a_2 m_2 + \cdots + a_t m_t.$$

Therefore

$$(1 - a_t) m_t = a_1 m_1 + a_2 m_2 + \cdots + a_{t-1} m_{t-1}.$$

But, since $1 - a_t \notin \mathfrak{m}$ and R is local then $1 - a_t$ is a unit and, then, m_t can be generate by $m_1, m_2, \ldots, m_{t-1}$. Hence, M is generated by $m_1, m_2, \ldots, m_{t-1}$, in contradiction with the minimality of generators of M. □

Corollary 1.1.6. *[6, Corollary 1.3.5] (Krull's Intersection Theorem). Let $(R, \mathfrak{m})$ be a Noetherian local ring, and M be a finitely generated R-module. Then*

$$\cap_{k \in \mathbb{N}} \mathfrak{m}^k M = (0).$$

Proof. We will assume that $M = R$. The general case is analogous. We write $I = \cap_{k \in \mathbb{N}} \mathfrak{m}^k$. By the Nakayama's Lemma, we just need to show that $\mathfrak{m}I = I$. We consider the set

$$A = \{J \text{ ideal in } R : J \cap I = \mathfrak{m}I\}.$$

Clearly $\mathfrak{m}I \in A$ and, since R is Noetherian then A has a maximal element, which we call J.

We claim that there exist a $k \in \mathbb{N}$ such that $\mathfrak{m}^k \subset J$. In fact, since $\mathfrak{m}$ is finitely generated, we just need to show that for each $f \in \mathfrak{m}$, there exists $\alpha \in \mathbb{N}$ such that $f^\alpha \in J$. For each $f \in \mathfrak{m}$ we consider the chain of ideals

$$J : f \subset J : f^2 \subset \ldots.$$

Since the ring is Noetherian, this chains stabilizes, that is, there exists α such that $J : f^\alpha \subset J : f^{\alpha+1}$. This is the α we were looking for:

$$x \in (J + (f^\alpha)) \cap I \Rightarrow x = y + af^\alpha \in I, \text{ with } y \in J,\ a \in R \Rightarrow af^{\alpha+1} = fx - fy \in \mathfrak{m}I + J = J,$$

hence $a \in J : f^{\alpha+1} = J : f^\alpha$and then $x \in I \cap J = \mathfrak{m}I$. Therefore, since the other inclusion is trivial, $(J + (f^\alpha)) \cap I = \mathfrak{m}I$. Since J is maximal in A, $(J + (f^\alpha)) = J$, which completes the proof of the claim.

To conclude the proof we observe that $I \subset \mathfrak{m}^k \subset J$ and then $I \subset J \cap I = \mathfrak{m}I$. □

Another important corollary of the Nakayama's Lemma is the following lemma on the number of generators of a module over a local ring.

Corollary 1.1.7. *[6, Corollary 1.3.6] Let M be a finitely generated module over a local ring $(R, \mathfrak{m})$.*

1. *Let $f_1, \ldots, f_s \in M$ such that the classes of the f_i generate $M/\mathfrak{m}M$ as $R/\mathfrak{m}$-vector space. Then $f_1, \ldots, f_s$ generate M.*

2. *The minimal number of generators of M is equal to* $\dim_{R/\mathfrak{m}}(M/\mathfrak{m}M)$.

In particular, if $(R, \mathfrak{m})$ be a Noetherian local ring. The *embedding dimension*, $\mathrm{edim}(R)$, of R is defined by

$$\mathrm{edim}(R) := \dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2).$$

Definition 1.1.8. An analytic algebra (also called analytic $\mathbb{C}$ - algebra) is a $\mathbb{C}$ - algebra of type $\mathbb{C}\{x_1, \dots, x_n\}/I$, where I is an ideal in $\mathbb{C}\{x_1, \dots, x_n\}$.

Lemma 1.1.9. *Let (R, m_R), (S, m_S) be analytic $\mathbb{C}$-algebras and $\varphi : R \to S$ a morphism of $\mathbb{C}$-algebras. Then $\varphi(m_R) \subset m_S$.*

Proof. In fact, if not then there exist $f \in \mathfrak{m}_R$ such that $\varphi(f)(0) = c \neq 0$. Hence $\varphi(f)(0)-c \in \mathfrak{m}_R$ is not a unit. Since it is a ring homomorphism, φ sends units in R to units in S, therefore $\varphi(f)(0) - c$ is not a unit. However, φ is also a $\mathbb{C}$-vector space homomorphism and then $\varphi(f) - c = \varphi(f - c)$ which is a unit. □

Definition 1.1.10. The height of a prime ideal P of a ring R, ht(P), is the maximum of the length, n, of a chain of strict inclusions

$$P_0 \subset P_1 \subset P_2 \cdots \subset P_n = P.$$

where all of the P_i are prime ideals of R. The Krull dimension of a ring R is the maximum of the heights of prime ideals in R. Usually this is simply called the dimension of R. The height of a not necessarily prime ideal I is defined to be the minimum of the heights of the prime ideals containing I. The (Krull) dimension of a module, M, dim M, is the Krull dimension of the ring $R/Ann_R(M)$, where $Ann_R(M) = \{r \in R : rm = 0, \ \forall m \in M\}$.

We say that a Noetherian local ring , R, is *regular* if edim$R$ = dim R.

Definition 1.1.11. [6, Definition 6.5.1] Let $(R, \mathfrak{m})$ be a local ring and M be an R-module.

1. A sequence $f_1, \dots, f_r$ of elements in $\mathfrak{m}$ is called a regular sequence of M if f_1 is not a zerodivisor of M, and f_i is not a zerodivisor of $M/(f_1, ..., f_{i-1})M$ for $i = 2, \dots, r$.

2. Let $I \subset IR$ be an ideal with $IM \neq M$. Then the I-depth of M, depth(I, M) is the maximal length of a regular sequence of M in I. If $IM = M$ we define depth$(I, M) = \infty$.

3. The depth of M, depth(M) is the maximal length of a regular sequence of M, that is, depth(M) = depth(m, M). If we want to emphasize the ring R, we will write depth$_R(M)$.

Over a Noetherian local ring, it is always true that

$$\text{depth}(M) \leq \dim M.$$

When the equality holds, we say that the module M is *Cohen-Macaulay*. The ring R is called Cohen-Macaulay, if R is a *Cohen-Macaulay* R-module.

Let R be a ring. An element $x \in R$ is called nilpotent if there exists an $n \in \mathbb{N}$ with $x^n = 0$. A ring is called *reduced* if it has no nonzero nilpotent elements.

One can show that a ring R is reduced if and only if it satisfies the Serre's conditions R_0 and S_1,

R_0: R_p is regular for every prime ideal p of R with height equals to zero.

S_1: depth $R_p \geq \min\{\text{ht}(P), 1\}$, for every prime ideal p of R.

Here R_p denotes the localization of R on $R - p$.

1.2 The local ring $\mathcal{O}_n$

Let N and P be real or complex manifolds of dimensions n and p respectively, and $x \in N$. In the set of smooth (C^∞ in real case or holomorphic in complex case) mappings defined in a neighborhood of x in N into P we introduce the following equivalence relation:

Definition 1.2.1. Two mappings $f_1 : U_1 \to P$ and $f_2 : U_2 \to P$ are equivalent if there exists a neighborhood $U \subset U_1 \cap U_2$ of x in N such that $f_1 |_U$ and $f_2 |_U$ coincide.

An equivalence class under this relation is called *germ of mapping or map-germ* at x and is denoted by $f : (N, x) \to (P, y)$, $y = f(x)$. An element of an equivalence class is called *representative of the germ.*

Let $f : (N, x) \to (P, y)$ be a map-germ at x. Its *derivative* $df_x : T_xN \to T_yP$ is defined as the derivative at x of any representative of the germ. We say that $f : (N, x) \to (P, y)$ is a *germ of a diffeomorphism* if one of its representative (and so any) is a local diffeomorphism. It follows from the Inverse Function Theorem that a map-germ at x is a germ of a diffeomorphism if and only if its derivative at x is an isomorphism.

Let $f : (N, x) \to (P, y)$ and $g : (P, y) \to (M, z)$ be two map-germs where M is a manifold and $z \in M$. We define the composition $g \circ f : (N, x) \to (M, z)$ as: take representatives $\tilde{f} : U \to P$ and $\tilde{g} : V \to M$, $\tilde{f}(U) \subset V$, of f and g respectively, the map-germ $g \circ f$ is the equivalence class of $\tilde{g} \circ \tilde{f}$.

The *rank* of a map-germ at x is defined as the rank of its derivative at x. When the rank is n the map-germ is *immersive* and when the rank is p the map-germ is *submersive*. We say that the map-germ is *singular* when it is neither immersive nor submersive.

Definition 1.2.2. Two map-germs $f_1 : (N_1, x_1) \to (P_1, y_1)$ and $f_2 : (N_2, x_2) \to (P_2, y_2)$ are $\mathscr{A}$-equivalent if there exist germs of diffeomorphisms $h : (N_2, x_2) \to (N_1, x_1)$ and $k : (P_2, y_2) \to (P_1, y_1)$ such that the following diagram commutes:

$$\begin{array}{ccc} (N_1, x_1) & \xrightarrow{f_1} & (P_1, y_1) \\ h \uparrow & & k \uparrow \\ (N_2, x_2) & \xrightarrow{f_2} & (P_2, y_2) \end{array}$$

that is, $f_1 = k \circ f_2 \circ h^{-1}$ (or $f_1 \circ h = k \circ f_2$).

Note that since any map-germ $f : (N, x) \to (P, y)$ is $\mathscr{A}$-equivalent to some germ $g : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we consider only smooth map-germs $(\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$.

We denote by $\mathcal{O}_{n,p}$ the set of smooth map-germs $f : (\mathbb{K}^n, 0) \to \mathbb{K}^p$. When $p = 1$ we denote it by $\mathcal{O}_n$.

$\mathcal{O}_n$ is a local ring whose maximal ideal is $\mathfrak{m}_n = \{f \in \mathcal{O}_n : f(0) = 0\}$.

1.3 Germ of Complex Analytic Sets

We present here a small introduction about germs of analytic spaces. The text is based on the Chapter 3 of [6]. See also [19].

Definition 1.3.1. 1. A set $X \subset \mathbb{C}^n$ is called *locally analytic* if, for each $p \in X$, there exists an open subset V of $\mathbb{C}^n$ with $p \in V$ and finitely many holomorphic functions $f_1, \ldots, f_s$ such that

$$X \cap V = \{x \in V : f_1(x) = \cdots = f_s(x) = 0\}.$$

2. Let U be an open subset of $\mathbb{C}^n$. A subset $X \subset U$ is called an analytic subset of U, if X is locally analytic and closed in U.

Given holomorphic functions $f_1, \ldots, f_s : V \subset \mathbb{C}^n \to \mathbb{C}$, with V open, we denote the analytic subset of V defined as the zeroset of $f_1, \ldots, f_s$ by $V(f_1, \ldots, f_s)$. That is

$$V(f_1, \ldots, f_k) = \{x \in V : f_1(x) = \cdots = f_s(x) = 0\}.$$

Example 1.3.2. 1. $\mathbb{C}^n = V(0)$ and $\emptyset = V(1)$ are analytic subsets of $\mathbb{C}^n$.

2. The set

$$X := \{(x, y) \in \mathbb{C}^2 : y = 0, \mathrm{Im}(x) \geq 0\}$$

is not locally analytic. In fact, assume that there exist a connected open subset V of $\mathbb{C}^2$ with $(0, 0) \in V$ and holomorphic functions $f_1, \ldots, f_s$ defined on V such that

$$X \cap V = \{(x, y) \in V : f_1(x, y) = \cdots = f_s(x, y) = 0\}.$$

Therefore $X \cap V \subset \{(x, y) \in V : f_1(x, y) = 0\}$. We denote $g_i(y) = f_i(x, 0)$, $i = 1, \ldots, s$. By the Identity Theorem ([6, Theorem 3.1.9]) in one variable, $g_i|_{V \cap \{y=0\}}$ is identically equal to 0. Hence $\{(x, y) \in V : y = 0\} \subset X \cap V$.

Definition 1.3.3. 1. Let $f_1, \ldots, f_n$ be holomorphic on an open subset $U \subset \mathbb{C}^n$. Let $p \in U$, and suppose $f_1(p) = \cdots = f_n(p) = 0$. The set $\{f_1, ..., f_n\}$ is called a set of coordinate functions at p if $\det\left(\frac{\partial f_j}{\partial x_i}(p)\right) \neq 0$.

2. A subset $X \subset \mathbb{C}^n$ is called a complex submanifold of $\mathbb{C}^n$ if for every $x \in X$ there exists an open subset U in $\mathbb{C}^n$ and coordinate functions $w_1, ..., w_n$ of x such that

$$X \cap U = \{y \in U : w_1(y) = \cdots = w_m(y) = 0\}$$

for some $m \leq n$.

Definition 1.3.4. Let $U \subset \mathbb{C}^n$ be an open set and $X \subset U$ an analytic subset. A point $x \in X$ is called regular, or X is called smooth at x, if there exists an open subset V in $\mathbb{C}^n$ with $x \in V$ such that $X \cap V$ is a complex submanifold of $\mathbb{C}^n$. If $x \in X$ is not regular, then x is called singular, or a singularity. The set of singular points of X is denoted by $\mathrm{Sing}(X)$.

In order to study local properties, we introduce the notion of germs of set and of analytic space.

Definition 1.3.5. 1. Let X be a topological space and $p \in X$. Two subsets A and B of X are called equivalent at p if there exist an open neighborhood U of p such that $A \cap U = B \cap U$.

The equivalence class of a subset A is called *germ* of A at p and denoted by (A, p).

2. Let (A, x) and (B, x) be germs. We define $(A, x) \subset (B, x)$ if there are representatives A of (A, x) and B of (B, x) such that $A \subset B$.

3. Let $(A, x) \subset (Y, x)$ and $(B, x) \subset (Y, x)$ be germs. Then we define $(A, x) \cap (B, x)$ to be the germ of $A \cap B$ at x for any representative A of (A, x) and B of (B, x). We define $(A, x) \cup (B, x)$ to be the germ of $A \cup B$ at x for any representative A of (A, x) and B of (B, x).

4. A germ of an complex space (X, x) is a germ at x of a locally analytic subset of $\mathbb{C}^n$.

In particular, if $f \in \mathcal{O}_{n,x}$ then we define the germ of analytic hypersurface $(V(f), x)$ as follows. Consider an open neighborhood U of x on which f converges and the analytic subset

$$V(f) = \{p \in U : f(p) = 0\}$$

of U. The germ $(V(f), x)$ is called the zero set of f.

Moreover, if $I = (f_1, \dots, f_s) \subset \mathcal{O}_{n,x}$ is an ideal then the germ of complex space of I is

$$(V(I), x) = \cap_{i=1}^{s} (V(f_i), x).$$

This definition is independent of the generators of the ideal I.

Since the ring $\mathcal{O}_{n,x}$ is Noetherian, any germ of analytic space is a germ of type $(V(I), x)$ for some ideal I of $\mathcal{O}_{n,x}$.

Let (X, x) be a germ of a complex space. The ideal of (X, x)

$$\mathscr{I}(X, x) = \{f \in \mathcal{O}_{n,x} : (X, x) \subset (V(f), x)\}.$$

The inclusion here is an inclusion of germs. So $f \in I(X, x)$ if there exists a representative X of (X, x) and an open neighborhood U of x such that X is an analytic subset of U, f converges on U, and its restriction to X is the zero function.

Remark 1.3.6. When we write germ of analytic space, we are usually referring to germ of set (X, x) together with its local algebra, $\mathcal{O}_{n,x}/I$, where I is the ideal which defines X.

A *germ of analytic space* is a germ of complex space $(X, 0)$ defined by a radical ideal. That is, the local algebra $\mathcal{O}_n/I$ is reduced, where I is the ideal which defines $(X, 0)$

For instance, let $I = (x^2) \subset \mathbb{C}\{x\}$. Then $(V(I), 0)$ is the point 0. In this case, I defines a germ of a complex space $\mathcal{O}_{X,x} = \mathbb{C}\{x\}/I$.

Associated to such a germ of a complex space (X, x) is its reduction (X_{red}, x), with local ring $\mathcal{O}_{X_{red},x}$ the reduction of $\mathcal{O}_{X,x}$, obtained by dividing out the nilpotent elements.

Example 1.3.7. Let $X = V(xy, y^2) \subset \mathbb{C}^2$. Then $(X, 0)$ is the analytic complex space with ring $\mathcal{O}_2/(xy, y^2)$. Here X is the union of the x-axis with the fat point with support $\{0\}$ (see [19, Example 1.38.2]. We associate to it the (reduced) analytic variety $(X, 0)$ with local ring $(\mathcal{O}_2/(xy, y^2))/(y) = \mathcal{O}_2/(y)$

Next, we see how the definitions the ideal of an analytic space and the zero set of an ideal relate.

Proposition 1.3.8. *Let (V, x) and (W, x) be germ of analytic spaces in $(\mathbb{C}^n, x)$, (A, x) be a germ of set in $(\mathbb{C}^n, x)$ and let I and J be ideals of $\mathcal{O}_{n,x}$.*

1. *$\mathscr{I}(A, x)$ is a radical ideal.*

2. *$(V, x) \subset (W, x) \Rightarrow \mathscr{I}(W, x) \subset \mathscr{I}(W, x)$.*

3. *$I \subset J \Rightarrow (V(J), x) \subset (V(I), x)$.*

4. *$(V(IJ), x) = (V(I \cap J), x) = (V(I), x) \cup (V(J), x)$. In particular, finite unions of analytic spaces are analytic spaces.*

5. *$(V(I + J), x) = (V(I), x) \cap (V(J), x)$. In particular, finite intersections of analytic spaces are analytic spaces.*

6. *$\mathscr{I}((V, x) \cup (W, x)) = \mathscr{I}(V, x) \cap \mathscr{I}(W, x)$.*

It is not difficult to see that for any ideal $I \subset \mathscr{O}_n$, $I \subset \mathscr{I}(V(I))$. The other inclusion is not true. However, there is a very important result called the **Nullstellensatz** theorem which says that $\mathscr{I}(V(I)) = \sqrt{I}$. The prove can be found at [6, Theorem 3.4.4] and [19, Theorem 1.72], for instance.

Definition 1.3.9. Let (X, x) be a germ of an analytic space. We say that (X, x) is *irreducible* if from $(X, x) = (X_1, x) \cup (X_2, x)$, with (X_1, x) and (X_2, x) germs of analytic spaces it follows that either $(X, x) = (X_1, x)$ or $(X, x) = (X_2, x)$.

Proposition 1.3.10. *Let (X, x) be a germ of an analytic space.*

1. *(X, x) is irreducible if and only if $\mathscr{I}(X, x)$ is a prime ideal.*

2. *There is a, up to permutation, unique decomposition $(X, x) = (X_1, x) \cup$ÂůÂůÂů$\cup (X_r, x)$, with (X_i, x) irreducible and $(X_j, x) \not\subset (X_i, x)$ for $i \neq j$. This is called the irreducible decomposition of (X, x). The (X_i, x) are called irreducible components of (X, x).*

Proof. 1. We assume that (X, x) is irreducible. Let $f, g \in \mathscr{O}_{n,x}$ such that $fg \in \mathscr{I}(X)$. Then

$$(X, x) = (V(\mathscr{I}(X)), x) \supset (V(fg), x) = (V(f), x) \cup (V(g), x).$$

Hence, by the hypothesis, $(X, x) = (X, x) \cap (V(f), x)$ or $(X, x) = (X, x) \cap (V(g), x)$. So either $(X, x) \subset (V(f), x)$ or $(X, x) =\subset (V(g), x)$, which means that either $f \in \mathscr{I}(X)$ or $g \in \mathscr{I}(X)$.

On the other hand, we assume that $\mathscr{I}(X, x)$ is a prime ideal. If there exist germs of analytic sets (X_1, x) and (X_2, x) such that $(X, x) = (X_1, x) \cup (X_2, x)$ then, by the Nullstellensatz theorem

$$\mathscr{I}(X) = \mathscr{I}(V(I(X))) = \mathscr{I}(X_1, x) \cap \mathscr{I}(X_2, x).$$

Since $\mathscr{I}(X, x)$ is a prime ideal, either $\mathscr{I}(X, x) = \mathscr{I}(X_1, x)$ or $\mathscr{I}(X, x) = \mathscr{I}(X_2, x)$. Therefore, either $(X, x) = (X_1, x)$ or $(X, x) = (X_2, x)$.

2. Let $\mathscr{I}(X, x) = I_1 \cap \cdots \cap I_r$ be an irredundant primary decomposition of $\mathscr{I}(X, x)$. Then

$$(X, x) = V(\mathscr{I}(X, x), x) = (V(I_1), x) \cup \cdots \cup (V(I_r), x)$$

and, for $i = 1, \ldots, r$, $(X_i, x) := (V(I_i, x), x)$ is irreducible by item 1. Moreover this decomposition is unique up to permutation by the uniqueness of the primary decomposition of ideals and $(X_j, x) \not\subset (X_i, x)$ for $i \neq j$ because the primary decomposition of the ideal was no redundant.

□

The following table summarizes the relationships between radical ideas and germs of analytic spaces.

Ideals		**analytic spaces**
radical ideals		analytic spaces
I	$\longrightarrow$	$V(I)$
$\mathscr{I}(V)$	$\longleftarrow$	V
inclusion of ideals		inclusion of analytic spaces
$I \subset J$	$\longrightarrow$	$V(I) \supset V(J)$
$\mathscr{I}(V) \supset \mathscr{I}(W)$	$\longleftarrow$	$V \subset W$
addition of ideals		intersection of analytic spaces
$I + J$	$\longrightarrow$	$V(I) \cap V(J)$
product of ideals		union of analytic spaces
IJ	$\longrightarrow$	$V(I) \cup V(J)$
$\sqrt{\mathscr{I}(V)\mathscr{I}(W)}$	$\longleftarrow$	$V \cup W$
intersection of ideals		union of analytic spaces
$I \cap J$	$\longrightarrow$	$V(I) \cup V(J)$
$\mathscr{I}(V) \cap \mathscr{I}(W)$	$\longleftarrow$	$V \cup W$
prime ideals	$\longleftrightarrow$	irreducible analytic spaces
minimal decomposition		minimal decomposition
$I = P_1 \cap P_2 \cdots \cap P_m$	$\longrightarrow$	$V(I) = V(P_1) \cup V(P_2) \cdots \cup V(P_m)$
$\mathscr{I}(V) = \mathscr{I}(V_1) \cap \mathscr{I}(V_2) \cap \cdots \cap \mathscr{I}(V_m)$	$\longleftarrow$	$V = V_1 \cup V_2 \cup \cdots \cup V_m$

Exercises

1. Prove Proposition 1.3.8.

1.3.1 The local algebra of a germ of analytic space

Definition 1.3.11. Let (X, p) and (Y, q) be two germs of topological spaces. A germ of a continuous map $f : (X, p) \to (Y, q)$ is defined as an equivalence class of maps $f : U \to W$, with $f(p) = q$, and where U and W are representatives of (X, p) and (Y, q) respectively. Two such maps $f_1 : U_1 \to W$ and $f_2 : U2 \to W$ are called equivalent if they agree on an open neighborhood of p contained in $U_1 \cap U_2$.

Let $(X, x) \subset (\mathbb{C}^n, x)$ and $(Y, y) \subset (\mathbb{C}^m, y)$ be germs of an analytic space. A *germ of an analytic map* $\varphi : (X, x) \to (Y, y)$ is a germ of a map $\varphi : (X, x) \to (Y, y)$ such that some representative is the restriction to X of an analytic function on an open neighborhood of x in $\mathbb{C}^n$. When $Y = \mathbb{C}$, we say that φ is a germ of analytic function.

We denote by $\mathscr{O}_{X,x}$ the set of the germ of analytic functions $f : (X, x) \to \mathbb{C}$. Germs of analytic functions can be added and multiplied, so $\mathscr{O}_{X,x}$ has the structure of a commutative $\mathbb{C}$-algebra. In fact, it is a local algebra whose maximal ideal, $\mathfrak{m}_{X,x}$, is the subset of germs $h \in O_{X,x}$ such that $h(x) = 0$. This is called the *local algebra* of (X, x).

Lemma 1.3.12. *1. Let $(X, x) \subset (\mathbb{C}^n, x)$ be a germ of an analytic space, and $\mathscr{I}(X, x)$ be the ideal of X. Then $\mathscr{O}_{X,x} = \mathscr{O}_{n,x}/\mathscr{I}(X, x)$.*

2. Let $(X, x) \subset (\mathbb{K}^n, x)$ be a germ of a submanifold. Then $O_{X,x} \approx \mathbb{K}\{x_1, ..., x_k\}$ for some k.

3. *Let $\varphi : (X, x) \to (Y, y)$ be a germ of an analytic mapping. Then by composition φ induces a map of $\mathbb{K}$-algebras*
$$\begin{array}{rccc} \varphi^* : & \mathscr{O}_{Y,y} & \to & \mathscr{O}_{X,x} \\ & f & \mapsto & f \circ \varphi \end{array}$$

We say that a germ of an analytic map $\varphi : (X, x) \to (Y, y))$ is a isomorphism if it has a two-side inverse which is also a germ of a analytic map. If there exists such a isomorphism, we say that the germs of analytic spaces (X, x) and (Y, y) are isomorphic. Our goal now is to show that two germs of analytic spaces are isomorphic if and only if their local algebra are isomorphic.

Theorem 1.3.13. *Let $(X, x) \subset (\mathbb{C}^n, x)$ and $(Y, y) \subset (\mathbb{C}^m, y)$ be germs of analytic spaces. Let $\alpha : \mathscr{O}_{(Y,y)} \to \mathscr{O}_{(X,x)}$ be a $\mathbb{C}$-algebra homomorphism. Then there exists a unique germ of an analytic mapping $\varphi : (X, x) \to (Y, y)$ with $\varphi^* = \alpha$.*

Proof. Without loss of generality, we may assume that $x = 0$ and $y = 0$.

By Lemma 1.1.9 $\alpha(\mathfrak{m}_{(Y,0)}) \subset \mathfrak{m}_{(X,0)}$. Therefore, $\alpha(\mathfrak{m}_{(Y,0)^k}) \subset \mathfrak{m}^k_{(X,0)}$, for all $k > 0$.

Take $w_1, \ldots, w_m$ generators of the maximal ideal of $\mathscr{O}_m$. Let $\overline{w_i}$ be the class of w_i in $\mathscr{O}_m/\mathscr{I}(Y, 0)$ and $f_i = \alpha(\overline{w_i})$, $i = 1, \ldots, m$. We define

$$\varphi = (f_1, \ldots, f_m) : (X, 0) \to (\mathbb{C}^m, 0).$$

Hence we get a map $\varphi^* : \mathscr{O}_m \to \mathscr{O}_{X,0}$.

1. $\varphi^* = \tilde{\alpha}$, where $\tilde{\alpha}$ is the composition $\mathscr{O}_m \to \mathscr{O}_{(Y,0)} \xrightarrow{\alpha} \mathscr{O}_{(X,0)}$. In fact, for each $i = 1, \ldots, k$, $\varphi^*(w_i) = \tilde{\alpha}(w_i)$. Since they are both $\mathbb{C}$-algebra homomorphism it follows that $\varphi^*(g) = \tilde{\alpha}(g)$ for all polynomial $g \in \mathbb{C}[w_1, \ldots, w_m]$. And given $g \in \mathscr{O}_m$, we can write
$$g = g_k + g'_k,$$
where g_k is a polynomial of degree smaller then k and $g'_k \in \mathfrak{m}^k_m$. Then
$$\varphi^*(g) - \tilde{\alpha}(g) = \varphi^*(g_k`) - \tilde{\alpha}(g'_k) \in \mathfrak{m}^k_{(X,0)}.$$
Since it holds for any $k > 0$, $\varphi^*(g) - \tilde{\alpha}(g)) \in \cap_k \mathfrak{m}^k_{(X,0)} = \{0\}$, by the Krull's intersection theorem.

2. $\varphi(X, 0) \subset (Y, 0)$. In fact, since $(Y, 0) = (V(\mathscr{I}(Y, 0)), 0)$, it is enough to observe that for all $g \in \mathscr{I}(Y, 0)$ the map $g \circ \varphi : (X, 0) \to \mathbb{C}$ is the zero map because $\mathscr{I}(Y, 0)$ is in the kernel of φ^* by the construction of $\tilde{\alpha}$.

□

Corollary 1.3.14. *Two germs of analytic spaces (X, x) and (Y, y) are isomorphic if and only if $\mathscr{O}_{(X,x)}$ and $\mathscr{O}_{(Y,y)}$ are isomorphic.*

Proof. The Nullstellensatz says that there is a 1-1 correspondence between reduced analytic algebras and germs of analytic spaces. The previous theorem implies that the isomorphism classes of reduced analytic algebras correspond to isomorphism classes of germs of analytic spaces □

1.3.2 Dimension

Definition 1.3.15. Let (X, x) be a germ of a complex space and $(\mathscr{O}_{X,x}, \mathfrak{m})$ be its local ring.

1. The *Krull dimension* of (X, x) is the Krull dimension of its local ring, that is, the maximal length k of chains of prime ideals
$$p_0 \subsetneq p_1 \subsetneq p_k$$
in $\mathscr{O}_{X,x}$.
2. The Chevalley dimension of (X, x) is the least number of generators for an $\mathfrak{m}$-primary ideal of $\mathscr{O}_{(X,x)}$.
3. The Weierstrass dimension of (X, x) is the least number k, such that there exists a Noether normalization $\mathscr{O}_k \subset \mathscr{O}_{(X,x)}$ of (X, x) (see [6, Corollary 3.3.19] for the definition of Noether normalization).

Remark 1.3.16. All the three dimensions in the previous definition coincide for a germ of analytic variety. See [6, Section 4.1].

By taking zero sets, it follows that the Krull dimension of (X, x) is the supreme of the length n of a chain of irreducible subvarieties of (X, x):
$$0 \subsetneq (X_1, x) \subsetneq \ldots \subsetneq (X_n, x) \subseteq (X, x).$$

Example 1.3.17. For the case $\mathscr{O}_{X,x} = \mathscr{O}_n$, consider the following chain of prime ideals:
$$(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \ldots \subsetneq (x_1, \ldots, x_n)$$
from which it follows that the Krull dimension of $(\mathbb{C}^n, 0)$ is at least n. It needs proof, and is in fact nontrivial that the dimension of $(\mathbb{C}^n, 0)$ is indeed equal to n.

Theorem 1.3.18. *Suppose* (X, x) *is a germ of a complex space. Let* $(X, x) = (X_1, x) \cup \cdots \cup (X_r, x)$ *be an irreducible decomposition of* (X, x)*. Then*
$$\dim(X, x) = \max\{\dim(X_i, x), i = 1, \ldots, r\}.$$

1.3.3 Finite germs of maps

In general, it is not possible to talk about the image of a germ of a map. For instance, let $f : \mathbb{R}^2 \to \mathbb{R}^2$ the map defined by $f(x, y) = (x, xy)$ and consider its germ at the origin. For each real positive number r, the sets $X_r = \{(x, y) \in \mathbb{R}^2 : -r \leq x \leq r, -r \leq y \leq r\}$ and $Y_r = \{(x, y) \in \mathbb{R}^2 : -r \leq x - y \leq r\}$ may be taken as representatives of $(\mathbb{R}^2, 0)$ but, for any open set U of $\mathbb{R}^2$, $U \cap f(X_r) \neq U \cap f(Y_r)$. For a special kind of maps, the finite maps, this problem does not occur.

Definition 1.3.19. Let $f : X \to Y$ be a continuous map between topological spaces.

1. f is *closed* if the image $f(A) \subset Y$ is closed for all closed subspaces $A \subset X$.

2. f is *quasi-finite* if for all $p \in Y$ the fiber $f^{-1}(p)$ consists of a finite number of points.

3. f is called *finite* if it is both closed and quasi-finite.

Example 1.3.20. [6, Lemma 3.4.10] Let

$$P = y^r + a_1 y^{r-1} + \cdots + a_{r-1} y + a_r$$

be a polynomial, with coefficients $a_i \in \mathscr{O}_k$. Consider an open neighborhood $U \subset \mathbb{C}^k$ of 0 such that the power series $a_1, \ldots, a_r$ converge. Define:

$$X := \{P = 0\} \cap (U \times \mathbb{C}).$$

Then the canonical projection $\pi : X \to U$ is finite.

Definition 1.3.21. Let (X, p) and (Y, q) be two germs of topological spaces. A germ of continuous map $f : (X, p) \to (Y, q)$ is called *finite* if it has a finite representative.

Lemma 1.3.22. *Let X and Y be topological spaces and $f : X \to Y$ be a closed map. Let $p \in X$ and $q = f(p)$. Assume that $f^{-1}(q) = \{p\}$. Let the $A, B \subset X$ with $p \in A \cap B$ such that $(A, p) = (B, p)$. Then $(f(A), q) = (f(B), q)$.*

Proof. Let W be an open set in X such that $p \in W$ and $A \cap W = B \cap W$. Since f is closed then the set $f(X \backslash W)$ is a closed subset of Y and $T := Y \backslash f(X \backslash W)$ is an open subset of Y. We will show that $q \in T$ and $A \cap T = B \cap T$.

Since f is continuous, the set $U = f^{-1}(T)$ is an open subset of X. Moreover

1. $U = f^{-1}(f(U))$,

2. $U \subset W$ and

3. $p \in U$.

Besides, $f(U \cap A) = T \cap f(A)$. In fact,

$$x \in U \cap A \Rightarrow f(x) \in f(U) \cap f(A) \subset T \cap f(A),$$

on the other hand,

$$a \in T \cap f(A) \Rightarrow a = f(b), \text{ with } b \in A \cap f^{-1}(T) = A \cap U \Rightarrow a \in f(U \cap A).$$

Similarly, $f(U \cap B) = T \cap f(B)$. And $U \cap A = U \cap B$. Hence,

$$T \cap f(B) = f(U \cap B) = f(U \cap A) = T \cap f(A).$$

□

We remark that if $f : (X, p) \to (Y, q)$ is a finite map germ we can always choose a finite representative $f : X \to Y$ such that the hypothesis $f^{-1}(f(p)) = \{p\}$ is satisfied. In fact, since for any representative $f^{-1}(f(p))$ is a finite number of points, just reduce the neighborhoods if necessary. With this, we are ready to define the image of a finite map germ.

Definition 1.3.23. Let $f : (X, p) \to (Y, q)$ be a finite germ of a continuous map. The image of f is defined by $\mathrm{Im}(f) := (f(X), q)$, where $f : X \to Y$ is a finite representative such that $f^{-1}(f(p)) = \{p\}$.

When f is finite, it is called surjective if $\mathrm{Im}(f) = (Y, p)$.

The following result is known as Remmert's proper mapping theorem. For a proof see [27, Chapter V].

Theorem 1.3.24. *The image $(f(X), y)$ of any finite analytic map-germ $f : (X, x) \to (Y, y)$ is analytic.*

Theorem 1.3.25. *[36, Lemma D3] Let $f : (X, x) \to (Y, y_0)$ be a finite surjective analytic map-germ with (Y, y_0) irreducible. There exists a hypersurface $(D, y) \subset (Y, y_0)$ such that for all small enough representatives Y of (Y, y_0) and D of (D, y_0), the fibre $f^{-1}(y)$ has constant cardinality for all $y \in D$.*

The constant cardinality from the preview theorem is called the *degree* of the map f and denoted $\deg(f)$.

Theorem 1.3.26. *Let $f : (X, x) \to (\mathbb{C}^d, 0)$ be a finite surjective map-germ. If (X, x) is Cohen-Macaulay then*

$$\deg(f) = \dim_{\mathbb{C}} \frac{\mathscr{O}_X}{(f_1, \dots, f_d)},$$

where $(f_1, \dots, f_d)$ is the ideal generated by the coordinates functions of f.

Proof. See, for instance, [36, Corollary D6]. □

1.3.4 The singular locus and the Jacobian Criterion

Definition 1.3.27. [36, Definition D1] Let (X, x) be a germ of complex space. We say that (X, x) is smooth or that x is a regular point of X if (X, x) is isomorphic (as a complex space-germ) to $(\mathbb{C}^d, 0)$, for some d. Otherwise, we say that (X, x) is singular.

The above definition of regular point means that $\mathscr{O}_{X,x}$ is isomorphic to $\mathscr{O}_d$ so we necessarily must have $d = \dim(X, x_0)$. We fix representatives of X and the functions f_i on some open subset $U \subset \mathbb{C}^n$ such that X is given by the vanishing of f_i on U. Then, it makes sense to consider the set Σ of points $x \in X$ such that X is not regular at x. The set-germ (Σ, x_0) is called the singular locus of (X, x_0) and is denoted by $\mathrm{Sing}(X, x_0)$.

We can characterize the regular points by means of the Jacobian Criterion.

Theorem 1.3.28. *Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a germ of complex space, and let the ideal of $(X, 0)$ be generated by $f_1, \dots, f_s \in C\{x_1, \dots, x_n\}$. We denote by $rank_0(f_1, \dots, f_s)$ the rank of the Jacobian matrix*

$$\left(\frac{\partial f_i}{\partial x_j}(0)\right)_{1 \le i \le s,\, 1 \le j \le n}.$$

Then

$$edim(X, 0) + rank_0(f_1, \dots, f_s) = n,$$

where $edim(X, x) := edim(\mathscr{O}_{(X,x)})$.

Remark 1.3.29. Let X be an analytic subset of an open subset U of $\mathbb{C}^n$, say locally defined by holomorphic functions $f_1, \ldots, f_s$ on U. Suppose that for all $x \in X$, the germ (X, x) has dimension $n - c$. It follows directly from the Jacobian Criterion that the singular locus of X is contained in the zero set of the c-minors of the Jacobian matrix, which is an analytic set. In fact, it is true that these sets are equal but the proof depends on the Coherence Theorem of Oka-Cartan, the which is very advanced for the purpose of these notes.

Example 1.3.30. Let $(X, 0) \subset (\mathbb{C}^4, 0)$ be the germ of analytic space generated by $xz - y^2, yw - z^2, xw - yz$. The dimension of $(X, 0)$ is two. Therefore,

$$\begin{aligned}
\mathrm{Sing}(X,0) &= V\left((xz-y^2, yw-z^2, xw-yz) + I_2\begin{bmatrix} z & -2y & x & 0 \\ 0 & w & -2z & y \\ w & -z & -y & x \end{bmatrix}\right) \\
&= V(xz-y^2, yw-z^2, -x^2, 2y^2+xz, -yz-xw, -xy, 4yz-xw, -2z^2, -2xy, \\
&\qquad z^2-2yw, -2y^2, -zw, -xz, -yz, 2z^2+yw, -2zw, y^2-2xz, -w^2, -yz-xw, yw) \\
&= V(x^2, xy, y^2-2xz, xz, yz, 2z^2+yw, xw, yw, zw, w^2) \\
&= \{0\}
\end{aligned}$$

1.4 Isolated Complete intersection Sigularities

Finally, we are ready for the definition of isolated complete intersection singularity (ICIS). The complete intersections are the natural generalization of hypersurfaces. That is, a complete intersection is a germ of analytic space defined by the zero set of a map germ from $\mathbb{C}^n$ to $\mathbb{C}^p$ which keeps the codimension: the codimension of the germ of analytic space is equal to the codimension of $\{0\}$ in $\mathbb{C}^p$.

Definition 1.4.1. Let $(X, 0) \subset (\mathbb{C}^n, 0)$ be a germ complex space defined by an ideal I in $\mathcal{O}_n$. Let k be the minimal number of generators of I. Then $(X, 0)$ is called a *complete intersection* if the dimension of $(X, 0)$ is $n - k$.

Example 1.4.2. We consider $(X, 0)$, the coordinate axes in $\mathbb{C}^3$, given by the zero set of the ideal $I = (xy, xz, yz)$. It is easy to see (using for example the Chevalley dimension), that the dimension of $(X, 0)$ is one. Therefore, $(X, 0)$ is not a complete intersection.

We remark here that $(X, 0)$ is a determinantal variety. The determinantal varieties are the natural generalization of complete intersection: they are germs of analytic spaces defined by the inverse image of the set of the matrices with a fixed rank by an analytic map germ from $\mathbb{C}^n$ to the set of the matrices with size $m \times k$ with the additional hypothesis that the codimension is the expected one. Here, I is generated by the minors of size two of the matrix

$$\begin{bmatrix} x & 0 & z \\ 0 & y & z \end{bmatrix}.$$

We say that a germ of analytic space is Cohen-Macaulay if its local ring is Cohen-Macaulay.

Theorem 1.4.3. *If* $(X, 0) \subset (\mathbb{C}^n, 0)$ *is an ICIS then it is Cohen-Macaulay.*

For a prove of this theorem see [19, Corollary B.8.10].

Definition 1.4.4. An *isolated complete intersection singularity* (ICIS) is a complete intersection $(X,0)$ such that $\mathrm{Sing}(X,0) = (0,0)$.

Example 1.4.5. 1. $(X,0) = (V(xy - z^2, x^2 + y^2 + z^2), 0)$ is an ICIS in $(\mathbb{C}^3, 0)$. In fact, $\dim(X,0) = 1$ and

$$\begin{aligned} \text{Sing(X,0)} &= V\left((xy - z^2, x^2 + y^2 + z^2) + I_2\begin{pmatrix} y & x & -2z \\ 2x & 2y & 2z \end{pmatrix}\right) \\ &= V(xy - z^2, x^2 + y^2 + z^2, y^2 - x^2, xz + 2yz, yz + 2xz) \\ &= \{0\} \end{aligned}$$

here $I_2(M)$ denotes the ideal generate by the minors of size two of the matrix M.

2. For germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with $n \le p$ of corank 1 and with finite left-right codimension, the germs of multiple points, $D^k(f)$, are ICIS (see [36, Chapter 9]).

3. If we consider the matrix $M = \begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix}$ and I the ideal generated by the minors of size two of M. Then $(V(I), 0)$ is an isolated singularity in $(\mathbb{C}^4, 0)$ which is not a complete intersection. In fact it has dimension 2 but the radical ideal I can not be generated by 2 elements.

Corollary 1.4.6. *If $(X,0)$ is an ICIS with dimension greater than or equal to two then it is reduced.*

Proof. In fact, we will show that $(X,0)$ satisfies Serre's R_0 and S_1 conditions. The condition R_0 is satisfied if the singular set of $(X,0)$ has codimension greater then or equal to one. The condition S_1 is satisfied if the germ is Cohen-Macaulay. □

Chapter 2

The contact group

2.1 The $\mathcal{K}$ group

Consider $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 2.1.1. The *contact group* $\mathcal{K}$ is the set of pairs of germs of diffeomorphisms (h, H), where $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$, $H : (\mathbb{K}^n \times \mathbb{K}^p, 0) \to (\mathbb{K}^n \times \mathbb{K}^p, 0)$ such that $\pi_1 \circ H = h$, $(\pi_2 \circ H)(x, 0) = 0$ where π_1 and π_2 are the projections into $\mathbb{K}^n$ and $\mathbb{K}^p$, respectively.

Notice that $H(x, y) = (h(x), H_2(x, y))$, $H_2(x, 0) = 0$.

The set of pairs $(h, H) \in \mathcal{K}$, such that h is the identity $I_{\mathbb{K}^n}$ form a subgroup of $\mathcal{K}$, usually denoted by $\mathcal{C}$.

Definition 2.1.2. Let $f, g \in \mathfrak{m}_n\mathcal{O}_{n,p}$. We say that f and g are contact equivalent, $f \sim_{\mathcal{K}} g$, if there is a pair $(h, H) \in \mathcal{K}$ such that $H(x, f(x)) = (h(x), g(h(x)))$.

Remark 2.1.3. Notice that if $f \sim_{\mathcal{K}} g$, then the diffeomorphism $H : (\mathbb{K}^n \times \mathbb{K}^p, 0) \to (\mathbb{K}^n \times \mathbb{K}^p, 0)$ sends graph(f) into graph(g), leaving $\mathbb{K}^n \times \{0\}$ invariant. This geometric viewpoint of contact equivalence was extended by Montaldi [37] as follows: two pairs of germs of submanifolds of $\mathbb{R}^m$ have the same contact type if there is a germ of diffeomorphism of $\mathbb{R}^m$ taking one pair to the other. Moreover, he proved in [37], that the contact type of a pair of germs of manifolds is completely characterized by the $\mathcal{K}$-equivalence class of a convenient map. This result is one the fundamental pieces of the applications of singularity theory to differential geometry (see Bruce and Giblin [2] and Izumiya, Romero-Fuster, Ruas and Tari, [23]).

The *tangent space* and the *extended tangent space* of $\mathcal{K}$-equivalence are, respectively

$$T\mathcal{K} f = tf(\mathfrak{m}_n\theta_n) + f^*(\mathfrak{m}_p)\theta_f$$

$$T\mathcal{K}_e f = tf(\theta_n) + f^*(\mathfrak{m}_p)\theta_f$$

We also define $\mathcal{K} - \operatorname{codim}(f) = \dim_{\mathbb{K}} \dfrac{\mathfrak{m}_n\theta_f}{T\mathcal{K} f}$ and $\mathcal{K}_e - \operatorname{codim}(f) = \dim_{\mathbb{K}} \dfrac{\theta_f}{T\mathcal{K}_e f}$.

The $\mathcal{K}_e$-codimension of f is also known as its *Tjurina number* and denoted by $\tau(f)$.

The following result was first proved by Mather in [33].

Proposition 2.1.4 (Gibson [18], Proposition 2.2, Mond and Nuño-Ballesteros [36], Section 4.4)**.** *The following statements are equivalent.*

(1) Two map-germs $f, g \in \mathfrak{m}_n \mathscr{O}_{n,p}$ are $\mathscr{K}$-equivalent.

(2) There exists a germ of diffeomorphism $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that

$$h^* f^*(\mathfrak{m}_p)\mathscr{O}_n = g^*(\mathfrak{m}_p)\mathscr{O}_n.$$

The local algebra we introduce now is an useful invariant of $\mathscr{K}$-equivalence. For a given map-germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ we define the *local algebra* of f as

$$Q(f) = \frac{\mathscr{O}_n}{f^*(\mathfrak{m}_p)\mathscr{O}_n}.$$

It follows from the previous proposition that the isomorphism class of $Q(f)$ is a $\mathscr{K}$-invariant. Furthermore, it is a complete invariant of $\mathscr{K}$-equivalence for germs f with finite $\mathscr{K}$-codimension. More precisely, we have

Theorem 2.1.1. *If f and g are map-germs with finite $\mathscr{K}$-codimension it follows that $f \sim_{\mathscr{K}} g$ if and only if the local algebras $Q(f)$ and $Q(g)$ are isomorphic.*

Remark 2.1.5. For complex analytic germs the hypothesis of finite $\mathscr{K}$-codimension in Theorem 2.1.1 is not needed.

Example 2.1.6. Let $F : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a germ of rank r. Then, up to $\mathscr{A}$-equivalence, we can take F in the normal form $F(x, y) = (x, f(x, y))$, $x \in \mathbb{K}^r$, $y \in \mathbb{K}^{n-r}$, with $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^{p-r}, 0)$ and $j^1 f(0, 0) \equiv 0$. Let $f_0 : (\mathbb{K}^{n-r}, 0) \to (\mathbb{K}^{p-r}, 0)$ be the rank zero germ $f_0(y) = f(0, y)$. Then $Q(F) = Q(f_0)$.

If $\mathscr{K}$-codim $(f_0) < \infty$ and $Q(F) \cong Q(f_0)$ it follows that F is $\mathscr{K}$-equivalent to the trivial unfolding $F_0(x, y) = (x, f_0(y))$ of f_0.

As we shall see in the next section, germs $f \in \mathfrak{m}_n \mathscr{O}_{n,p}$ of finite $\mathscr{K}$-codimension are finitely $\mathscr{K}$-determined, and in this case $\mathscr{K}(f) = \mathscr{K}(z)$, where $z = j^k f(0)$ for some k.

Now, for each positive integer k, we set

$$Q_k(f) = \frac{\mathscr{O}_n}{f^*(\mathfrak{m}_p)\mathscr{O}_n + \mathfrak{m}_n^{k+1}}.$$

$Q_k(f)$ is the local algebra of $z = j^k f(0)$. We can also write $Q_k(f) = Q(z)$.

It is not hard to show that $z \sim_{\mathscr{K}^k} z'$ if and only if $Q_k(z)$ and $Q_k(z')$ are isomorphic.

Exercises

1. Go to the first part of the mini-course (2021) and recall the definition of right-left equivalence ($\mathscr{A}$-equivalence) and show that if $f, g : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ are $\mathscr{A}$-equivalent then they are $\mathscr{K}$-equivalent.This was a proposed exercise of first part.

2. Let $f \in \mathfrak{m}_n \mathscr{O}_{n,p}$. Prove that (see Lemma 4.1 in [36])

$$\{\frac{d\Phi_t \cdot f}{dt}|_{t=0} : \Phi_t \in \mathscr{K} \text{ is smooth}, \Phi_0 = Id\} = tf(\mathfrak{m}_n\theta_n) + f^*\mathfrak{m}_p\theta_f = T\mathscr{K} f.$$

Part II

Lecture 2

Chapter 3

$\mathscr{K}$-Determinacy

Let $\mathscr{G}$=$\mathscr{R}$, $\mathscr{L}$, $\mathscr{A}$, $\mathscr{C}$ or $\mathscr{K}$ be one of Mather's groups and $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ a smooth map-germ.

Definition 3.0.1. The map-germ f is k-$\mathscr{G}$-determined if any smooth map-germ $g : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ with $j^k g = j^k f$ is $\mathscr{G}$-equivalent to f. If f is k-$\mathscr{G}$-determined for some $k < \infty$, then it is finitely-$\mathscr{G}$-determined, and the least such k is the degree of determinacy.

3.0.1 Infinitesimal criteria of finite $\mathscr{K}$-determinacy

The following result has a similar statement for all Mather's groups. Its proof can be found in [36] and [47].

Theorem 3.0.1. *Let $f \in \mathfrak{m}_n\mathscr{O}_{n,p}$. The following are equivalent:*

(a) f is finitely-$\mathscr{K}$-determined,

(b) for some k, $T\mathscr{K} f \supset \mathfrak{m}_n^k \mathscr{O}_{n,p}$,

(c) $\mathscr{K} - codim(f) = \dim_{\mathbb{K}} \dfrac{\mathfrak{m}_n\mathscr{O}_{n,p}}{T\mathscr{K} f} < \infty$,

(d) $\mathscr{K}_e - codim(f) = \dim_{\mathbb{K}} \dfrac{\mathscr{O}_{n,p}}{T\mathscr{K}_e f} < \infty$,

More precisely,

(i) If f is r-$\mathscr{K}$-determined, $T\mathscr{K} f \supset \mathfrak{m}_n^{r+1}\mathscr{O}_{n,p}$.

(ii) If $T\mathscr{K} f \supset \mathfrak{m}_n^{r+1}\mathscr{O}_{n,p}$ then f is $r+1$-$\mathscr{K}$-determined.

(iii) If $\mathscr{K} - codim(f) = d < \infty$, then $T\mathscr{K} f \supset \mathfrak{m}_n^{d+1}\mathscr{O}_{n,p}$.

Corollary 3.0.2. *Let $f \in \mathfrak{m}_n\mathscr{O}_{n,p}$. Suppose*

$$T\mathscr{K} f + \mathfrak{m}_n^{r+2}\mathscr{O}_{n,p} \supset \mathfrak{m}_n^{r+1}\mathscr{O}_{n,p},$$

then f is $r+1$-$\mathscr{K}$-determined.

3.0.2 Geometric criterion of finite $\mathscr{K}$-determinacy

Let $f : N \to P$ be a smooth map between manifolds. We recall that the critical set $C(f)$ is by definition the set of points $x \in N$ such that the derivative of f at x is not surjective. The following result can be found in [36].

Theorem 3.0.3 (Mather-Gaffney Criterion)**.** *A holomorphic map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ has finite $\mathscr{K}$-codimension if and only if there is a representative which is finite-to-one on its critical set.*

Exercises

1. Show that $f(x_1, x_2) = (x_1^2, x_2^2)$ and $g(x_1, x_2) = (x_1^2 - x_2^2, x_1 x_2)$ are both 2-$\mathscr{K}$-determined.

2. Show that $f(x_1, x_2) = (x_1^2 \pm x_2^k, x_1 x_2)$ is k-$\mathscr{K}$-determined.

3.1 Classification of stable germs

Let $f \in \mathfrak{m}_n \mathscr{O}_{n,p}$ such that $\mathscr{K}_e - \operatorname{codim}(f) < \infty$. Let

$$N\mathscr{K}_e f = \frac{\mathscr{O}_{n,p}}{T\mathscr{K}_e f}$$

be the normal space. Consider $\phi_i \in \mathscr{O}_{n,p}$, $i = 1, \dots, r$, whose images in $N\mathscr{K}_e f$ together with those of the $\frac{\partial}{\partial y_j}$ span $N\mathscr{K}_e f$ as $\mathbb{K}$-vector space.

Theorem 3.1.1 ([18])**.** *Let $F : (\mathbb{K}^n \times \mathbb{K}^r, 0) \to (\mathbb{K}^p \times \mathbb{K}^r, 0)$ be the map-germ given by*

$$F(x, u) = (f(x) + \sum_{i=1}^{r} u_i \phi_i(x), u).$$

Then f is a stable germ.

Example 3.1.1. Let $f \in \mathfrak{m}_n$ be given by $f(x_1, \dots, x_n) = x_1^{k+1} \pm x_2^2 \pm \dots \pm x_n^2$ (A_k-singularity). Then $T\mathscr{K}_e f$ is the ideal of $\mathscr{O}_n$ generated by $\{x_1^k, x_2, \dots, x_n\}$ and $\{1, x_1, x_1^2, \dots, x_1^{k-1}\}$ is a basis for the $\mathbb{K}$-vector space $N\mathscr{K}_e f$. Therefore,

$$F(x, u) = (x_1^{k+1} \pm x_2^2 \pm \dots \pm x_n^2 + u_1 x_1 + \cdots + u_{k-1} x_1^{k-1}, u)$$

is a stable map-germ. For example:

- $n = 1$, $k = 2$, $F(x, u) = (x^3 + ux, u)$: cusp singularity.

- $n = 1$, $k = 3$, $F(x, u) = (x^4 + u_1 x + u_2 x^2, u_1, u_2)$: swallowtail singularity.

3.2 Complete transversal method

The references for this sections are [3], [4].

Proposition 3.2.1. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ acting smoothly on an affine space A and let W be a subspace of V_A such that*

$$\mathfrak{g}(x+w) = \mathfrak{g}x,$$

for all $x \in A$ and $w \in W$. Then

1. *For any $x \in A$ we have*
$$x + \mathfrak{g}x \cap W \subset Gx \cap \{x + W\}.$$

2. *If $x_0 \in A$ and T is a vector subspace of W satisfying*
$$W \subset T + \mathfrak{g}x_0,$$
then for any $w \in W$ there exist $g \in G$ and $t \in T$ such that $g(x_0 + w) = x_0 + t$.

For each k, let H^k be the vector subspace of $J^k(n, p)$ of homogeneous k-jets of degree k, that is,

$$H^k = \frac{\mathfrak{m}_n^k \mathscr{O}_{n,p}}{\mathfrak{m}_n^{k+1}\mathscr{O}_{n,p}}.$$

Let $\mathscr{G}$ be any one of Mather's groups $\mathscr{R}$, $\mathscr{L}$, $\mathscr{A}$, $\mathscr{C}$ or $\mathscr{K}$. Consider $\mathscr{G}_k$ as the normal subgroup of $\mathscr{G}$ consisting of those germs whose k-jet is equal to that of the identity.

A subgroup $\mathscr{H}$ of $\mathscr{G}$ is called a Lie subgroup of $\mathscr{G}_1$ if for each k the k-jet group $\mathscr{H}^{(k)}$ is a Lie subgroup of $\mathscr{G}_1^{(k)}$.

Proposition 3.2.2. *Let $\mathscr{H}$ be a Lie subgroup of $\mathscr{G}_1$ and let σ be a k-jet in $J^k(n, p)$. If $T \subset H^{k+1}$ is a vector subspace such that*

$$H^{k+1} \subset T + T\mathscr{H}^{(k+1)}\sigma,$$

then for each $(k+1)$-jet τ with $j^k\tau = \sigma$ there exists $t \in T$ such that τ is $\mathscr{H}^{(k+1)}$-equivalent to $\sigma + \tau$.

Proof. We apply Proposition 3.2.1 with $A = J^{k+1}(n, p)$, $W = H^{k+1}$ and $G = \mathscr{H}^{(k+1)}$. Since $\mathscr{H}^{(k+1)} \subset \mathscr{G}_1^{(k+1)}$, we have $\varphi(h) = h$, for all $h \in H^{k+1}$ and $\varphi \in \mathscr{H}^{(k+1)}$. But this implies that $\eta(\sigma + h) = \eta\sigma$, for all $h \in H^{k+1}$ and $\eta \in \mathfrak{g}$, which gives the necessary condition. □

Definition 3.2.3. A subspace $T \subset H^{k+1}$ satisfying the conditions of above proposition is called a complete transversal for $\sigma \in J^k(n, p)$.

Let $f \in \mathfrak{m}_n\mathscr{O}_{n,p}$. We recall that $J(f) = \mathscr{O}_n\{\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\} \subset \mathscr{O}_{n,p}$ is called Jacobian submodule of f. Since we are interested in the $\mathscr{K}$ group, we shall consider the following result.

Proposition 3.2.4. *The tangent space to the $\mathscr{K}^k$-orbit of a k-jet $f \in J^k(n, p)$ is given as k-jets of elements of*

$$\mathfrak{m}_n J(f) + \mathscr{O}_n.f^*\mathfrak{m}_p\{e_1, \ldots, e_p\}.$$

For the $\mathscr{K}_1^k$-orbit of the k-jet $f \in J^k(n, p)$, the tangent space is given as k-jets of elements of

$$\mathfrak{m}_n^2 J(f) + \mathfrak{m}_n.f^*\mathfrak{m}_p\{e_1, \ldots, e_p\}.$$

Example 3.2.5. Consider the case of map-germs $\mathbb{R}^2 \to \mathbb{R}^2$ with zero 1-jet. The classification of homogeneous 2-jets reduces to that of pencils of binary quadratic forms. Using a change of coordinates in source we can reduce the first form of the pair to $x_1^2 \pm x_2^2, x_1^2$ or 0. Now the other change of coordinates allows us to subtract multiples of the first entry from the second. Using this reduce any pencil to one of the forms

$$(x_1^2, x_2^2),\ \ (x_1^2 - x_2^2, x_1x_2),\ \ (x_1^2, x_1x_2),\ \ (x_1^2 \pm x_2^2, 0),\ \ (x_1^2, 0),\ \ (0, 0).$$

We have:

1. (x_1^2, x_2^2) and $(x_1^2 - x_2^2, x_1x_2)$ are both 2-$\mathscr{K}$-determined (exercise above).
2. Consider 2-jets $\sigma = (x_1^2, x_1x_2)$. Thinking of σ as a $k-1$-jet we find that a complete transversal is given by $(x_1^2 + ax_2^k, x_1x_2)$. In fact, since

   $$T\mathscr{K}_1\,\sigma = \mathfrak{m}_2^2\{(2x_1, x_2), (0, x_1)\} + \mathfrak{m}_2\{(x_1^2, 0), (x_1x_2, 0), (0, x_1^2), (0, x_1x_2)\},$$

   we have that $H^k \subset \mathbb{R}\{(x_2^k, 0)\} + T\mathscr{K}_1^{(k)}\,\sigma$. If $a \neq 0$ we obtain $(x_1^2 \pm x_2^k, x_1x_2)$ which is k-$\mathscr{K}$-determined (exercise above).
3. Consider 2-jets $\sigma = (x_1^2 \pm x_2^2, 0)$. In the minus case we can use the alternative normal form $(x_1x_2, 0)$. We have:

   $$T\mathscr{K}_1\,\sigma = \mathfrak{m}_2^2\{(x_1, 0), (x_2, 0)\} + \mathfrak{m}_2\{(x_1x_2, 0), (0, x_1x_2)\},$$

   so a complete transversal is given by $(x_1x_2, ax_1^k + bx_2^k)$. If $ab \neq 0$ we reduce to $(x_1x_2, \pm x_1^k \pm x_2^k)$, which are k-$\mathscr{K}$-determined. If one of a and b is zero, say b, we reduce to (x_1x_2, x_1^k), which regarded as an $l-1$-jet has complete transversal $(x_1x_2, x_1^k + bx_2^l)$. We reduce to the normal forms $(x_1x_2, x_1^k \pm x_2^l)$, which is $(l+1)$-determined.

Exercises

1. Let $f \in \mathfrak{m}_n\mathscr{O}_{n,p}$. Suppose

   $$T\mathscr{K}_1\, f + \mathfrak{m}_n^{k+2}\mathscr{O}_{n,p} \supset \mathfrak{m}_n^{k+1}\mathscr{O}_{n,p},$$

   then f is k-$\mathscr{K}$-determined. Hint: regarding f as a k-jet, this shows that the $k+1$-transversal of f is empty, and then all subsequent transversal are also empty.
2. Fill in all the details of the complete transversal in the case (x_1x_2, x_1^k) in Example 3.2.5.
3. Investigate the germs emerging from $(x_1^2 + x_2^2, 0)$ in Example 3.2.5.

Chapter 4

$\mathscr{K}$-Versal unfoldings of map germs (real and complex)

4.1 Introduction

Since the beginning of singularity theory it has been clear that in order to understand a singularity you have to understand what happens when you deform it into less degenerate types of singularities. For instance, when looking at a bent wire from the tangent direction at a point of the wire where it does not have 0 torsion you see a cusp. If you move your head slightly to the left and to the right, from one side you will see a regular piece of wire and from the other side you will see a kind of loop. We have deformed the cusp and by seeing what happens near the cusp we have understood how this singularity appears. In a certain sense we need a family (a 1-paramater family in this case) of views in order to grasp the full nature of the singularity. In this part of the lecture we will give the definition of unfolding and deformation of a smooth function germ $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$. When working with map germs $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, $p > 1$ it is natural to consider $\mathscr{A}$-equivalence (i.e. smooth changes of coordinates in source and target) and the notion of unfolding is crucial. In this chapter we want to set up the theory of deformations of germs under $\mathscr{K}$-equivalence. This theory plays a fundamental role in the classification of stable map-germs. Furthermore, $\mathscr{K}$-equivalence is an important tool to study ICIS, as we will see in the following chapter. We will follow the approach in [30] and [36].

4.2 Basic definitions and examples

We recall from the previous lecture, that in the analytic case, studying $\mathscr{K}$-classes of mapgerms is equivalent to studying isomorphism classes of the analytic spaces, then we define. Let $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a smooth function germ, where smooth means C^∞ when $\mathbb{K} = \mathbb{R}$ or holomorphic when $\mathbb{K} = \mathbb{C}$.

Definition 4.2.1. A *r-parameter unfolding* of f is a map-germ

$$F'\colon (\mathbb{K}^r \times \mathbb{K}^n, 0) \to (\mathbb{K}^r \times \mathbb{K}^p, 0)$$

of the form $F'(u, x) = (u, F(u, x))$, such that $F(0, x) = f(x)$. If we denote $F(u, x)$ by $f_u(x)$, the above condition becomes $f_0 = f$. The map-germ

$$F\colon (\mathbb{K}^r \times \mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$$

is a *r-parameter deformation* of f.

In this chapter we want to set up the theory of deformations of germs under $\mathscr{K}$−equivalence. This theory plays a fundamental role in the classification of stable map germs, as we discuss in section 4.4.

Definition 4.2.2. ii) Two r-parameter deformations F, G of f are *$\mathscr{K}$-equivalent* if there exists a germ of diffeomorphism

$$\Phi\colon (\mathbb{K}^r \times \mathbb{K}^n, 0) \to (\mathbb{K}^r \times \mathbb{K}^n, 0)$$

of the form $\Phi(u, x) = (u, \varphi_u(x))$ where $\varphi(0, x) = x$ (i.e. Φ is a deformation of the identity in $\mathbb{K}^n$), such that

$$\Phi^*(G^*(\mathcal{M}_p)) = F^*(\mathcal{M}_p).$$

In this case we call Φ a *$\mathscr{K}$-equivalence of deformations*. It is clear that this relation implies that F, G are $\mathscr{K}$-equivalent as germs, notice however that the change of coordinates at the source respects the product structure on $\mathbb{K}^r \times \mathbb{K}^n$.

iii) A deformation F is called *$\mathscr{K}$-trivial* if it is equivalent to the constant deformation $\tilde{F}(x, u) = f(x)$.

iv) A map-germ is called *$\mathscr{K}$-stable* if any deformation of it is trivial.

Example 4.2.3. Consider the germ $f\colon (\mathbb{K}, 0) \to (\mathbb{K}, 0)$ given by $f(x) = x$ and the deformations $F(u, x) = x$ and $G(u, x) = x + ux^2$. Taking the diffeomorphism $\Phi(u, x) = (u, x + ux^2)$ we get $\Phi^*(\langle F\rangle) = (\langle G\rangle)$, where $\langle F\rangle$ denotes the ideal generated by the coordinate functions $F_1, \ldots, F_p$ of F. It follows that F and G are $\mathscr{K}$-equivalent. Furthermore, since F is the constant deformation, this means that G is $\mathscr{K}$-trivial.

In fact, given any deformation $H(u, x)$, since $H(0, x) = x$, by considering the diffeomorphism $\Phi(u, x) = (u, H(x, u))$ we see that H is trivial. We have shown that $f(x) = x$ is stable.

Remark 4.2.4. A function germ $f\colon (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ is $\mathscr{K}_e$-stable if and only if $\dim_{\mathbb{K}} \frac{\theta(f)}{T\mathscr{K}_e(f)} = 0$, and, thus, if and only if df_0 is surjective, i.e. the function is regular.

Deformations allow us to see what happens around a singularity, but in order to understand the singularity completely we want to know what are *all* the possible phenomena that appear around it. The idea of a versal deformation is that it captures all the possible less degenerate singularities into which a certain singularity can be deformed.

Definition 4.2.5. i) Let $F : (\mathbb{K}^r \times \mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a deformation of a map-germ f and let $h : (\mathbb{K}^s, 0) \to (\mathbb{K}^r, 0)$ be a map-germ. The *pull-back* of F by h is the deformation

$$h^*F : (\mathbb{K}^n \times \mathbb{K}^s, 0) \to (\mathbb{K}^p, 0)$$

given by

$$h^*F(v, x) = F(h(v), x).$$

The map-germ h is called the *base change map*.

ii) An r-parameter deformation F of f is *versal* if for any s-parameter deformation G there is a map-germ $h : (\mathbb{K}^s, 0) \to (\mathbb{K}^r, 0)$ such that G is equivalent to h^*F. It is called *miniversal* if there is no versal deformation with less than r parameters.

iii) Two r-parameter deformations F and G of f are *isomorphic* if there exists a diffeomorphism $h : (\mathbb{K}^r, 0) \to (\mathbb{K}^r, 0)$ such that G is equivalent to h^*F.

Example 4.2.6. Consider the function $f(x, y) = x^2 + y^3$ and the deformations $F(x, y, u_1, u_2) = x^2 + y^3 + u_1 y + u_2$ and $G(x, y, v) = x^2 + y^3 + 3vy^2$. Given the map-germ $h(v) = (-3v^2, 2v^3)$, we get $h^*F(x, y, v) = x^2 + y^3 - 3v^2 y + 2v^3$. Using the diffeomorphism $\Phi(v, x, y) = (v, x, y + v)$ we see that $G = h^*F \circ \Phi$, i.e. G is equivalent to h^*F. We will see in the next section that F is, in fact, versal.

4.3 Characterizations of versality

The definition of versality is not very useful to verify if a certain deformation F is versal, as we cannot find a map germ h for any other possible unfolding G, so we need certain characterizations to find or prove when an unfolding is versal.

Remember that we have defined

$$T\mathscr{K}_e(f) = tf(\theta(n)) + f^*(\mathcal{M}_p)\theta(f)$$

and the $\mathscr{K}_e$-codimension of f is $\dim_{\mathbb{K}} \frac{\theta(f)}{T\mathscr{K}_e(f)}$. Let $F(u, x) = f_u(x)$ be a $\mathscr{K}$-deformation of a function germ f. We denote $\frac{\partial f_u}{\partial u_i}|_{u=0}$ by $\dot{F}_i$. The following is an infinitesimal criterion for versality due to Martinet ([29]). We will prove only the necessity of the criterion, the proof of sufficiency is longer and exceeds the reach of this lecture notes. The proof is based on a fundamental geometrical lemma of the theory of deformations concerning the existence of a certain liftable vector fields, and also relies on the Preparation Theorem. A detailed account can be found in Chapters IV and XI in [30] or Theorem 5.1 in [36] for the $\mathscr{A}$-equivalence version.

Theorem 4.3.1. *The r-parameter deformation F of f is versal if and only if*

$$T\mathscr{K}_e(f) + Sp_{\mathbb{K}}\{\dot{F}_1, \dots, \dot{F}_r\} = \theta(f),$$

i.e. the map germs $\dot{F}_1, \dots, \dot{F}_r$ generate $\frac{\theta(f)}{\mathscr{K}_e(f)}$ as a $\mathbb{K}$-vector space.

Remark 4.3.2. An unfolding F satisfying the condition

$$T\mathscr{K}_e(f) + Sp_{\mathbb{K}}\{\dot{F}_1, \dots, \dot{F}_d\} = \theta(f)$$

is called *transversal.*

Proof. (Proof of Necessity) Suppose F is $\mathscr{K}-$ versal. We will show that F is $\mathscr{K}$-transversal, that is

$$T\mathscr{K}_e(f) + Sp_{\mathbb{K}}\{\dot{F}_1, \dots, \dot{F}_r\} = \theta(f).$$

Let $g \in \theta(f)$ and consider a 1-parameter deformation $G(v, x) = f(x) + vg(x)$ of f. Notice that $\dot{G} = g$. Since $F(u, x) = f_u(x)$ is versal, there exists $h : (\mathbb{K}, 0) \to (\mathbb{K}^r, 0)$, $h = (h_1, \dots, h_r)$ such that $G(v, x)$ is equivalent to $h^*F(v, x) = f_{h(v)}(x)$. We write $H(v, x) = f_{h(v)}(x)$. Applying the chain rule we get

$$\dot{h^*F} = \frac{d(f_{h(v)})}{dv}\, |_{v=0} = \sum\nolimits_{i=1}^{r} h_i'(0) \frac{\partial f_u}{\partial u_i}|_{u=0} = \sum\nolimits_{i=1}^{r} h_i'(0)\dot{F}_i \in Sp_{\mathbb{K}}\{\dot{F}_1, \dots, \dot{F}_r\}.$$

Since G is equivalent to H, there exists a diffeomorphism $\Phi(v, x) = (v, \varphi_v(x))$ such that $G^*(\mathcal{M}_p) = \Phi^*(H^*(\mathcal{M}_p))$ or, alternatively,

$$\left\langle G_1, \dots, G_p \right\rangle = \left\langle f_{1\,h(v)} \circ \varphi_v, \dots f_{p\,h(v)} \circ \varphi_v \right\rangle.$$

Applying the chain rule again and taking into account that $f_{h(0)} = f$ and that φ_0 is the identity we get

$$\dot{G} - \dot{H} = g - \frac{d}{dv}(f_{h(v)} \circ \varphi_v)|_{v=0} \in T\mathscr{K}_e(f).$$

Then,

$$g \in T\mathscr{K}_e(f) + Sp_{\mathbb{K}}\{\dot{F}_1, \ldots, \dot{F}_d\}.$$

□

Example 4.3.3. i) The deformation $F(u_1, u_2, x, y) = x^2 + y^3 + u_1 y + u_2$ of Example 4.2.6 is $\mathscr{K}$-versal since $\frac{\mathscr{O}_n}{J(f)+\langle f\rangle} = \frac{\mathscr{O}_n}{J(f)}$ is generated by y and 1. This explains why $G(x, y, v) = x^2+y^3+3vy^2$ is equivalent to h^*F for some h. In fact, any other deformation H will be equivalent to a pull-back of F.

ii) Consider $f(x) = x^4$, $\frac{\mathscr{O}_n}{J(f)}$ is generated by $\{1, x, x^2\}$ so a versal deformation is $F(x, u_1, u_2, u_3) = x^4 + u_1x^2 + u_2x + u_3$. The parameter u_3 is just a translation. If you consider the plane u_1, u_2, for every point in the plane you get a different function. It is interesting to see how this function varies and what singularities appear. For instance, along the curve $(-6s^2, 8s^3)$, the function has an inflection point at the origin. On one side of this curve the function has two local minima and one local maximum, on the other side there is just one local minimum. This is called a *bifurcation diagram*, we refer the reader to [36] for more details on this set.

If we consider the unfolding $G(x, u_1) = x^4 + u_1x^2$, as u_1 varies we will appreciate changes in the function, namely it has 3 critical points when $u_1 < 0$ and 1 critical point otherwise. However, this deformation is not versal, in particular it does not show how in any neighbourhood of the function f there are functions with inflection points.

On the other hand, the unfolding $H(x, y, u_1, u_2, u_3, u_4) = x^4 + u_1x^2 + u_2x + u_3 + u_4x^3$ is also versal but it is not miniversal, since F has less parameters than H. In fact, H can be seen as a trivial deformation of F.

Corollary 4.3.4. *A map germ f admits a $\mathscr{K}$-versal deformation if and only if its $\mathscr{K}_e$-codimension is finite. Moreover, the $\mathscr{K}_e$-codimension is equal to the number of parameters in a miniversal deformation.*

Proof. Given a versal r-parameter deformation F, by the versality criterion, $\dot{F}_1, \ldots, \dot{F}_r$ generate $\frac{\theta(f)}{T\mathscr{K}_e(f)}$ as a $\mathbb{K}$-vector space, so $\mathscr{K}_e$-cod$(f) = \dim_{\mathbb{K}} \frac{\theta(f)}{T\mathscr{K}_e(f)} \leq r$. Converserly, if $\mathscr{K}_e$-cod$(f) = r$, there exist $g_1, \ldots, g_r \in \theta(f)$ whose classes generate $\frac{\theta(f)}{T\mathscr{K}_e(f)}$ over $\mathbb{K}$, so $F(u, x) = f(x) + \sum_{i=1}^{r} u_i g_i(x)$ is a miniversal deformation of f. □

Corollary 4.3.5. *Let F, G be $\mathscr{K}$-versal deformations of a germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ of finite $\mathbb{K}$-codimension r. Then F, G are $\mathscr{K}$- isomorphic deformations.*

Proof. Suppose first that F, G are two r-parameter miniversal deformations. Since F is $\mathscr{K}$-versal, there exists $h : (\mathbb{K}^r, 0) \to (\mathbb{K}^r, 0)$ such that G is equivalent to h^*F. Since G is versal, h^*F is versal too. Applying the chain rule to $h^*F = f_h$ we get

$$(\dot{h^*F})_i = \sum_{i=1}^{r} \frac{\partial h_j}{\partial u_i}(0)\dot{F}_j$$

for $i = 1, \ldots, r$. Since both F and h^*F are miniversal, $\{\dot{F}_1, \ldots, \dot{F}_r\}$ and $\{(\dot{h^*F})_1, \ldots, (\dot{h^*F})_r\}$ are bases of $\frac{\theta(f)}{T\mathscr{K}_e(f)}$, and so $(\frac{\partial h_j}{\partial u_i}(0))$ is an invertible matrix. This means that h is a diffeomorphism and so G and F are isomorphic.

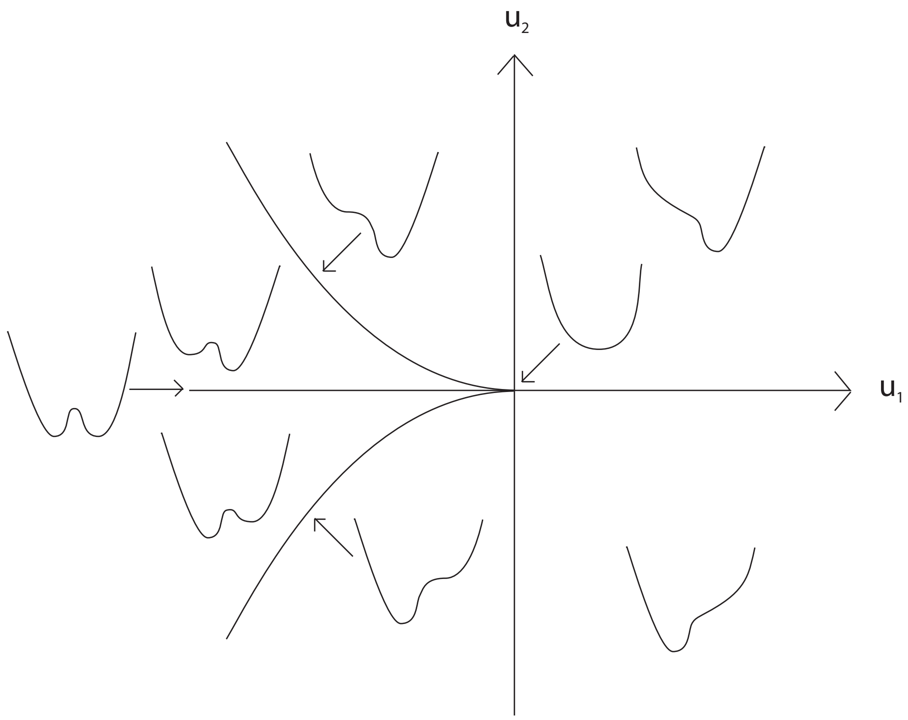


Figure 4.1: Different functions for different values of u_1 and u_2 represented in the $\{u_1, u_2\}$-plane.

Now suppose F, G are versal m-parameter deformations with $m > r$. We have $\dim_{\mathbb{K}} Sp_{\mathbb{K}}\{\dot{F}_1, \ldots, \dot{F}_m\} = r$, so there are $m - r$ linear combinations of the $\dot{F}_i$ which give 0. This means that there exists a linear change of parameters $h_1 : (\mathbb{K}^m, 0) \to (\mathbb{K}^m, 0)$ such that $h_1^* F$ verifies that there exists $m - r$ partials $(h_1^* F)_i$ which are 0, i.e. $h_1^* F$ is a constant deformation of a miniversal deformation. Similarly, there exists h_2 such that $h_2^* G$ is a constant deformation of a miniversal deformation. Since h_1 and h_2 are diffeomorphisms, F and G are isomorphic. □

In fact, we can obtain a little more information from the above proof, as we can see in the next corollary.

Corollary 4.3.6. *Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a germ of finite $\mathscr{K}-$ codimension r, and let F be a $\mathscr{K}-$ versal deformation of f. For $m \geq r$ any m-parameter $\mathscr{K}$-versal deformation F' of f is $\mathscr{K}$-isomorphic to the $(m - r)$-parameter constant deformation of F. Hence, any two m-parameter $\mathscr{K}$-versal deformations of f are $\mathscr{K}$-isomorphic.*

4.4 Relation between $\mathscr{K}$-equivalence and $\mathscr{A}$-equivalence

The contact group $\mathscr{K}$ defined by Mather in [32] plays a fundamental role in the classification of stable mappings. In this section we discuss the relation between $\mathscr{K}$-equivalence and $\mathscr{A}$-equivalence, as an important tool to classify stable singularities.

Proposition 4.4.1. *Let $G : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a germ of rank r. Then, there exists an invertible germ $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ for which $F = G \circ h$ is an r-parameter unfolding of a germ of rank* 0.

Proof. With linear changes of coordinates in source and target, we can assume that $g : (\mathbb{K}^n, 0) \to (\mathbb{K}^r, 0)$, is a rank r map-germ, where $g = \pi \circ G$, and $\pi : \mathbb{K}^n \to \mathbb{K}^r$ is the usual projection. Hence, applying the implicit function theorem, we can find a diffeomorphism $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ such that $g \circ h = \pi$. Then, $F = G \circ h$ is the required germ. □

Let $F : (\mathbb{K}^r \times \mathbb{K}^n, 0) \to (\mathbb{K}^r \times \mathbb{K}^p, 0)$ be a r-parameter unfolding of a germ of rank 0, $f_F : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$. For the purposes of this section, we can assume that f_F is $\mathscr{K}$-finitely determined. We now consider the correspondence

$$F \mapsto f_F.$$

Proposition 4.4.2. *Let $F, F' : (\mathbb{K}^r \times \mathbb{K}^n, 0) \to (\mathbb{K}^r \times \mathbb{K}^p, 0)$ be r-parameter unfoldings of germs $f_F, f_{F'}$, of rank* 0. *If F, F' are $\mathscr{A}$-equivalent then $f_F, f_{F'}$ are $\mathscr{K}$-equivalent.*

Proof. Since $F(u, x) = (u, f(u, x))$ and $F'(u, x) = (u, f'(u, x))$ and $F \simeq_{\mathscr{A}} F'$, it follows that the local algebras $Q(F)$ and $Q(F')$ are isomorphic. Now,

$$Q(F) = \frac{\mathscr{O}_{r+n}}{\langle u, f(u, x)\rangle} \backsimeq \frac{\mathscr{O}_n}{\langle f_F(x)\rangle} = Q(f_F).$$

Similarly, we get that $Q(F') \backsimeq Q(f_{F'})$, and then $Q(f_F) \backsimeq Q(f_{F'})$, which imply that $f_F \simeq_{\mathscr{K}} f_{F'}$, and it follows that f_F and $f_{F'}$ are $\mathscr{K}$-equivalent.

□

It follows from above that the correspondence $F \mapsto f_F$ induces a mapping from $\mathscr{A}$-orbits to $\mathscr{K}$-orbits. We want to understand this mapping in detail.

Let $f : (\mathbb{K}^s \times \mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a s-deformation which is submersive. We denote by V_f the germ of smooth manifold of $\mathbb{K}^s \times \mathbb{K}^n$ defined by $f^{-1}(0)$. Let

$$\pi_f : (V_f, 0) \to (\mathbb{K}^s, 0)$$

be the germ at 0 of the restriction to V_f of the projection $\pi : \mathbb{K}^s \times \mathbb{K}^n \to \mathbb{K}^s$.

Proposition 4.4.3. *Let $f, g : (\mathbb{K}^s \times \mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be $\mathscr{K}$-versal s-parameter deformations of the germs $f_0, g_0 : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ of rank* 0. *If f_0 and g_0 are $\mathscr{K}$-equivalent then π_f, π_g are $\mathscr{A}$-equivalent.*

Proof. We first notice that a $\mathscr{K}$-versal deformation of a germ of rank 0 is always submersive, so that V_f is a non singular manifold in $(\mathbb{K}^s \times \mathbb{K}^n, 0)$. It follows from ???? that f, g are $\mathscr{K}$-isomorphic deformations.

As in [18], (4.3), the proof is given in two steps.

Step 1. We first consider the case $f_0 = g_0$. Since f, g are $\mathscr{K}$-isomorphic deformations, there is a commutative diagram

$$\begin{array}{ccc} (\mathbb{K}^s \times \mathbb{K}^n, 0) & \xrightarrow{\Phi} & (\mathbb{K}^s \times \mathbb{K}^n, 0) \\ \pi \downarrow & & \pi \downarrow \\ (\mathbb{K}^s, 0) & \xrightarrow{h} & (\mathbb{K}^s, 0) \end{array}$$

such that Φ, h are invertible germs for which

$$(f \circ \Phi)^*(\mathcal{M}_p) = g^*(\mathcal{M}_p).$$

Then, it follows that Φ induces a mapping from V_g onto V_f, yelding a commuting diagram of germs

$$\begin{array}{ccc} (V_g, 0) & \xrightarrow{\Phi} & (V_f, 0) \\ \pi_g \downarrow & & \pi_f \downarrow \\ (\mathbb{K}^s, 0) & \xrightarrow{h} & (\mathbb{K}^s, 0) \end{array}$$

expressing the fact that π_f, π_g are $\mathscr{A}$-equivalent.
Step 2. We now consider the general case, when $f_0 \simeq_{\mathscr{K}} g_0$. Then, there exist an invertible germ $h : (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ and an invertible $p \times p$-matrix $M(x)$ with entries in $\mathscr{O}_n$ such that $g_0(x) = M(x).f_0(h(x))$. The s-parameter deformation $g'(u, x) = M(x).f(u, h(x))$ of g_0 is $\mathscr{K}$-versal as well. It follows from Step 1 that $\pi_g, \pi_{g'}$ are $\mathscr{A}$-equivalent. To finish the proof, notice that π_f and $\pi_{g'}$ are $\mathscr{A}$-equivalent, since $1 \times h$ maps $V_{g'}$ onto V_f.

□

Let $F : (\mathbb{K}^r \times \mathbb{K}^n, 0) \to (\mathbb{K}^r \times \mathbb{K}^p, 0)$ be an r-parameter unfolding of a $\mathscr{K}$-finitely determined germ $f_F : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ of rank 0, given by $F(u, x) = (u, f(u, x))$.

To the unfolding F we can associate the germ $D_F : (\mathbb{K}^r \times \mathbb{K}^p \times \mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ given by

$$(u, w, x) \mapsto -w + f(u, x),$$

thus D_F is an $(r + p)$-parameter submersive deformation of f_F. Notice that there is a geometric connection between F and D_F, since V_{D_F} is the graph of f and π_{D_F} can be identified with F.

Theorem 4.4.4. *The map-germ F is $\mathscr{A}$-stable if and only if D_F is a $\mathscr{K}$-versal deformation of f_F.*

Studying the fibers of the map π_f , if D_F is a $\mathscr{K}$-versal deformation of f_F is equivalent to studying all the “nearby” analytic sets to $f_F = 0$. The theorem shows these nearby analytic sets fit into a stable map. Thus when $n \geq p$, the study of the geometry of stable maps is inseparable from studying the deformations of the analytic space defined by f_F.

The proof of this theorem relies on the following well known results (see [31], [36].)

Theorem 4.4.5 (Preparation Theorem)**.** *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a C^∞ map-germ, E a finitely generated $\mathcal{E}_n$-module. If* $\dim_{\mathbb{R}}(\frac{E}{f^*(\mathcal{M}_p)\cdot E}) < \infty$, *then E is finitely generated as $\mathcal{E}_p$-module (via f).*

Remark 4.4.6. When $\mathbb{K} = \mathbb{C}$, we replace Malgrange Preparation Theorem by Weierstrass Preparation Theorem.

Proposition 4.4.7. *Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a $\mathscr{K}$-finite map-germ. The following conditions are equivalent:*

1. *f is stable.*
2. *$T\mathscr{A}_e(f) = \theta(f)$*

3. $T\mathscr{A}_e f + f^*(\mathcal{M}_p)\theta(f) = \theta(f)$

Proof. Proof of Theorem 4.4.4

Let $F(u, x) = (u, f(u, x))$ be a rank r unfolding of a map-germ $f_0(x)$ of rank 0.

Let E be the finitely generated $\mathscr{O}_{r+n}$-module given by

$$E = \frac{\theta(F)}{T\mathscr{A}_e(F) + F^*(\mathcal{M}_{r\times p})\theta(F)}. \tag{4.1}$$

Then

$$\frac{E}{\mathcal{M}_u E} = \frac{\theta(f_0)}{T\mathscr{K}_e(f_0) + Sp_{\mathbb{K}}\{\dot{F}_1, \ldots, \dot{F}_r, e_1, \ldots, e_p, \}} \tag{4.2}$$

where $e_1, \ldots, e_p$ is the canonical basis in $\mathbb{K}^p$.

We now suppose F is stable. Then, it follows from Proposition **??** that $E = 0$. Since (4.1) implies (4.2), we get that

$$\theta(f_0) = T\mathscr{K}_e(f_0) + Sp_{\mathbb{K}}\{\dot{F}_1, \ldots, \dot{F}_r, e_1, \ldots, e_p\},$$

and hence $D_F = -w + f(u, x)$ is a $\mathscr{K}$-versal deformation of f_0.

To prove the converse, it suffices to prove that (4.2) imply (4.1) and apply again Proposition **??**. The proof uses Theorem 4.4.5. We refer to [29], chapter XIV for details of the proof.

□

Let

$$S(r, n, p) \;=\; \{\mathscr{A} - \text{orbits of stable germs}\, F : (\mathbb{K}^r \times \mathbb{K}^n, 0) \to (\mathbb{K}^r \times \mathbb{K}^p, 0)\},$$

of rank r.

$$K(r, n, p) = \{\mathscr{K} - \text{orbits of germs}\, (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)\}$$

of rank 0 and $\mathscr{K}$-codimension $\leq r + p$.

We are now prepared to state the main result of this section.

Theorem 4.4.8. *The mapping*

$$S(r, n, p) \to K(r, n, p) \tag{4.3}$$

induced by the correspondence $F \to f_F$ is a bijection.

Proof. It follows from Proposition 4.4.2 that the map is injective. We only need to prove that the map is surjective. Let $f_0 : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a a germ of rank 0 and $\mathscr{K}$-codimension $\leq r + p$. Then, we can construct an $(r + p) - \mathscr{K}$-versal deformation of the form $-w + f(u, x)$ with f an r-parameter deformation of f_0. This is precisely the deformation D_F associated to the r-parameter unfolding $F : (\mathbb{K}^r \times \mathbb{K}^n, 0) \to (\mathbb{K}^r \times \mathbb{K}^p, 0)$ given by $F(u, x) = (u, f(u, x))$. It follows from Theorem that D_F is $\mathscr{K}$-versal, and we can apply Theorem 4.4.4 to get that F is stable. The trivial observation that F has rank r concludes the proof.

□

We now define the Kodaira-Spencer map of f, which can be seen as the infinitesimal counterpart of the map 4.3.

Definition 4.4.9. The *Kodaira-Spencer map* of a map-germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ is defined as

$$\bar{\rho}_f\colon \theta_p \longrightarrow \frac{\theta(f)}{T\mathscr{K}_e f},$$

given by $\bar{\rho}_f(\xi) = [\xi \circ f]$. The reduced Kodaira-Spencer map is

$$\rho_f\colon T_0\mathbb{C}^p \longrightarrow \frac{\theta(f)}{T\mathscr{K}_e f},$$

given by $\rho_f(v) = [\xi \circ f]$, such that $\xi_0 = v$.

Obviously, ρ_f is surjective if and only if so is $\bar{\rho}_f$. Moreover, the following holds.

Theorem 4.4.10. *The map ρ_f is surjective if and only if f is $\mathscr{A}$-stable.*

Proof. The image of $\bar{\rho}_f(\theta_p)$ in $\frac{\theta(f)}{T\mathscr{K}_e f}$ is the submodule

$$M = \frac{\bar{\rho}_f(\theta_p)}{\bar{\rho}_f(\theta_p) \cap T\mathscr{K}_e(f)} \simeq \frac{\bar{\rho}_f(\theta_p) + T\mathscr{K}_e(f)}{T\mathscr{K}_e(f)}$$

Then the quotient of these two modules is given by

$$E = \frac{\frac{\theta(f)}{T\mathscr{K}_e f}}{M} \simeq \frac{\theta(f)}{T\mathscr{K}_e f + \bar{\rho}_f(\theta_p)} \tag{4.4}$$

Notice that

$$T\mathscr{K}_e f + \bar{\rho}_f(\theta_p) = T\mathscr{A}_e f + f^*(\mathcal{M}_p)\theta(f).$$

Then it follows that $\bar{\rho}_f$ is surjective if and only if $\dim E = 0$, and we apply Proposition 4.4.7 to get the result. □

We end this section with the following result due to Mather which is the fundamental tool to classify stable singularities.

Theorem 4.4.11. *Let F and G stable map-germs. Then*

$$F \simeq_{\mathscr{K}} G \text{ if and only if } F \simeq_{\mathscr{A}} G.$$

Proof. The proof follows from Theorem 4.4.8. □

Example 4.4.12. Normal forms for (real) stable singularities whose local algebra are $B^{\pm}_{2,2} = (x^2 \pm y^2, xy)$ (We use here du Plessis and Wall notation [7]. They are denoted $\mathrm{I}_{2,2} = (x^2 + y^2, xy)$ and $\mathrm{II}_{2,2} = (x^2 - y^2, xy)$ by Mather [33].)

$$\begin{aligned} F : (\mathbb{R}^2 \times \mathbb{R}^2, 0) &\to (\mathbb{R}^2 \times \mathbb{R}^2, 0) \\ (x, y, u, v) &\mapsto F(x, y, u, v) = (x^2 \pm y^2 + ux + vy, xy, u, v). \end{aligned}$$

Exercises

1. Let $f(x,y) = x^3 + y^2 x$.

   i) Find a $\mathscr{K}$-miniversal deformation.

   ii) Show that $F(x,y,u) = x^3 + y^2 x + 2ux^2 y$ is a trivial deformation of f.

2. Let $f(x,y) = (x^2, y^2)$. Find a $\mathscr{K}$-miniversal deformation.

3. Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}^2,)\ f(x,y,z) = (x, z^2 + y^3 \pm x^k y)$, $k \geq 2$. Find a $\mathscr{K}$-miniversal deformation.

4. Show that if two deformations are equivalent, then they are isomorphic.

5. Show that if h is a diffeomorphism in the parameter space, then F and $h^* F$ are isomorphic.

6. Show that f is $\mathscr{K}$-stable if and only if the $\mathscr{K}_e$-codimension is 0.

7. Show that if f is $\mathscr{K}$- stable, then all deformations are $\mathscr{K}$-versal.

8. Prove Corollary 4.3.6.

Part III

Lecture 3

Chapter 5

Deformations of ICIS

5.1 Basic invariants of ICIS

We recall that an ICIS is a germ of complex space (X, x) such that:

1. it is a complete intersection, that is, if $\dim(X, x) = n$, it can be embedded in $(\mathbb{C}^N, x)$ as the zero locus of an ideal $I \subseteq \mathscr{O}_{N,x}$ generated by $N - n$ functions $I = (f_1, \dots, f_{N-n})$;

2. it has isolated singularity, that is, there exists a representative X such that $X \setminus \{x\}$ is smooth.

The local ring of (X, x) is $\mathscr{O}_{X,x} = \mathscr{O}_{N,x}/I$.

It follows that (X, x) can be seen as the fibre of a holomorphic map germ

$$f = (f_1, \dots, f_{N-n}) \colon (\mathbb{C}^N, x) \to (\mathbb{C}^{N-n}, 0),$$

which is $\mathscr{K}$-finite, by the Geometric Criterion (see Theorem **??**). Conversely, the fibre (X, x) of any $\mathscr{K}$-finite map germ $f \colon (\mathbb{C}^N, x) \to (\mathbb{C}^{N-n}, 0)$ is an ICIS of dimension n. Moreover, $\mathscr{O}_{X,x} = \mathscr{O}_{N,x}/f^*\mathfrak{m}_{N-n,x} = Q(f)$, the local algebra of f.

We now from Theorem 2.1.1 that two holomorphic map germs $f, g \colon (\mathbb{C}^N, x) \to (\mathbb{C}^{N-n}, 0)$ are $\mathscr{K}$-equivalent if and only if $Q(f) \cong Q(g)$. Thus, we have a bijection between the set of isomorphism classes of ICIS (X, x) embedded in $(\mathbb{C}^N, x)$ and the set of $\mathscr{K}$-classes of $\mathscr{K}$-finite map germs $f \colon (\mathbb{C}^N, x) \to (\mathbb{C}^{N-n}, 0)$.

Remark 5.1.1. Suppose that (X, x) is embedded in $(\mathbb{C}^N, x)$ and (Y, y) is embedded in $(\mathbb{C}^M, y)$, with $N \leq M$. After taking translations, we can assume, for simplicity, that $x = 0$ in $\mathbb{C}^N$ and $y = 0$ in $\mathbb{C}^M$. Then $(X, 0)$ is the fibre of some $f \colon (\mathbb{C}^N, 0) \to (\mathbb{C}^{N-n}, 0)$ and $(Y, 0)$ is the fibre of some $g \colon (\mathbb{C}^M, 0) \to (\mathbb{C}^{M-n}, 0)$. Instead of f we take the constant unfolding $f \times \mathrm{id}_{\mathbb{C}^{M-N}}$ and now we have that $(X, 0) \cong (Y, 0)$ if and only if $f \times \mathrm{id}_{\mathbb{C}^{M-N}}$ and g are $\mathscr{K}$-equivalent.

Let (X, x) be an ICIS given as the fibre of a $\mathscr{K}$-finite map germ $f \colon (\mathbb{C}^N, x) \to (\mathbb{C}^{N-n}, 0)$. We define the $\mathscr{O}_{X,x}$-module

$$T^1_{X,x} = \frac{\mathscr{O}^{N-n}_{X,x}}{JM_X(f)},$$

where $JM_X(f)$ is the *Jacobian submodule*, that is, the submodule of $\mathscr{O}^{N-n}_{X,x}$ generated by the residue classes of the partial derivatives $\partial f/\partial x_i$, $i = 1, \dots, n$. We have an isomorphism of $\mathscr{O}_{X,x}$-modules

$$T^1_{X,x} \equiv \frac{\theta(f)}{T\mathscr{K}_e f},$$

so $T^1_{X,x}$ has finite dimension over $\mathbb{C}$. Its $\mathbb{C}$-dimension is called the *Tjurina number*

$$\tau(X,x) := \dim_{\mathbb{C}} T^1_{X,x} = \dim_{\mathbb{C}} \frac{\theta(f)}{T\mathscr{K}_e f},$$

which coincides with the $\mathscr{K}_e$-codimension of f.

If M is an R-module and N is an S-module, an isomorphism between M and N will be a pair (φ, L), where $\varphi\colon R \to S$ is a ring isomorphism and $L\colon M \to N$ is an R-module isomorphism (where we consider N as an R-module via φ).

Lemma 5.1.2. *If $(X,x) \cong (Y,y)$, there exists an isomorphism (φ, L) between $T^1_{X,x}$ and $T^1_{Y,y}$, where $\varphi : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$. In particular, $\tau(X,x) = \tau(Y,y)$.*

Proof. We assume for simplicity that $x = 0$ and $y = 0$. Suppose first that $(X,x), (Y,y)$ are the fibres of $\mathscr{K}$-equivalent map germs $f, g\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^{N-n}, 0)$, respectively. There exist a pair (ϕ, A), where $\phi\colon (\mathbb{C}^N,0) \to (\mathbb{C}^N, 0)$ is a diffeomorphism and $A \in Gl_{N-n}(\mathscr{O}_{\mathbb{C}^N,0})$ such that $g = A \cdot (f \circ \phi)$. We consider the pair $(\phi^*, \tilde{A})$, where $\phi^*\colon \mathscr{O}_N \to \mathscr{O}_N$ and $\tilde{A}\colon \theta(f) \to \theta(g)$ is given by $\tilde{A}(\xi) = A \cdot (\xi \circ \phi)$. This is an isomorphism between $\theta(f)$ and $\theta(g)$.

Moreover, we now we know that $\phi^*(f^*\mathfrak{m}_{N-n}) = g^*\mathfrak{m}_{N-n}$ and $\tilde{A}(T\mathscr{K}_e f) = T\mathscr{K}_e g$. Hence $(\phi^*, \tilde{A})$ induces an isomorphism (φ, L), where now $\varphi : \mathscr{O}_{X,x} \to \mathscr{O}_{Y,y}$ and

$$L\colon T^1_{X,x} \equiv \frac{\theta(f)}{T\mathscr{K}_e f} \longrightarrow \frac{\theta(g)}{T\mathscr{K}_e g} \equiv T^1_{Y,y}. \tag{5.1}$$

Suppose now that $(X,x), (Y,y)$ are the fibres of map germs $f\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^{N-n}, 0)$ and $g\colon (\mathbb{C}^M, 0) \to (\mathbb{C}^{M-n}, 0)$, respectively, with $M \geq N$. By the previous case, we can assume that $g = f \times \mathrm{id}_{\mathbb{C}^r}$, with $r = M - N$. Here we consider $i\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^N \times \mathbb{C}^r, 0)$ given by $i(x) = (x, 0)$. We have a pair $(i^*, \tilde{A})$, where $i^*\colon \mathscr{O}_M \to \mathscr{O}_N$, $\tilde{A}\colon \theta(g) \to \theta(f)$ is given by $\tilde{A}(\xi) = \pi \circ \xi \circ i$ and $\pi(y,u) = y$.

In this case, $i^*(g^*\mathfrak{m}_{N-n}) = f^*\mathfrak{m}_{N-n}$ and $\tilde{A}(T\mathscr{K}_e g) = T\mathscr{K}_e f$. So $(i^*, \tilde{A})$ induces an isomorphism (φ, L), with $\varphi : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ and

$$L\colon T^1_{Y,y} \equiv \frac{\theta(g)}{T\mathscr{K}_e g} \longrightarrow \frac{\theta(f)}{T\mathscr{K}_e f} \equiv T^1_{X,x}. \tag{5.2}$$

□

Remark 5.1.3. We see in the proof of Lemma 5.1.2 that when $(X,x) \cong (Y,y)$ are the fibres of $\mathscr{K}$-equivalent map germs $f, g\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^{N-n}, 0)$, then there exists an isomorphism (φ, L) between $T^1_{X,x}$ and $T^1_{Y,y}$, for some isomorphism between the ambient local rings $\varphi : \mathscr{O}_N \to \mathscr{O}_N$. We will see in Section 6.4 a theorem due to Gaffney and Hauser [15] which shows the converse. That is, if exists an isomorphism (φ, L) between $T^1_{X,x}$ and $T^1_{Y,y}$, for some isomorphism $\varphi : \mathscr{O}_N \to \mathscr{O}_N$, then $(X,x) \cong (Y,y)$.

Finally, we recall that the Kodaira-Spencer map of a map germ $f, g\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^{N-n}, 0)$ is

$$\bar{\rho}_f\colon \theta_{N-n} \longrightarrow \frac{\theta(f)}{T\mathscr{K}_e f} \equiv T^1_{X,x},$$

given by $\bar{\rho}_f(\xi) = [\xi \circ f]$, where (X, x) is the fibre of f. The reduced Kodaira-Spencer map is

$$\rho_f\colon T_0\mathbb{C}^{N-n} \longrightarrow \frac{\theta(f)}{T\mathscr{K}_e f} \equiv T^1_{X,x},$$

given by $\rho_f(v) = [\xi \circ f]$, such that $\xi_0 = v$. Obviously, ρ_f is surjective if and only if so is $\bar{\rho}_f$. Moreover, ρ_f is surjective if and only if f is $\mathscr{A}$-stable (see Theorem 4.4.10).

We remark that the Kodaira-Spencer maps depend on the choice of f. Nevertheless, if $f, g\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^{N-n}, 0)$ are $\mathscr{A}$-equivalent then we have some kind of uniqueness (see Exercise 3).

Exercises

1. Consider the germs $f, g\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ given by $f(x, y) = (x, y^3 + xy)$ and $g(x, y) = (x, y^3)$. Show:

   (a) f, g are $\mathscr{K}$-finite and they define the same 0-dimensional ICIS with $\tau = 2$.

   (b) ρ_f is an isomorphism, so f is stable.

   (c) ρ_g is not surjective, so g is not stable. In particular, $\rho_f \neq \rho_g$.

2. Let $f\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^{N-n}, 0)$ be holomorphic and take $F(x, u) = (f_u(x), u)$ an r-parameter unfolding. Show that there exists an $\mathscr{O}_{N+r}$-isomorphism

$$L\colon \frac{\theta(F)}{T\mathscr{K}_e F} \longrightarrow \frac{\theta(f)}{T\mathscr{K}_e f}$$

   sending the class of $\partial/\partial y_i$ into the class of $\partial/\partial y_i$, for $i = 1, \dots, N - n$ and the class of $\partial/\partial u_j$ into the class of $\partial f_u/\partial u_j|_{u=0}$, for $j = 1, \dots, r$.

3. Show that if $f, g\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^{N-n}, 0)$ are $\mathscr{A}$-equivalent, we have commutative diagrams

$$\begin{array}{ccc} \theta_{N-n} & \xrightarrow{\bar{\rho}_f} & T^1_{X,x} \\ \downarrow & & \downarrow \\ \theta_{N-n} & \xrightarrow{\bar{\rho}_g} & T^1_{Y,y} \end{array} \qquad \begin{array}{ccc} T_0\mathbb{C}^{N-n} & \xrightarrow{\rho_f} & T^1_{X,x} \\ \downarrow & & \downarrow \\ T_0\mathbb{C}^{N-n} & \xrightarrow{\rho_g} & T^1_{Y,y} \end{array}$$

   where the rows are isomorphisms induced by the $\mathscr{A}$-equivalence and (X, x), (Y, y) are the fibres of f, g, respectively.

4. Show that any ICIS (X, x) is isomorphic to the fibre of a $\mathscr{K}$-finite map germ $f\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^{N-n}, 0)$ of rank 0 (in that case N is called the embedding dimension of (X, x)).

5.2 Versal deformations of ICIS

In this section, we study versal deformations of ICIS, which will be closely related to the notion of $\mathscr{K}$-versal unfoldings of $\mathscr{K}$-finite map germs $f\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^{N-n}, 0)$.

Definition 5.2.1. Let (X_0, x_0) be an ICIS of dimension n. A *deformation* of (X_0, x_0) is a pair (ι, f), where $f\colon (X, x) \to (S, s)$ is a holomorphic map germ between smooth spaces with $\dim(X, x) - \dim(S, s) = n$ and ι is an isomorphism from (X_0, x_0) to the fibre (X_s, x) of f, so ι induces an isomorphism between $\mathscr{O}_{X,x}/\mathfrak{m}_{S,s}\mathscr{O}_{X,x}$ and $\mathscr{O}_{X_0,x_0}$.

A *morphism* from a deformation (ι', f') to another (ι, f) is a pair $(g, \tilde{g})$ such that the diagram

$$\begin{tikzcd} (X', x') \arrow[r, "\tilde{g}"] \arrow[d, "f'"] & (X, x) \arrow[d, "f"] \\ (S', s') \arrow[r, "g"] & (S, s) \end{tikzcd} \tag{5.3}$$

is Cartesian and $\tilde{g} \circ \iota' = \iota$.

A deformation (ι, f) is called *versal* if for any other deformation (ι', f') there exists a morphism $(g, \tilde{g})$ from (ι', f') to (ι, f). In general, we do not require the morphism $(g, \tilde{g})$ to be unique in any sense. However, if the differential $dg_{s'} : T_{s'}S' \to T_sS$ is unique, we say that the deformation is *miniversal*.

Remark 5.2.2. Let (ι, f) be a deformation, with $f\colon (X, x) \to (S, s)$. Assume we have an $\mathscr{A}$-equivalence (ϕ, ψ) between f and $f'\colon (X', x) \to (S', s)$. Then (ι', f'), with $\phi \circ \iota$, is also a deformation isomorphic to (ι, f) (see Exercise 1). In particular, (ι, f) is isomorphic to a deformation of the form (ι', f'), with $f'\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^{N-n}, 0)$.

Lemma 5.2.3. *Let $(g, \tilde{g})$ be a morphism from (ι', f') to (ι, f). Then we have a commutative diagram*

$$\begin{tikzcd} T_{s'}S' \arrow[r, "\rho_{f'}"] \arrow[d, "dg_{s'}"] & T^1_{X'_{s'},x'} \arrow[d, "L"] \\ T_sS \arrow[r, "\rho_f"] & T^1_{X_s,x} \end{tikzcd} \tag{5.4}$$

for some isomorphism (φ, L).

Proof. We split the morphism $(g, \tilde{g})$ as the composition of two morphisms:

$$\begin{tikzcd} (X', x') \arrow[r] \arrow[d, "f'"] & (X \times S', (x, s')) \arrow[r] \arrow[d, "f \times \mathrm{id}_{S'}"] & (X, x) \arrow[d, "f"] \\ (S', s') \arrow[r, "(g, \mathrm{id}_{S'})"] & (S \times S', (s, s')) \arrow[r, "\pi_1"] & (S, s) \end{tikzcd} \tag{5.5}$$

where π_1 is the projection onto the first component. It is enough to show that for each morphism we have a commutative diagram as in (5.4).

In the left hand side of (5.5), $\varphi = (g, \mathrm{id}_{S'})$ is an embedding. The fact that (X', x') is smooth implies that φ is transverse to $F = f \times \mathrm{id}_{S'}$. Thus, we can choose coordinates in such a way that such Cartesian square is transformed into another one of the form

$$\begin{tikzcd} (\mathbb{C}^N, 0) \arrow[r, "j"] \arrow[d, "h"] & (\mathbb{C}^N \times \mathbb{C}^r, 0) \arrow[d, "H"] \\ (\mathbb{C}^{N-n}, 0) \arrow[r, "i"] & (\mathbb{C}^{N-n} \times \mathbb{C}^r, 0) \end{tikzcd}$$

where $i(y) = (y, 0)$, $j(x) = (x, 0)$ and H is an unfolding of h. We have a commutative diagram

$$\begin{tikzcd} T_0\mathbb{C}^{N-n} \arrow[r, "\rho_h"] \arrow[d, "di_0"] & \frac{\theta(h)}{T\mathscr{K}_e h} \\ T_0(\mathbb{C}^{N-n}\times\mathbb{C}^r) \arrow[r, "\rho_H"] & \frac{\theta(H)}{T\mathscr{K}_e H} \arrow[u, "L"'] \end{tikzcd}$$

where L is the isomorphism of Exercise 5.1.2. In fact, for $i = 1, \dots, N-n$,

$$L \circ \rho_H \circ di_0 \left(\left.\frac{\partial}{\partial y_i}\right|_0 \right) = L \circ \rho_H \left(\left.\frac{\partial}{\partial y_i}\right|_0 \right) = L \left[\frac{\partial}{\partial y_i} \right] = \left[\frac{\partial}{\partial y_i} \right] = \rho_h \left(\left.\frac{\partial}{\partial y_i}\right|_0 \right).$$

Now we look at the Cartesian square in the right hand side of (5.5). By taking coordinates again, we can see it as a Cartesian square of the form

$$\begin{tikzcd} (\mathbb{C}^N \times \mathbb{C}^r, 0) \arrow[r, "q"] \arrow[d, "H"] & (\mathbb{C}^N, 0) \arrow[d, "h"] \\ (\mathbb{C}^{N-n} \times \mathbb{C}^r, 0) \arrow[r, "p"] & (\mathbb{C}^{N-n}, 0) \end{tikzcd}$$

where $p(y, u) = y$, $q(x, u) = x$ and $H = h \times \mathrm{id}_{\mathbb{C}^r}$. Now it is obvious that we have a commutative diagram

$$\begin{tikzcd} T_0(\mathbb{C}^{N-n}\times\mathbb{C}^r) \arrow[r, "\rho_H"] \arrow[d, "dp_0"] & \frac{\theta(H)}{T\mathscr{K}_e H} \arrow[d, "L"] \\ T_0\mathbb{C}^{N-n} \arrow[r, "\rho_h"] & \frac{\theta(h)}{T\mathscr{K}_e h} \end{tikzcd}$$

where L is the isomorphism given in (5.2).

□

Lemma 5.2.4. *Let (X_0, x_0) be an ICIS. There exists a deformation whose reduced Kodaira-Spencer map is an isomorphism. Any deformation admits a morphism to a deformation with surjective Kodaira-Spencer map.*

Proof. The first part is just Mather's method to construct stable map germs. Assume that (X_0, x_0) is the fibre of a $\mathscr{K}$-finite map germ $f : (\mathbb{C}^N, 0) \to (\mathbb{C}^{N-n}, 0)$ of rank 0 (see Exercise 4). It follows that $T\mathscr{K}_e f \subseteq \mathfrak{m}_N \theta(f)$. Since f is $\mathscr{K}$-finite,

$$\mathscr{K}_e - \mathrm{codim}(f) = \dim_{\mathbb{C}} \frac{\theta(f)}{T\mathscr{K}_e f} < \infty,$$

so $\mathfrak{m}_N\theta(f)/T\mathscr{K}_e(f)$ has also finite dimension as a $\mathbb{C}$-vector space. We choose $g_1, \dots, g_d \in \theta(f)$ whose residue classes give a $\mathbb{C}$-basis of $\mathfrak{m}_N\theta(f)/T\mathscr{K}_e(f)$. So, $\partial/\partial y_1, \dots, \partial/\partial y_{N-n}$ together with $g_1, \dots, g_d$ give a $\mathbb{C}$-basis of $\theta(f)/T\mathscr{K}_e(f)$ and $\mathscr{K}_e - \mathrm{codim}(f) = N - n + d$.

We consider the d-parameter unfolding

$$F(x, u) = \left(f(x) + \sum\nolimits_{i=1}^{d} u_i g_i(x), u \right),$$

so (ι, F) is a deformation of (X_0, x_0), with $\iota(x) = (x, 0)$. Its reduced Kodaira-Spencer map

$$\rho_F : T_0(\mathbb{C}^{N-n} \times \mathbb{C}^d) \longrightarrow \frac{\theta(f)}{T\mathscr{K}_e f}$$

now sends $\partial/\partial y_i|_0$ to the class of $\partial/\partial y_i$ and $\partial/\partial u_j|_0$ to the class of g_j. Therefore, it is an isomorphism.

To prove the second part, let (ι', f') be any deformation of (X_0, x_0). By taking charts in (X, x) and (S, s) we can assume that $f' : (\mathbb{C}^M, 0) \to (\mathbb{C}^{M-n}, 0)$, for some $M \geq N$. Now, we can proceed as before, but in this case we get a d-parameter unfolding F' whose Kodaira-Spencer map

$$\rho_{F'} : T_0(\mathbb{C}^{M-n} \times \mathbb{C}^d) \longrightarrow \frac{\theta(f)}{T\mathscr{K}_e f'}$$

is only surjective. The required deformation is (ι'', F'), where $\iota''(x) = (\iota'(x), 0)$ and the morphism from (ι', f') to (ι'', F') is defined in the obvious way. □

Theorem 5.2.5. *Let (X_0, x_0) be an ICIS. Then:*

1. *A deformation of (X_0, x_0) is versal if and only if its Kodaira-Spencer map is surjective.*

2. *Two versal deformations of (X_0, x_0) are isomorphic if their base germs have the same dimension.*

3. *(X_0, x_0) admits a miniversal deformation and any two are isomorphic.*

4. *A deformation of (X_0, x_0) is miniversal if and only if its reduced Kodaira-Spencer map is an isomorphism.*

Proof. 1. Let (ι, f) be a versal deformation. By Lemma 5.2.4, there exists a deformation (ι', f') whose Kodaira-Spencer map $\rho_{f'}$ is surjective. By versality, there exists a morphism $(g, \tilde{g})$ from (ι', f') to (ι, f). Now Lemma 5.2.3 gives $\rho_f \circ dg_s = L \circ \rho_{f'}$, for some isomorphism L. Hence, ρ_f is also surjective.

Suppose now that (ι, f) is a deformation such that ρ_f is surjective and let (ι', f') be any deformation. By Lemma 5.2.4, there exists another deformation (ι'', f'') such that $\rho_{f''}$ is surjective and a morphism $(\iota', f') \to (\iota'', f'')$. By Theorem 4.4.10, f and f'' are $\mathscr{A}$-stable. Suppose that $f : (X, x) \to (S, s)$ and $f'' : (X'', x'') \to (S'', s'')$, with $\dim S = p$ and $\dim S'' = q$. Take $k = \max\{p, q\}$ and consider $f \times \mathrm{id}_{\mathbb{C}^{k-p}}$ and $f'' \times \mathrm{id}_{\mathbb{C}^{k-q}}$. These two maps are also $\mathscr{A}$-stable map germs between spaces of the same dimension and they are also $\mathscr{K}$-equivalent. By Theorem 4.4.11, $f \times \mathrm{id}_{\mathbb{C}^{k-p}}$ and $f'' \times \mathrm{id}_{\mathbb{C}^{k-q}}$ are $\mathscr{A}$-equivalent and hence, the corresponding deformations are isomorphic, by Exercise 1. Now it is obvious that we have a morphism $(\iota'', f'') \to (\iota, f)$.

2. Let (ι, f) and (ι', f') be versal deformations such that the base spaces have the same dimension. By 1, the Kodaira-Spencer mappings are surjective, so f and f' are $\mathscr{A}$-stable, by Theorem 4.4.10. Since f and f' are $\mathscr{K}$-equivalent, they are $\mathscr{A}$-equivalent by Theorem 4.4.11 and hence, (ι, f) and (ι', f') are isomorphic (see Exercise 1).

3. and 4. Let (ι, f) be a deformation such that ρ_f is an isomorphism. By 1, it is versal and hence, for any deformation (ι', f') there exists a morphism $(g, \tilde{g})$ from (ι', f') to (ι, f). By Lemma 5.2.3, $\rho_f \circ dg_s = L \circ \rho_{f'}$, for some isomorphism L. Since ρ_f is invertible, dg_s is unique, so (ι, f) is miniversal.

Conversely, let (ι, f) be a miniversal deformation. Then ρ_f is surjective, by 1. By Lemma 5.2.4, there exists a deformation (ι', f') such that $\rho_{f'}$ is an isomorphism. By 2, (ι, f) is isomorphic to $(\iota', f' \times \mathrm{id}_{\mathbb{C}^\ell})$, for some $\ell \geq 0$. We must show that $\ell = 0$.

In fact, let $(g, \tilde{g})$ be the isomorphism $(\iota, f) \to (\iota', f' \times \mathrm{id}_{\mathbb{C}^\ell})$. We have another morphism $(g', \tilde{g}')$ given as a composition

$$(\iota, f) \longmapsto (\iota', f') \longmapsto (\iota', f' \times \mathrm{id}_{\mathbb{C}^\ell}).$$

By miniversality, dg_s and dg'_s must coincide, but this is only possible if $\ell = 0$.

□

Remark 5.2.6. The proof of Lemma 5.2.4 gives an algorithm to construct a minimersal deformation (this is Mather's algorithm to construct stable germs).

Example 5.2.7. Let $f\colon (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$ be given by

$$(x, y, z) \mapsto (x^2 + y^2 + z^2, xy).$$

Then $T\mathscr{K}_e f$ is the submodule of $\mathscr{O}_3^2$ generated by

$$(2x, y), (2y, x), (2z, 0)$$

plus the submodule $(x^2+y^2+z^2, xy)\mathscr{O}_3^2$. A $\mathbb{C}$-basis of $\mathfrak{m}_3\mathscr{O}_3^2/T\mathscr{K}_e f$ is (computed with Singular):

$$(y, 0), (0, y), (0, z)$$

A miniversal deformation of the fibre $(X, 0)$ of f is (ι, F), where $\iota(x, y, z) = (x, y, z, 0)$ and $F\colon (\mathbb{C}^3 \times \mathbb{C}^3, 0) \to (\mathbb{C}^2 \times \mathbb{C}^3, 0)$ is given by

$$F(x, y, z, u) = (f_1(x, y, z) + u_1 y, f_2(x, y, z) + u_2 y + u_3 z, u).$$

In this case, we have $\tau = 5$.

The following corollary gives the relationship between versality of deformations of ICIS and $\mathscr{K}$-versality of unfoldings of map germs.

Corollary 5.2.8. *Let (X_0, x_0) be an ICIS given as the fibre of a $\mathscr{K}$-finite map germ $f\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^{N-n}, 0)$. Let $F(x, u) = (f_u(x), u)$ be an r-parameter unfolding of f. The following statements are equivalent:*

1. *(ι, F) is a versal deformation of (X_0, x_0), where $\iota(x) = (x, 0)$,*
2. *F is $\mathscr{A}$-stable,*
3. *$\tilde{F}(x, u, v) = (f_u(x) + v, u, v)$ is a $\mathscr{K}$-versal deformation of f.*

Exercises

1. Let (ι, f) be a deformation, with $f\colon (X,x) \to (S,s)$. Assume we have an $\mathscr{A}$-equivalence (ϕ,ψ) between f and $f'\colon (X',x) \to (S',s)$. Then (ι', f'), with $\phi\circ\iota$, is also a deformation isomorphic to (ι, f).

2. Compute a versal deformation and the Tjurina number of the ICIS defined by the following map germs:

   $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^2,0)$, $f(x,y) = (x^3, y^2)$,

   $f\colon (\mathbb{C}^3,0) \to (\mathbb{C}^2,0)$, $f(x,y) = (x^3 + y^2 + z^2, xy)$.

5.3 Some analytic properties of versal deformations

Definition 5.3.1. Let $(Z,0)$ be a germ of analytic subset of $(\mathbb{C}^N,0)$. A vector field $\xi \in \theta_N$ is called *logarithmic* for $(Z,0)$ if $\xi(h) \in I(Z,0)$, for all $h \in I(Z,0)$. This is equivalent to the fact that ξ is tangent to Z at any regular point z of Z. The subset of all logarithmic vector fields is denoted by $\mathrm{Der}(-\log Z)$ and it is not difficult to see that $\mathrm{Der}(-\log Z)$ is an $\mathscr{O}_N$-submodule of θ_N.

Definition 5.3.2. A *free divisor* is a hypersurface $(Z,0) \subset (\mathbb{C}^N,0)$ such that $\mathrm{Der}(-\log Z)$ is a free $\mathscr{O}_N$-module of rank N.

Definition 5.3.3. Consider a holomorphic map germ $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$. A vector field $\xi \in \theta_p$ is called *liftable* if there exists $\eta \in \theta_n$ such that $df\circ\eta = \xi\circ f$. The subset of θ_p of liftable vector fields is denoted by $\mathrm{Lift}(f)$. We observe that $\mathrm{Lift}(f)$ is the kernel of the morphism $\overline{\omega f}$ defined as the composition

$$\theta_p \xrightarrow{\omega f} \theta(f) \longrightarrow \frac{\theta(f)}{T\mathscr{R}_e f},$$

so $\mathrm{Lift}(f)$ is a $\mathscr{O}_p$-submodule of θ_p.

We recall that the *discriminant* of a $\mathscr{K}$-finite map germ $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$, with $n \geq p$ is $D := f(C)$, the image of the critical locus C of f. By the geometric criterion of $\mathscr{K}$-finiteness (Theorem **??**), the restriction to the critical locus $f\colon C \to (\mathbb{C}^p,0)$ is finite and hence its image D is analytic in $(\mathbb{C}^p,0)$.

Lemma 5.3.4. *The discriminant D of a $\mathscr{K}$-finite map germ $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ with $n \geq p$ is a hypersurface of $(\mathbb{C}^p,0)$.*

Proof. On one hand, C is defined as the zero locus of the ideal in $\mathscr{O}_n$ generated by the $p\times p$-minors of the Jacobian matrix of f. This implies that $\dim D = \dim C \geq n - (n-p+1) = p-1$. On the other hand, D is a null subset in $(\mathbb{C}^p,0)$, by Sard's Theorem and thus $\dim D = p-1$. □

Proposition 5.3.5. *Let $f\colon (\mathbb{C}^n,0) \to (\mathbb{C}^p,0)$ be $\mathscr{A}$-stable, $n \geq p \geq 2$. Then, the restriction $f\colon C \to D$ is the normalisation of D.*

Proof. We know that C is the zero locus of the ideal in $\mathscr{O}_n$ generated by the $p\times p$-minors of the Jacobian matrix of f and $\dim C = n - (n-p+1) = p-1$, by Lemma 5.3.4. This implies that

C is determinantal and hence, Cohen-Macaulay. Moreover, since f is stable, the differential $df\colon (\mathbb{C}^n, x) \to \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^p)$ is multitransverse to the rank stratification of $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^p)$.

On one hand, the singular locus $\Sigma(C)$ of C is the inverse image of the subset of linear maps of rank $\leq p-2$, which has dimension $n-2(n-p+2) \leq p-4$. By Serre's conditions S2 and R1, C is normal.

On the other hand, $f\colon C \to D$ is finite by the geometric criterion of $\mathscr{K}$-finiteness (Theorem **??**). Moreover, the multitransversality also implies that the set of pairs $(x, x') \in C \times C$ such that $x \neq x'$ and $f(x) = f(x')$ has dimension $p-2$, so $f\colon C \to D$ is generically one-to-one. Hence, $f\colon C \to D$ is the normalisation of D.

□

Proposition 5.3.6. *Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be $\mathscr{A}$-stable, $n \geq p$, with discriminant D. Then,*

$$\mathrm{Lift}(f) = \mathrm{Der}(-\log D).$$

Proof. Suppose $\xi \in \mathrm{Lift}(f)$, so $df \circ \eta = \xi \circ f$, for some $\eta \in \theta_n$. By integrating η and ξ we get flows ϕ_t and ψ_t, respectively, such that $f \circ \phi_t = \psi_t \circ f$. Given any point $y \in D$, with $y = f(x)$ and $x \in C$, we have a commutative diagram

$$\begin{array}{ccc} (\mathbb{C}^n, x) & \xrightarrow{f} & (\mathbb{C}^p, y) \\ \downarrow{\scriptstyle \phi_t} & & \downarrow{\scriptstyle \psi_t} \\ (\mathbb{C}^n, \phi_t(x)) & \xrightarrow{f} & (\mathbb{C}^p, \psi_t(y)) \end{array}$$

which gives $\phi_t(x) \in C$ and $\psi_t(y) \in D$. This shows that ξ is tangent to D at any regular point $y \in D$, hence $\xi \in \mathrm{Der}(-\log D)$.

Let us see the converse. The case $p = 1$ is trivial, because the only stable singularities are regular points or Morse critical points and in both cases the equality $\mathrm{Lift}(f) = \mathrm{Der}(-\log D)$ is obvious. We can assume $p \geq 2$ and hence, the restriction $f\colon C \to D$ is the normalisation of D, by Lemma 5.3.5.

Let $\xi \in \mathrm{Der}(-\log D)$. By integrating ξ we get a flow $\Psi\colon (\mathbb{C}^p \times \mathbb{C}, 0) \to (\mathbb{C}^p, 0)$ with the property that $\psi_t(D) \subset D$. We consider the diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ \uparrow{\scriptstyle \Phi} & & \uparrow{\scriptstyle \Psi} \\ (C \times \mathbb{C}, 0) & \xrightarrow{f \times \mathrm{id}_{\mathbb{C}}} & (D \times \mathbb{C}, 0) \end{array}$$

Since C is normal, $(C \times \mathbb{C}, 0)$ is also normal. By the universal property of the normalisation, there exists a unique analytic mapping $\Phi\colon (C \times \mathbb{C}, 0) \to C$ such that $f \circ \Phi = \Psi \circ (f \times \mathrm{id}_{\mathbb{C}})$. We take $\bar{\Phi}\colon (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}^n, 0)$ an analytic extension of Φ and define $\eta \in \theta_n$ as $\eta_x = \frac{\partial \bar{\Phi}}{\partial t}(x, 0)$.

By construction, $tf(\eta) = \omega f(\xi)$ on C. Since C is normal, it is reduced and hence the ideal of functions vanishing on C is $J(f)$, the ideal in $\mathscr{O}_n$ generated by the $p \times p$-minors of the Jacobian matrix of f. This gives

$$tf(\eta) - \omega f(\xi) \in J(f)\theta(f).$$

Observe that $J(f)$ is the 0th-Fitting ideal of $\theta(f)/T\mathscr{R}_e f$. By Exercise 2 we have $J(f)\theta(f) \subseteq T\mathscr{R}_e f$, hence

$$tf(\eta) - \omega f(\xi) = tf(\eta_1),$$

for some $\eta_1 \in \theta_n$. This implies

$$\omega f(\xi) = tf(\eta - \eta_1),$$

so $\xi \in \mathrm{Lift}(f)$. □

Theorem 5.3.7. *Let $f: (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be $\mathscr{A}$-stable, $n \geq p$. Then its discriminant D is a free divisor.*

Proof. By Proposition 5.3.6, $\mathrm{Lift}(f) = \mathrm{Der}(-\log D)$, so we must show that $\mathrm{Lift}(f)$ is free of rank p. The module $\theta(f)/T\mathscr{R}_e f$ has a matrix presentation over $\mathscr{O}_n$

$$\mathscr{O}_n^n \xrightarrow{df} \mathscr{O}_n^p \longrightarrow \frac{\theta(f)}{T\mathscr{R}_e f} \longrightarrow 0$$

and the support of $\theta(f)/T\mathscr{R}_e f$ is the critical locus C, which has dimension $p-1$. By a theorem of Buchsbaum and Rim [5], $\theta(f)/T\mathscr{R}_e f$ is Cohen-Macaulay. In particular,

$$\operatorname{depth} \frac{\theta(f)}{T\mathscr{R}_e f} = \dim \frac{\theta(f)}{T\mathscr{R}_e f} = p - 1.$$

But the depth of $\theta(f)/T\mathscr{R}_e f$ is the same when regarded as an $\mathscr{O}_p$-module via $f^*: \mathscr{O}_p \to \mathscr{O}_n$.

Consider the sequence of $\mathscr{O}_p$-modules

$$0 \longrightarrow \mathrm{Lift}(f) \longrightarrow \theta_p \xrightarrow{\overline{\omega f}} \frac{\theta(f)}{T\mathscr{R}_e f} \longrightarrow 0 \tag{5.6}$$

Here $\overline{\omega f}$ is surjective because f is $\mathscr{A}$-stable and by definition, $\mathrm{Lift}(f)$ is the kernel of $\overline{\omega f}$. So, the sequence (5.6) is exact. By the depth Lemma,

$$\operatorname{depth} \mathrm{Lift}(f) \geq \min\left\{\operatorname{depth} \theta_p, \operatorname{depth} \frac{\theta(f)}{T\mathscr{R}_e f} + 1\right\} = p.$$

But also $\operatorname{depth} \mathrm{Lift}(f) \leq \dim \mathrm{Lift}(f) \leq \dim \theta_p = p$, so $\operatorname{depth} \mathrm{Lift}(f) = p$. By the Auslander-Buchsbaum formula, the projective dimension of $\mathrm{Lift}(f)$ is 0, which means that it is a free $\mathscr{O}_p$-module.

Finally, the rank of $\theta(f)/T\mathscr{R}_e f$ over $\mathscr{O}_p$ is zero (see Exercise 3). Again the exactness of (5.6) implies that $\mathrm{Lift}(f)$ must have rank p.

□

Exercises

1. Let M be an R-module. Show that $F_0(M) \subseteq \operatorname{Ann} M$, where $F_0(M)$ is the 0-th Fitting ideal of M, that is, $F_0(M)$ is the ideal $I_p(\varphi)$ in R generated by the $p \times p$-minors of a matrix presentation

$$R^n \xrightarrow{\varphi} R^p \longrightarrow M \longrightarrow 0$$

and $\operatorname{Ann} M = \{a \in R \mid aM = 0\}$ is the annihilator of M. (Hint: Use the Cramer's rule).

2. Let $f: (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be holomorphic. Show that $J(f)\theta(f) \subseteq T\mathscr{R}_e f$, where $J(f)$ is the ideal generated by the $p \times p$ minors of the Jacobian matrix of f.

3. Let $f: (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be $\mathscr{K}$-finite. Show that $\theta(f)/T\mathscr{R}_e f$ has rank zero over $\mathscr{O}_p$. (Hint: Let $h \in \mathscr{O}_p$ be a reduced equation of D and show that $h \in \operatorname{Ann}_{\mathscr{O}_p}(\theta(f)/T\mathscr{R}_e f)$.)

Chapter 6

Topology of the generic fibres

6.1 The link of an isolated singularity

Let X be a closed analytic subset of an open set $U \subseteq \mathbb{C}^N$ and $x \in X$ such that $X \setminus \{x\}$ is smooth of constant dimension n. The main result of this section will be that X is homeomorphic at x to a cone on a C^∞-manifold, which is unique up to diffeomorphism.

Lemma 6.1.1 (Curve Selection Lemma)**.** *Let V be an open neighbourhood of p in $\mathbb{R}^m$ and let $f_1, \ldots, f_k, g_1, \ldots . g_\ell$ be real analytic functions on V such that p is in the closure of*

$$Z := \{x \in V \mid f_i(x) = 0,\ i = 1, \ldots, k,\ g_j(x) > 0,\ j = 1, \ldots, \ell\}.$$

Then there exists a real analytic curve $\gamma\colon [0, \delta) \to V$ with $\gamma(0) = 0$ and $\gamma(t) \in Z$ for $t \in (0, \delta)$.

Proof. [35, Lemma 3.1] for f_i, g_j polynomials, but the proof works also for real analytic functions. □

Lemma 6.1.2. *Let $r\colon X \to [0, \infty)$ be the restriction of a real analytic function $\tilde{r}\colon U \to \mathbb{R}$ such that $r^{-1}(0) = \{x\}$. Then 0 is not an accumulation point of critical values of $r|_{X\setminus\{x\}}$.*

Proof. See [28, Lemma 2.2]. □

Definition 6.1.3. Let $r\colon X \to [0, \infty)$ be as in Lemma 6.1.2. Then we say that r *defines the point x in X*. We use the following notation

$$X_{r\leq\epsilon} := \{x' \in X \mid r(x') \leq \epsilon\}$$

and similarly $X_{r<\epsilon}$, $X_{r=\epsilon}$, $X_{0<r<\epsilon}$, etc.

Remark 6.1.4. Usually we take $r\colon X \to [0, \infty)$ as $r(x') = \|x' - x\|^2$. Then

$$X_{r\leq\epsilon} = X \cap \overline{B}_\epsilon, \quad X_{r<\epsilon} = X \cap B_\epsilon, \quad X_{r=\epsilon} = X \cap S_\epsilon,$$

where $\overline{B}_\epsilon$, B_ϵ and S_ϵ are the closed ball, the open ball and the sphere of radius ϵ centered at x, respectively.

We recall that the cone on a topological space Z is the quotient space

$$CZ = \frac{Z \times [0, 1]}{Z \times \{0\}}.$$

Proposition 6.1.5. *Let $r\colon X \to [0,\infty)$ define x in X and let $\epsilon > 0$ such that $X_{r\le\epsilon}$ is compact and $r|_{X\setminus\{x\}}$ has no critical values in $(0,\epsilon]$. Then $X_{r=\epsilon}$ is a compact real analytic submanifold of U and there exists a homeomorphism H from the cone on $X_{r=\epsilon}$ onto $X_{r\le\epsilon}$ such that*

$$\begin{tikzcd} C(X_{r=\epsilon}) \arrow[r, "H"] \arrow[dr, "\pi"'] & X_{r\le\epsilon} \arrow[d, "\frac{1}{\epsilon}r"] \\ & {[0,1]} \end{tikzcd}$$

commutes, where π is induced by the projection into the second component.

Proof. See [28, Proposition 2.4]. □

Definition 6.1.6. Let $r\colon X \to [0,\infty)$ and $\epsilon > 0$ be as in Proposition 6.1.5. The submanifold $X_{r=\epsilon}$ is called the *link* of X at x.

When $r(x') = \|x' - x\|^2$, then $\epsilon > 0$ is called a *Milnor radius* for X at x and $\overline{B}_\epsilon$ (resp. B_ϵ, S_ϵ) is called a closed Milnor ball (resp. open Milnor ball, Milnor sphere) (see Remark 6.1.4).

In the next proposition we show that, up to isotopy, the link does not depend on the choice of r and ϵ.

Proposition 6.1.7. *Let $r, r'\colon X \to [0,\infty)$ define x in X. Then there exists $\epsilon > 0$ such that*

1. *the hypotheses of 6.1.5 are satisfied for r and ϵ;*

2. *if $\epsilon' > 0$ is such that $X_{r'\le\epsilon'} \subseteq X_{r\le\epsilon}$, then the hypotheses of 6.1.5 are satisfied for r' and ϵ' and there exists a diffeomorphism of $X_{r\le\epsilon, r'\ge\epsilon'}$ onto $[0,1]\times X_{r=\epsilon}$ which maps $X_{r=\epsilon}$ (resp. $X_{r'=\epsilon'}$) onto $\{0\}\times X_{r=\epsilon}$ (resp. $\{1\}\times X_{r'=\epsilon'}$).*

Proof. See [28, Proposition 2.5]. □

Corollary 6.1.8. *The diffeomorphism type of the link of X at x only depends on the abstract analytic set germ (X,x) (i.e., on the $\mathbb{C}$-algebra $\mathscr{O}_{X,x}$).*

Proof. Suppose that (X,x) and (X',x') are analytic set germs with isolated singularity such that $\mathscr{O}_{X,x} \cong \mathscr{O}_{X',x'}$. This implies that there exists a biholomorphism $\phi\colon (X,x) \to (X',x')$. If $r\colon X \to [0,\infty)$ defines x in X, then $r' = r\circ\phi^{-1}\colon X' \to [0,\infty)$ defines x' in X' and for $\epsilon > 0$ small enough, we have $\phi(X_{r=\epsilon}) = X'_{r'=\epsilon}$. □

6.2 The Milnor fibration

We assume that X is closed analytic set of pure dimension $n+k$ in some open $U \subseteq \mathbb{C}^N$, $x \in X$ and $F\colon U \to \mathbb{C}^k$ is an analytic map with the property that at each point of $F^{-1}F(x)\setminus\{x\}$, X is nonsingular and $F|_X$ is a submersion. As this is clearly a property of the restriction $f := F|_X$, we say that f *defines an isolated singularity* at x (even when f is a submersion at x). For convenience we assume that $f(x) = 0$. We also suppose that $r\colon X \to [0,\infty)$ is a real analytic function such that $r|_{f^{-1}(0)}$ defines $f^{-1}(0)$ at x. We take $\epsilon > 0$ such that the hypotheses of 6.1.5 are statisfied for $r|_{f^{-1}(0)}$ and ϵ.

Since $f^{-1}(0)_{r=\epsilon}$ is compact, there exists S a neighbourhood of 0 in $\mathbb{C}^k$ such that $f|_{X_{r=\epsilon}}$ is a submersion along $f^{-1}(S)_{r=\epsilon}$. We take S contractible. We introduce the following notation

$$X := f^{-1}(S)_{r<\epsilon}, \quad \overline{X} := f^{-1}(S)_{r\leq\epsilon}, \quad \partial\overline{X} := f^{-1}(S)_{r=\epsilon},$$

so that X is open in $f^{-1}(S)$ and $\overline{X}$ (resp. $\partial\overline{X}$) is its closure (resp. boundary) in $f^{-1}(S)$.

We also denote by C_f (or just C) the *critical locus* of f (i.e., the set of points of X which are singular or where f is not a submersion). If $k = 0$, then C_f is just the singular locus of X, which we denote by X_{sing} and we write $X_{reg} = X \setminus X_{sing}$. For each $s \in S$, let $\overline{X}_s$ (resp. X_s) be the intersection of $f^{-1}(s)$ with $\overline{X}$ (resp. X). Similarly, if $A \subseteq S$, we put

$$\overline{X}_A := \overline{X} \cap f^{-1}(A), \quad X_A := X \cap f^{-1}(A).$$

The image $D_f := f(C_f)$ is called the *discriminant* of f. We call the restriction $f\colon X \to S$ (resp. $f\colon \overline{X} \to S$) a *good representative* (reps. *good proper representative*) of f.

Remark 6.2.1. As in Remark 6.1.4, usually we take $r(x') = \|x' - x\|^2$ and it also common to take $S = B_\eta$, the open ball in $\mathbb{C}^k$ centered at 0 of radius η. In such a case, a good proper representative has the form

$$f\colon X \cap \overline{B}_\epsilon \cap f^{-1}(B_\eta) \longrightarrow B_\eta, \tag{6.1}$$

where $0 < \eta \ll \epsilon \ll 1$ and analogously for a good representative taking B_ϵ instead of $\overline{B}_\epsilon$. The expression $0 < \eta \ll \epsilon \ll 1$ means that we have to choose first $\epsilon > 0$ small enough and once we have fixed ϵ, then we choose $\eta > 0$ small enough, depending of ϵ.

Theorem 6.2.2. *With the above notation, we have:*

1. $f : \overline{X} \to S$ *is proper and* $f\colon \partial\overline{X} \to S$ *is a* C^∞*-trivial bundle.*
2. C_f *is analytic in* X *and closed in* $\overline{X}$*. Moreover,* $f|_{C_f}$ *is finite (i.e., proper with finite fibres).*
3. X_{sing} *has dimension* $\leq k$ *and* $C_f \setminus X_{sing}$ *has pure dimension* $k-1$.
4. D_f *is an analytic subset of* S *of the same dimension as* C_f*. It is a hypersurface in* S *(or void) if* $C_f \setminus X_{sing}$ *is dense in* C_f.
5. *The mapping* $f\colon (\overline{X}_{S\setminus D_f}, \partial\overline{X}_{S\setminus D_f}) \to S \setminus D_f$ *is a* C^∞*-fibre bundle pair, of which each fibre pair* $(\overline{X}_s, \partial\overline{X}_s)$ *is a complex analytic* n*-manifold with boundary.*
6. f *defines an ICIS at every point of* $\overline{X}_{reg}$.

Proof. See [28, Theorem 2.8]. □

Definition 6.2.3. With the notation of Theorem 6.2.2, the fibre X_s (resp. $\overline{X}_s$), with $s \in S \setminus D$ is called a *Milnor fibre* (resp. a *compact Milnor fibre*) and the fibre bundle of item 5 is referred as the *Milnor fibration*.

Remark 6.2.4. When we take $r(x') = \|x' - x\|^2$ and $S = B_\eta$ as in (6.1), the Milnor fibration is given by

$$f\colon X \cap \overline{B}_\epsilon \cap f^{-1}(B_\eta \setminus D_f) \longrightarrow B_\eta \setminus D_f, \tag{6.2}$$

and the Milnor fibre (resp. compact Milnor fibre) is $X \cap B_\epsilon \cap f^{-1}(s)$ (resp. $X \cap \overline{B}_\epsilon \cap f^{-1}(s)$), with $s \in B_\eta \setminus D_f$.

The Milnor fibration was considered for the first time by Milnor in the case $k = 1$ and X smooth [35]. The next proposition shows that the Milnor fibration only depends, up to diffeomorphism, on the germ $f\colon (X, x) \to (\mathbb{C}^k, 0)$.

Proposition 6.2.5. *Let $\overline{f}\colon \overline{X} \to S$ and $\overline{f}'\colon \overline{X}' \to S'$ be good proper representatives of the germ $f\colon (X, x) \to (\mathbb{C}^k, 0)$. Then there exist a neighbourhood T of 0 in $S \cap S'$ and a C^∞-diffeomorphism $H\colon (\overline{X}_T, \partial \overline{X}_T) \to (\overline{X}'_T, \partial \overline{X}'_T)$ which is the identity on a neighbourhood of $C_f \cap f^{-1}(T)$ and commutes with the projection onto T. In particular, H induces a diffeomorphism $\overline{X}_s \to \overline{X}'_s$, for all $s \in T$.*

Proof. See [28, Proposition 2.9]. □

6.3 The homotopy type of the Milnor fibre

In this section we consider the compact Milnor fibre $\overline{X}_s$ of an n-dimensional ICIS (X_0, x) given as the fibre of a $\mathscr{K}$-finite map germ $f\colon (\mathbb{C}^{n+k}, x) \to (\mathbb{C}^k, 0)$. The following theorem was proved first by Milnor [35] for the hypersurface case ($k = 1$) and later by Hamm [21] in the general case.

Theorem 6.3.1. *Let (X_0, x) be an ICIS of dimension n and take a good proper representative $f\colon \overline{X} \to S$. For any $s \in S$, $\overline{X}_s$ has the homotopy type of a wedge of a finite number of n-spheres $\overline{X}_s \simeq S^n \vee \cdots \vee S^n$.*

Proof. See [28, 5.7 and 5.9]. □

We use this theorem in the particular case of a compact Milnor fibre $\overline{X}_s$, with $s \in S \setminus D$, in order to define the Milnor number of an ICIS.

Definition 6.3.2. Let (X_0, x) be an ICIS of dimension n. The number of n-spheres in the compact Milnor fibre $\overline{X}_s \simeq S^n \vee \cdots \vee S^n$ is called the *Milnor number* of (X_0, x) and is denoted by $\mu(X_0, x)$.

The first important property of the Milnor number is that it only depends on the isomorphism class of the ICIS.

Proposition 6.3.3. *Let (X_0, x) and (Y_0, y) be ICIS such that $(X_0, x) \cong (Y_0, y)$, then $\mu(X_0, x) = \mu(Y_0, y)$.*

Proof. We know from Proposition 6.2.5 that $\mu(X_0, x)$ only depends on the $\mathscr{K}$-finite map germ $f\colon (\mathbb{C}^{n+k}, x) \to (\mathbb{C}^k, 0)$ which defines (X_0, x). We assume $x = 0$ for simplicity.

By Corollary 5.2.8, there exists $F(x, u) = (f_u(x), u)$ a stable r-parameter unfolding of f. We take a good proper representative fo F,

$$F\colon \overline{B}'_\epsilon \cap F^{-1}(B'_\eta) \longrightarrow B'_\eta,$$

where $0 < \eta \ll \epsilon \ll 1$, where $\overline{B}'_\epsilon$ is a closed ball in $\mathbb{C}^{n+k} \times \mathbb{C}^r$ and B'_η is an open ball in $\mathbb{C}^k \times \mathbb{C}^r$. After shrinking ϵ and η if necessary, we can assume that

$$f\colon \overline{B}_\epsilon \cap f^{-1}(B_\eta) \longrightarrow B_\eta$$

is also a good proper representative of f where now $\overline{B}_\epsilon$ is a closed ball in $\mathbb{C}^{n+k}$ and B_η is an open ball in $\mathbb{C}^k$. If $s \in B_\eta$ is a regular value of f, then $(s, 0) \in B'_\eta$ is a regular value of F and moreover,

$$(\overline{B}_\epsilon \cap f^{-1}(s)) \times \{0\} = \overline{B}'_\epsilon \cap F^{-1}(s, 0).$$

This shows that f and F define the same Milnor number.

Analogously, (Y_0, y) is the fibre of a $\mathscr{K}$-finite map germ $g\colon (\mathbb{C}^{n+\ell}, 0) \to (\mathbb{C}^\ell, 0)$ and there exists $G(x, v) = (g_v(x), v)$ a stable s-parameter unfolding of g, so g and G define the same Milnor number.

After multiplying F or G by the identity, we can assume that $F, G\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^p, 0)$. Since $(X_0, x) \cong (Y_0, y)$, F and G are $\mathscr{K}$-equivalent and hence $\mathscr{A}$-equivalent, by Theorem **??**. We have a commutative diagram

$$\begin{array}{ccc} (\mathbb{C}^N, 0) & \xrightarrow{F} & (\mathbb{C}^p, 0) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\ (\mathbb{C}^N, 0) & \xrightarrow{G} & (\mathbb{C}^p, 0) \end{array}$$

where ϕ and ψ are diffeomorphisms. We can choose good proper representatives of F and G so that we have a commutative diagram

$$\begin{array}{ccc} \overline{X} & \xrightarrow{F} & S \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\ \overline{X}' & \xrightarrow{G} & S' \end{array}$$

and we have $\psi(D_F) = D_G$. For all $s \in S \setminus D_F$, $\phi(\overline{X}_s) = \overline{X}'_{\psi(s)}$ and hence, F and G define the same Milnor number. □

Let (X_0, x) be an ICIS of dimension n, defined as the fibre of a $\mathscr{K}$-finite map germ $f\colon (\mathbb{C}^{n+k}, x) \to (\mathbb{C}^k, 0)$. By Theorem 6.2.2, the discriminant $(D, 0)$ is a hypersurface in $(\mathbb{C}^k, 0)$. Hence, there exists a line L in $\mathbb{C}^k$ such that $L \cap D = \{0\}$. After a linear change of coordinates in $\mathbb{C}^k$ we can assume that L is the line $y_k = 0$. This is equivalent to that $f' := (f_1, \dots, f_{k-1})\colon (\mathbb{C}^{n+k}, x) \to (\mathbb{C}^{k-1}, 0)$ defines an ICIS (X'_0, x) of dimension $n + 1$. The following result is known as the Lê-Greuel formula and was proved independently by Lê [44] and Greuel [1].

Theorem 6.3.4 (Lê-Greuel formula)**.** *With the above notation we have*

$$\mu(X_0, x) + \mu(X'_0, x) = \dim_{\mathbb{C}} \frac{\mathscr{O}_{n+k,x}}{I(f') + J(f)}, \tag{6.3}$$

where $I(f') = (f_1, \dots, f_{k-1})$ and $J(f)$ is the ideal in $\mathscr{O}_{n+k,x}$ generated by the maximal minors of the Jacobian matrix of f.

Proof. See [28, 5.10]. □

The Lê-Greuel formula (6.3) allows us to compute the Milnor number of any ICIS by means of a recursive formula. In fact, we can choose generic linear coordinates in $\mathbb{C}^k$ such that for each $\ell = 1, \dots, k$, the map germ $f^{(\ell)} := (f_1, \dots, f_\ell)\colon (\mathbb{C}^{n+k}, x) \to (\mathbb{C}^\ell, 0)$ defines an ICIS $(X_0^{(\ell)}, x)$ of dimension $n - \ell + k$.

Corollary 6.3.5. *With the above notation we have*

$$\mu(X_0,x)=\sum\nolimits_{\ell=1}^{k}(-1)^{k-\ell}\dim_{\mathbb{C}}\frac{\mathscr{O}_{n+k,x}}{I(f_{\ell-1})+J(f_\ell)}.$$

Observe that in the hypersurface case $k=1$, we recover the well known formula for the Milnor number

$$\mu(X_0,x)=\dim_{\mathbb{C}}\frac{\mathscr{O}_{n+1,x}}{J(f)},$$

where now $J(f)$ is the Jacobian ideal, generated by the partial derivatives of f. Another interesting particular case is when (X_0,x) is 0-dimensional, in which case the formula for $\mu(X_0,x)$ can be simplified as follows:

Proposition 6.3.6. *Let (X_0,x) be a 0-dimensional ICIS. Then,*

$$\mu(X_0,x)=\dim_{\mathbb{C}}\mathscr{O}_{X_0,x}-1.$$

Proof. See [28, Proposition 5.13]. □

Example 6.3.7. Consider $(X,0)=V(x^2+y^2+z^2,xy)\subset(\mathbb{C}^3,0)$. We have $(X',0)=V(x^2+y^2+z^2)$, so $\mu(X',0)=1$. By the Lê-Greuel formula

$$\mu(X,0)+\mu(X',0)=\dim_{\mathbb{C}}\frac{\mathscr{O}_3}{(x^2+y^2+z^2)+I_2\begin{pmatrix}2x & 2y & 2z\\ y & x & 0\end{pmatrix}}=6$$

(computed with Singular). Hence $\mu(X,0)=5$.

Let (X_0,x) be an ICIS of dimension n, defined as the fibre of a $\mathscr{K}$-finite map germ $f\colon(\mathbb{C}^{n+k},x)\to(\mathbb{C}^k,0)$. We take a good proper representative $f\colon\overline{X}\to S$. The following property is known as the conservation of the Milnor number for ICIS. It was proved, more generally, for deformations of isolated determinantal singularities in [38].

Theorem 6.3.8 (Conservation of the Milnor number)**.** *For any $s\in S$,*

$$\mu(X_0,x)=\sum\nolimits_{x'\in S(X_s)}\mu(X_s,x')+\beta_n(\overline{X}_s),$$

where $S(X_s)$ is the singular locus of X_s and $\beta_n(\overline{X}_s)$ is the nth-Betti number of $\overline{X}_s$ (that is, the number of n-spheres in the wedge $\overline{X}_s\simeq S^n\vee\cdots\vee S^n$).

Corollary 6.3.9 (Upper semicontinuity of the Milnor number)**.** *For any $s\in S$ and for any $x'\in S(X_s)$, $\mu(X_s,x')\le\mu(X_0,x)$.*

Theorem 6.3.10. *Let (X_0,x) be an ICIS. Then $\mu(X_0,x)=0$ if and only if (X_0,x) is smooth.*

Proof. If (X_0,x) is smooth then it is obvious that $\mu(X_0,x)=0$. Conversely, assume that (X_0,x) is not smooth. This is equivalent to that f has corank $\ell>0$ at x. We must show that $\mu(X_0,x)>0$ and we will prove it by induction on ℓ.

Let $\ell=1$ and assume $x=0$ for simplicity. We can choose coordinates in $(\mathbb{C}^{n+k},0)$ and in $(\mathbb{C}^k,0)$ such that

$$f(x)=(x_1,\dots,x_{k-1},f_k(x)),$$

for some function $f_k \in \mathfrak{m}_{n+k}^2$. By the Lê-Greuel formula (6.3),

$$\mu(X_0, 0) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x_1, \dots, x_{n+k}\}}{(x_1, \dots, x_{k-1}) + J(x_1, \dots, x_{k-1}, f_k)} = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x_k, \dots, x_{n+k}\}}{J(g)} = \mu(g),$$

where $g(x_k, \dots, x_{n+k}) = f_k(0, \dots, 0, x_k, \dots, x_{n+k})$. Since $g \in \mathfrak{m}_{n+1}^2$, $J(g) \subseteq \mathfrak{m}_{n+1}$ and hence, $\mu(g) > 0$.

Suppose now the result is true for corank ℓ and assume that the corank is $\ell + 1$. Again we put $x = 0$ for simplicity. We choose coordinates in $(\mathbb{C}^{n+k}, 0)$ and in $(\mathbb{C}^k, 0)$ such that

$$f(x) = (x_1, \dots, x_{k-\ell-1}, f_{k-\ell}(x), \dots, f_k(x)),$$

for some functions $f_{k-\ell}, \dots, f_k \in \mathfrak{m}_{n+k}^2$. We take the 1-parameter unfolding $F(x,t) = (f_t(x), t)$, where

$$f_t(x) = f(x) + t(0, \dots, 0, x_{k-\ell}, 0, \dots, 0).$$

For all $t \neq 0$, f_t has corank ℓ at 0. By hypothesis induction, $\mu(X_t, 0) > 0$, where $X_t = f_t^{-1}(0)$. On the other hand, by the upper semicontinuity 6.3.9, $\mu(X_0, 0) \geq \mu(X_t, 0) > 0$ and we are done. □

The following corollary is now a direct consequence of Theorems 6.3.8 and 6.3.10.

Corollary 6.3.11. *Suppose that $\mu(X_s, x') = \mu(X_0, x)$, for some $s \in S$ and $x' \in X_s$. Then $X_s \setminus \{x'\}$ is smooth.*

Exercises

1. Compute the Milnor number of the ICIS defined by the following map germs:
   $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, $f(x, y) = (x^p, y^q)$,
   $f\colon (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$, $f(x, y) = (x^3 + y^2 + z^2, xy)$.

6.4 The Gaffney-Hauser theorem

In [15], Gaffney and Hauser showed a very general theorem which states that if two germs of complex spaces (X, x) and (Y, y) have isomorphic singular loci (in some specific sense), then $(X, x) \cong (Y, y)$. Here we present a simplified version in the case that (X, x) and (Y, y) are ICIS. The case of hypersurfaces with isolated singularities was obtained previously by Mather and Yau in [34].

Theorem 6.4.1. *Let (X, x) and (Y, y) be ICIS given as the fibres of $\mathscr{K}$-finite map germs $f\colon (\mathbb{C}^N, x) \to (\mathbb{C}^p, 0)$ and $g\colon (\mathbb{C}^N, y) \to (\mathbb{C}^p, 0)$, respectively. Assume that there exists an isomorphism (φ, L) between $T^1_{X,x}$ and $T^1_{Y,y}$, for some $\varphi : \mathscr{O}_{N,y} \to \mathscr{O}_{N,x}$. Then $(X, x) \cong (Y, y)$.*

Proof. For simplicity we assume $x = y = 0$ in $\mathbb{C}^N$. By hypothesis we have an isomorphism (φ, L) between $\theta(f)/T\mathscr{K}_e f$ and $\theta(g)/T\mathscr{K}_e g$, for some $\varphi : \mathscr{O}_N \to \mathscr{O}_N$. We must show that f, g are $\mathscr{K}$-equivalent.

Step 1. We can assume that $T\mathcal{K}_e f = T\mathcal{K}_e g$.

We have $\varphi = \phi^*$ for some diffeomorphism $\phi : (\mathbb{C}^N, 0) \to (\mathbb{C}^N, 0)$. We can substitute f by $\tilde{f} = L \cdot (f \circ \phi)$, so $\tilde{f}$ and f are $\mathcal{K}$-equivalent, but now $T\mathcal{K}_e \tilde{f} = T\mathcal{K}_e g$.

Step 2. Let $f_t = (1-t)f + tg$, with $t \in \mathbb{C}$. Then $T\mathcal{K}_e f_t = T\mathcal{K}_e f$, for all $t \in C$.

For all $i = 1, \dots, n$ and for all $j, k = 1, \dots, p$, we have:

$$\frac{\partial f_t}{\partial x_i} = (1-t)\frac{\partial f}{\partial x_i} + t\frac{\partial g}{\partial x_i} \in T\mathcal{K}_e f,$$

and

$$(f_t)_k \frac{\partial}{\partial y_j} = (1-t) f_k \frac{\partial}{\partial y_j} + t g_k \frac{\partial}{\partial y_j} \in T\mathcal{K}_e f,$$

therefore $T\mathcal{K}_e f_t \subseteq T\mathcal{K}_e f$. The opposite inclusion is analogous, since for $t \neq 1$, we have $f = \frac{1}{1-t}(f_t - tg)$ and for $t = 1$ is obvious.

Step 3. For each $t_0 \in \mathbb{C}$, we consider the unfolding $F\colon (\mathbb{C}^n \times \mathbb{C}, (0, t_0)) \to (\mathbb{C}^p \times \mathbb{C}, 0)$ given by $F(x,t) = (f_t(x), t)$. Then F is $\mathcal{K}_e$-trivial, that is, there exist $\Phi\colon (\mathbb{C}^n \times \mathbb{C}, (0, t_0)) \to (\mathbb{C}^n \times \mathbb{C}, (0, t_0))$ unfolding of the identity and $\mathcal{L} = L_t \in Gl_p(\mathcal{O}_{\mathbb{C}^n \times \mathbb{C}, (0, t_0)})$ such that $L_0 = I_p$ and

$$L_t \cdot (f_t \circ \phi_t) = f_{t_0}.$$

Let $\mathcal{O}_{n+1} = \mathcal{O}_{\mathbb{C}^n \times \mathbb{C}, (0, t_0)}$. We have

$$\frac{\partial f_t}{\partial t} = g - f \in T\mathcal{K}_e f_t \subseteq \mathcal{O}_{n+1}\left\{\frac{\partial f_t}{\partial x_1}, \dots, \frac{\partial f_t}{\partial x_n}\right\} + ((f_t)_1, \dots, (f_t)_p)\mathcal{O}_{n+1}^p,$$

where the right hand side is the relative version of the $\mathcal{K}_e$-tangent space of the unfolding F. That is, we can write

$$-\frac{\partial f_t}{\partial t} = \sum\nolimits_{i=1}^n \xi_i \frac{\partial f_t}{\partial x_i} + \sum\nolimits_{j,k=1}^p a_{jk}(f_t)_j \frac{\partial}{\partial y_k},$$

for some functions $\xi_i, a_{jk} \in \mathcal{O}_{n+1}$. We consider the funcions ξ as the components of a time dependent vector field

$$\xi_t = \sum\nolimits_{i=1}^n \xi_i \frac{\partial}{\partial x_i}.$$

and the functions a_{jk} as the component of a time dependent matrix $A_t = (a_{jk}) \in M_{p\times p}(\mathcal{O}_{n+1})$. The above equality can be rewritten now as

$$\frac{\partial f_t}{\partial t} + df_t \cdot \xi_t + A_t \cdot f_t = 0.$$

We consider the following system of differential equations

$$\begin{cases} \frac{\partial \phi_t}{\partial t} = \xi_t \circ \phi_t, & \phi_{t_0} = \mathrm{id}, \\ \frac{\partial L_t}{\partial t} = L_t \cdot (A_t \circ \phi_t), & L_{t_0} = I_p, \end{cases}$$

By integrating this we get $\Phi = (\phi_t, t)$ and unfolding of the identity in $(\mathbb{C}^n \times \mathbb{C}, (0, t_0))$ and $\mathcal{L} = L_t \in Gl_p(\mathcal{O}_{\mathbb{C}^n \times \mathbb{C}, (0, t_0)})$. Let us see that

$$L_t \cdot (f_t \circ \phi_t) = f_{t_0}.$$

In fact,

$$\begin{aligned}
\frac{\partial}{\partial t}\left(L_t \cdot (f_t \circ \phi_t)\right) &= \frac{\partial L_t}{\partial t} \cdot (f_t \circ \phi_t) + L_t \cdot \left((df_t \circ \phi_t) \cdot \frac{\partial \phi_t}{\partial t} + \frac{\partial f_t}{\partial t} \circ \phi_t\right) \\
&= L_t \cdot (A_t \circ \phi_t) \cdot (f_t \circ \phi_t) + L_t \cdot \left((df_t \circ \phi_t) \cdot (\xi_t \circ \phi_t) + \frac{\partial f_t}{\partial t} \circ \phi_t\right) \\
&= L_t \cdot \left(A_t \cdot f_t + df_t \cdot \xi_t + \frac{\partial f_t}{\partial t}\right) \circ \phi_t = 0.
\end{aligned}$$

This shows that $L_t \cdot (f_t \circ \phi_t)$ does not depend on t. But for $t = t_0$ we have $L_{t_0} \cdot (f_{t_0} \circ \phi_{t_0}) = f_{t_0}$.

Step 4. By Step 3, the germ of f_{t_0} at 0 is $\mathscr{K}$-equivalent to the germ of f_t at $\phi_t^{-1}(0)$, for all t in a neighbourhood of t_0. On one hand, we have $(X_{t_0}, 0) \cong (X_t, \phi_t^{-1}(0))$, hence $\mu(X_{t_0}, 0) = \mu(X_t, \phi_t^{-1}(0))$ by Proposition 6.3.3) and $X_t \setminus \{\phi_t^{-1}(0)\}$ is smooth by Corollary 6.3.11. On the other hand, by Step 2, $T\mathscr{K}_e f_t = T\mathscr{K}_e f$, for all $t \in C$, which implies that 0 is a singular point of X_t. Therefore, $\phi_t^{-1}(0) = 0$.

This shows that F is not only $\mathscr{K}_e$-trivial, but also $\mathscr{K}$-trivial. The $\mathscr{K}$-class of the germ f_t is locally constant and since $\mathbb{C}$ is connected, it is globally constant. In particular, $f_0 = f$ and $f_1 = g$ are $\mathscr{K}$-equivalent.

□

Part IV

Lecture 4

Chapter 7

Equisingularity and ICIS

7.1 Introduction and some basic examples

As mentioned earlier in these lectures, to understand a singularity X we want to understand the "nearby" singularities, that is the singularities that appear in deformations of X. A first question in studying suchdeformations is: given a family of sets or maps, when are all the members the same? When are some of the members different? Equisingularity is the study of these questions. It is easier to say when a member of family is different, than it is to say when two sets or two maps are the same. Often the change in a single invariant suffices to pick out the members which are out of step with the rest.

As we shall see in this lecture, studying families of sets makes the introduction of infinitesimal methods natural and powerful, even when the integration of the resulting vectorfields only give homeomorphisms.

A basic question is what do we mean by "the same"? And how do we tell when a family of sets are the same using invariants of the members of the family? These questions are explored in this lecture for families of ICIS; we shall see that the invariants typically have both a topological/ geometric and infinitesimal character.

We start with some notation to describe a family of sets. In the diagram:

$$\begin{array}{ccccc} X^d(0) & \subset & \mathcal{X}^{d+k} & \subset & Y \times \mathbb{C}^N \\ \Big\downarrow{\scriptstyle p_Y} & & \Big\downarrow{\scriptstyle \pi_Y} & & \\ 0 \in & & Y = \mathbb{C}^k & & \end{array}$$

the parameter space is Y, $X(0)$ denotes the fiber of the family over $\{0\}$, $\mathcal{X}^{d+k}$ denotes the total space of the family which is contained in $Y \times \mathbb{C}^N$. We usually assume $Y \subset \mathcal{X}^{d+k}$, and $\mathcal{X} = F^{-1}(0)$, $X(y) = {f_y}^{-1}(0)$, where $f_y(z) = F(y, z)$.

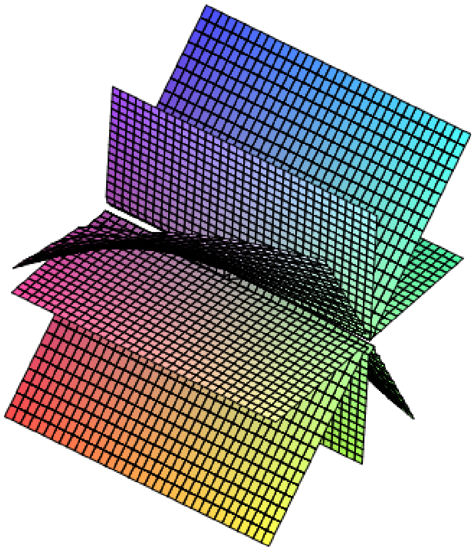
Given a family of map germs as above, we say the family is *smoothly trivial* if there exists a smooth family of origin preserving bi-holomorphic germs r_y such that $r_y(X(0)) = X(y)$. If the map-germs are only homeomorphisms we say the family is C^0 *trivial*.

Example 7.1.1. *Let $\mathcal{X}$ be the family of two moving lines in the plane with equation $F(y, z_1, z_2) = z_1(z_2 - yz_1) = 0$. Here y is the parameter, the z_2 axis is fixed, a component of every member of the family while the line $z_2 - yz_1 = 0$ moves with y. Our intuition says that all of these sets are the* *"same", and we know the family of functions $F(y, z_1, z_2) = z_1(z_2 - yz_1)$ are all right equivalent to $f_0(z_1, z_2) = z_1z_2$, because they are all Morse functions. Hence the family is smoothly trivial.*

Problem 7.1.2. *Show that the family of 3 moving lines in $\mathbb{C}^2$ is smoothly trivial, by showing that the family of functions $F(y, z_1, z_2) = z_1z_2(z_1 - (1 + y)z_2)$ is right equivalent to $f_0(z_1, z_2) = z_1z_2(z_1 - z_2)$ for $y \neq -1$. This will be a good review of some of the ideas of course A lecture 2.*

In fact, our intuition suggests that the family of n moving distinct lines should be "equisingular". The next example shows that we must use a weaker notion of equisingularity than smooth triviality if we want a notion that agrees with our intuition about the n moving lines.

Example 7.1.3. *Let X be the family of four moving lines in the plane with equation $F(x, y, z) = z_1z_2(z_2 + z_1)(z_2 - (1 + y)Z_1) = 0$. Here y is the parameter, the z_1 and z_2 axis are fixed, as is the line $z_2 + z_1 = 0$ while the line $z_2 - (1 + y)z_1 = 0$ moves with y. Here is a picture of the total space of the family:*



This family is not smoothly trivial as the next exercise shows,

Problem 7.1.4. *Show that the family of 4 lines is not smoothly trivial by following the hints and proving them: If r_y is a trivialization of the family of sets, $Dr_y(0)$ must carry the tangent lines of $X(0)$ to $X(y)$. If a linear map preserves the lines defined by $z_1 = 0, z_2 = 0, z_2 = -z_1$ then the linear map must be a multiple of the identity. Hence r_y can't map $z_2 = z_1$ to $z_2 = (1 + y)z_1$, $y \neq 0$.*

Even though the family of four lines is not smoothly trivial, we would like to use infinitesimal methods as the foundation of our theory of equisingularity. The infinitesimal approach using vectorfields, promises to reduce equisingularity problems to algebra, just as Mather's work does for smooth equivalence. We discuss the kind of vectorfields we will use in the next section.

Problem 7.1.5. *Show that an analytic set defined by a homogeneous polynomial of degree d in 2 variables in $\mathbb{C}^2$ consists of d lines counted with multiplicity.*

7.2 Rugose vectorfields and Verdier's condition *W*

Given a family of hypersurfaces $\mathcal{X}$ over Y^1, defined by $F(y,z)$, the family $\mathcal{X}$ is smoothly trivial if we can find a smooth vectorfield of the form $V = (\frac{\partial}{\partial y} - \xi)$ defined on $\mathcal{X}$, $\xi(y,0) = 0$, such that

$$DF(V) = 0, \text{on } \mathcal{X}.$$

Geometrically, this means that V is tangent to Y and to $\mathcal{X}$ on $\mathcal{X}_0$, the set of smooth points of $\mathcal{X}$, so the flow induced by V must preserve Y and $\mathcal{X}$.

If we only ask V to be real analytic at points of $\mathcal{X}_0$, then, there is a canonical way to define V, which works for every F. Here is the ξ that works.

Let

$$\xi(y,z) = \frac{\sum_{i=1}^n \frac{\partial F}{\partial y}(y,z)\overline{\frac{\partial F}{\partial z_i}}(y,z)\frac{\partial}{\partial z_i}}{\sum_{i=1}^n \frac{\partial F}{\partial z_i}(y,z)\overline{\frac{\partial F}{\partial z_i}}(y,z)}.$$

This means that any 1-parameter family of hypersurfaces has a canonical tangent vectorfield. This is not true in general, but nonetheless, for any 1-parameter family of equidimensional analytic sets, there does exist a cover $\{U_i\}$ of $\mathcal{X}_0$ and a collection of vectorfields V_i real analytic on U_i tangent to U_i.

Problem 7.2.1. *Show that with this definition of $\xi(y,z)$, $DF(V) = 0$ and V is real analytic whenever $D_z(F(y,z)) \neq 0$. (Here $D_z(F(y,z))$ is the vector of partial derivatives with respect to the z variables.)*

Verdier showed that the vectorfield V could be integrated to give a family of homeomorphisms which trivialized $\mathcal{X}$ provided the inequality

$$\|\xi(y,z)\| \leq C\|z\|$$

held on a neighborhood of the origin in $\mathcal{X}$, for some $C > 0$ [46]. Verdier called a vector field satisfying such an inequality a *rugose vectorfield*. He also defined a stratification condition, condition W, which ensured, that if it held between all pairs of incident strata, smooth vectorfields on the smallest stratum lifted to rugose vectorfields on larger strata.

The basic pair of strata is the case where X_0 is the set of smooth points of a complex analytic set X, and Y is a smooth subset of X at a point $y \in Y$. Condition W says that the distance between between the tangent space to X at a point x_i of X_0 and the tangent space to Y at y goes to zero as fast as the distance between x_i and Y. We first need to define what we mean by the distance between two linear spaces.

Suppose A, B are linear subspaces at the origin in $\mathbb{C}^N$, then define the distance from A to B as:

$$\operatorname{dist}(A,B) = \sup_{\substack{u \in B^\perp - \{0\} \\ v \in A - \{0\}}} \frac{\|(u,v)\|}{\|u\|\,\|v\|}.$$

In the applications B is the "big" space and A the "small" space. The inner product is the Hermitian inner product when we work over $\mathbb{C}$. The same formula also works over $\mathbb{R}$.

Example 7.2.2. *For this example, we work with linear subspaces of* $\mathbb{R}^3$. *Let* A = *x-axis,* B *a plane with unit normal* u_0, *then the formula for the distance from* A *to* B *reduces to* $\cos\theta$, *where* θ *is the small angle between* u_0 *and the x-axis, in the plane they determine. So when the distance is* 0, B *contains the x-axis.*

We recall Verdier's condition W.

Definition 7.2.3. *Suppose* $Y \subset \bar{X}$, *where* X, Y *are strata in a stratification of an analytic space, and* $\operatorname{dist}(TY_0, TX_x) \leq C\operatorname{dist}(x, Y)$ *for all* x *close to* Y. *Then the pair* (X, Y) *satisfies* **Verdier's condition** *W at* $0 \in Y$ *([46]).*

Example 7.2.4. *For a family of n-lines, the pair* $\mathcal{X}_0, Y$ *is easily seen to satisfy this condition, because* $\mathcal{X}$ *is made up of n smooth surfaces, intersecting along* Y, *and the intersection of their tangent spaces, at points of* Y *is just* Y. *Since each component of* $\mathcal{X}_0$ *satisfies W over* Y, *so does* $\mathcal{X}_0$.

We use the W condition for the definition of equisingularity which we will study.

Definition 7.2.5. *A family* $\mathcal{X}$ *is W-equisingular (or just equisingular) if* $\mathcal{X}$ *has a stratification in which adjacent pair of strata satisfy condition W, and the parameter space* Y *is a stratum.*

Verdier introduced condition W after the Whitney conditions were introduced; these played a central role in the development of the topological equisingularity of sets and maps developed by Thom and Mather. Here is their definition in the analytic case.

If X is an analytic set, X_0 the set of smooth points on X, Y a smooth subset of X, then the pair (X_0, Y) satisfies **Whitney's condition A** at $y \in Y$ if for all sequences $\{x_i\}$ of points of X_0,

$$\begin{array}{c}\{x_i\} \to y \\ \{TX_{x_i}\} \to T\end{array} \Rightarrow T \supset TY_y$$

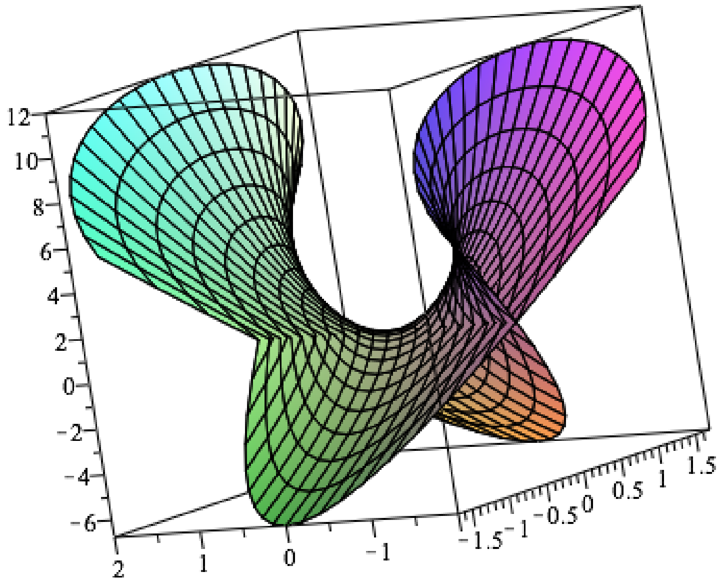
The pair (X_0, Y) satisfies **Whitney's condition B** at $y \in Y$ if for all sequences $\{x_i\}$ of points of X_0,

$$\begin{array}{r}\{x_i\} \to y \\ \{TX_{x_i}\} \to T \\ sec(x_i, \pi_Y(x_i)) \to L\end{array} \quad \Rightarrow T \supset L$$

Problem 7.2.6. *Show that the family of 4 lines satisfies the Whitney conditions. (Hint: The family consists of submanifolds meeting pairwise transversely.)*

Teissier showed that in the complex analytic case, conditions W and the Whitney conditions are equivalent. (See [42].) As we shall see, W connects fairly easily with vectorfields, Jacobian ideals, and modules. This is not so true for the Whitney conditions.

Example 7.2.7. *This is a famous example used in many singularities talks.* $\mathcal{X}$ *is defined by* $F(y, z_1, z_2) = z_2^3 + z_1^2 - y^2 z_2^2 = 0$. *The members of the family* $X(y)$ *consist of node singularities where the loop is pulled smaller and smaller as y tends to zero, becoming a cusp at* $y = 0$. *Here is a picture:*



Our intuition says that for $y = 0$ there is a drastic change in the family, and this family should not be equisingular for any reasonable definition.

Problem 7.2.8. *Show that W fails for Teissier's example for $\mathcal{X}_0, Y$ where Y is the y-axis at the origin, by following these hints. If W holds, then the analytic inequality must hold along every curve. Consider the curve $\phi(t) = (t, 0, t^2)$. Check that the image of ϕ lies in $\mathcal{X}$. Now compute each side of the inequality restricted to the image of ϕ. You should end up looking at*

$$\frac{\|\frac{\partial F}{\partial y}(t, 0, t^2)\|}{\|DF(t, 0, t^2)\|} \leq C\|t^2\|.$$

Show that this cannot hold by comparing orders in t on each side of the inequality.

Problem 7.2.9. *Show that Whitney's condition b also fails for Teissier's example, directly from the definition. (Hint: use the curve $\phi(t) = (t, 0, t^2)$ again.)*

If we project the surface to the y, z_2 plane, the critical set of the projection is the closure of the smooth points of the surface where the line $y = 0, z_2 = 0$ is tangent to the surface; this happens when F and F_x are 0, and is the curve ϕ. The curve ϕ is the *polar curve* of $\mathcal{X}$ for the projection onto the y, z_2 plane. Later on we will see that a family of plane curves is W-equisingular if and only if the polar curve at the origin is empty.

As a first step to understanding the W condition, we consider the case where X is a hypersurface in $\mathbb{C}^n$. We would like to re-write this condition in terms of F where F defines X. This will allow us to develop an algebraic formulation of the W condition.
Set-up: We use the basic set-up with $\mathcal{X}^{k+n}$ a family of hypersurfaces in $Y^k \times \mathbb{C}^{n+1}$.

Proposition 7.2.10. *Condition W holds for* $(\mathcal{X}_0, Y)$ *at* $(0,0)$ *if and only if there exists U a neighborhood of* $(0,0)$ *in* $\mathcal{X}$ *and* $C > 0$ *such that*

$$\|\frac{\partial F}{\partial y_l}(y,z)\| \leq C \sup_{i,j} \|z_i \frac{\partial F}{\partial z_j}(y,z)\|$$

for all $(y,z) \in U$ *and for* $1 \leq l \leq k$.

Proof. In this set-up, Y is a k-plane, so we will set $A = Y$, and calculate the distance between $\overline{Y \text{ and a}}$ tangent plane to $\mathcal{X}_0$ at (y,z) which is our B. At a smooth point of $\mathcal{X}^{k+n}$, we can use $\overline{DF(y,z)}/\|DF(y,z)\|$ for $u \in B^{\perp}$, and the standard basis for the vectors from A.

Then the distance formula says that condition W holds if and only if

$$\sup_{1 \leq l \leq k} \frac{\|\frac{\partial F}{\partial y_l}(y,z)\|}{\|DF(y,z)\|} \leq C'' \text{dist}((y,z), Y) = C' \sup_{1 \leq i \leq n+1} \|z_i\|$$

This is equivalent to

$$\|\frac{\partial F}{\partial y_l}(y,z)\| \leq C \sup_{1 \leq i \leq n+1} \|z_i\| \sup_{1 \leq j \leq n+1} \|\frac{\partial F}{\partial z_j}(y,z)\|$$

From which the desired result follows.

□

Denote the ideal generated by the partial derivatives of F with respect to the z variables by $J_z(F)$, and the ideal generated by z_j by m_Y. Then $z_i \frac{\partial F}{\partial z_j}$ are a set of generators for $m_Y J_z(F)$. The inequality above says that the partial derivatives of F with respect to y_l go to zero as fast as the ideal $m_Y J_z(F)$. We will examine the implications of this in the next section.

7.3 The Theory of Integral Closure of Ideals and Modules

Many operations on ideals and submodules of a free module come from operations on rings. (For other examples of this, see [13], [12], [17].)

We illustrate this idea by reviewing the notions of the integral closure of a ring and the normalization of an analytic space, then relating these to the integral closure of an ideal in the next section.

Definition 7.3.1. *Let A, B be commutative Noetherian rings with unit,* $A \subset B$ *a subring. Then* $h \in B$ *is integrally dependent on A if there exists a monic polynomial* $f(T) = T^n + \sum_{i=0} f_i T^i$, $f_i \in A$ *such that* $f(h) = 0$. *The integral closure of A in B consists of all elements of B integrally dependent on A.*

Example 7.3.2. *Let A be the ring of convergent power series in the germs* t^2 *and* t^3, *denoted* $\mathbb{C}\{t^2, t^3\}$, $B = \mathbb{C}\{t\}$. *Then if* $f(T) = T^2 - t^2$ *we have* $f(t) = 0$, *so t is integrally dependent on A. In fact, B is the integral closure of A in B.*

Definition 7.3.3. *Let A be the local ring of an analytic space X, x, B the ring of meromorphic functions on X at x; the space associated with the integral closure of A in B is the normalization of X.*

Example 7.3.4. *Let $A = \mathbb{C}\{t^2, t^3\}$, $B = \mathbb{C}\{t\}$. Then A is the local ring at the origin of the cusp $x^3 - y^2 = 0$, and since $t^3/t^2 = t$, the ring of meromorphic functions on X at the origin is $\mathbb{C}\{t\}$. So by the previous example the normalization of the cusp is a line.*

In this context a ring A is normal if the integral closure of A in its quotient field is A. A space germ is normal if its local ring is normal. Normal spaces have nice properties–they are non-singular in codimension 1 and the Riemann removable singularities theorem is true for them. Given a space germ X, we always have a map π_{NX} from the normalization of X, denoted NX, to X which is finite and generically 1-1. NX and π_{NX} are unique up to smooth right equivalence. You can read proofs of these facts in [20] p 154-163, working backwards as necessary.

The following exercise is easy assuming the facts in the last paragraph.

Problem 7.3.5. *Show that the normalization of an irreducible curve germ X, x is $\mathbb{C}, 0$.*

If you know a little bit about singularities of maps, the next exercise is also easy.

Problem 7.3.6. *Suppose $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, $n < p$ and f is an $\mathcal{A}$-finitely determined map-germ. Show $(\mathbb{C}^n, 0), f$ is a normalization of the image of f.*

Basic Results from the Theory of Integral Closure for Ideals

The operation of integral closure of rings creates, as we shall see, an operation on ideals, the operation of forming the integral closure of I, which is an ideal, denoted $\overline{I}$. The integral closure of $m_n J(f)$ in O_n plays the same role in a theory of equisingularity of functions built on condition W as $T\mathcal{R}(f)$ does for right equivalence, and $\overline{m_n J(f)}$ in $O_{X,x}$ plays a similar role for the theory of equisingularity of hypersurfaces based on condition W.

Assume I is an ideal in $O_{X,x}$, $f \in O_{X,x}$. In discussing the properties of integral closure, sometime we work on a small neighborhood of X. In this case, I refers to the coherent sheaf I generates on U.

List of Basic Properties ([26]) f is integrally dependent on I if one of the following equivalent conditions obtain:

(i) There exists a positive integer k and elements a_j in I^j, so that f satisfies the relation $f^k + a_1 f^{k-1} + \cdots + a_{k-1} f + a_k = 0$ in $O_{X,0}$.

(ii) There exists a neighborhood U of 0 in $\mathbb{C}^N$, a positive real number C, representatives of the space germ $X, 0$ the function germ f, and generators $g_1, \ldots, g_m$ of I on U, which we identify with the corresponding germs, so that for all x in U we have: $\|f(x)\| \le C \max\{\|g_1(x)\|, \ldots, \|g_m(x)\|\}$.

(iii) For all analytic path germs $\phi : (\mathbb{C}, 0) \to (X, 0)$ the pull–back $\phi^* f = f \circ \phi$ is contained in the ideal generated by $\phi^*(I)$ in the local ring of $\mathbb{C}$ at 0. If for all paths $\phi^* f$ is contained in $\phi^*(I) m_1$, then we say f *is strictly dependent on I* and write $f \in I^\dagger$.

Let NB denote the normalization of the blowup of X by I, $\bar{D}$ the pullback of the exceptional divisor of the blowup of X by I to NB by the normalization map. Then we have:

(iv) For any component C of the underlying set of $\bar{D}$, the order of vanishing of the pullback of f to NB along C is no smaller than the order of the divisor $\bar{D}$ along C. This implies that the pullback of f lies in the ideal sheaf generated by the pullback of I.

The set of all elements of $O_{X,x}$ which are integrally dependent on I is the *integral closure of* I and is denoted $\overline{I}$.

Proposition 7.3.7. *If I is an ideal in $O_{X,x}$, then so is $\overline{I}$.*

Proof. We use property iii). Let $\phi : (\mathbb{C}, 0) \to (X, 0)$ be any analytic curve, $g \in O_{X,x}$, f_1, f_2 in $\overline{I}$. Then $(gf_1 + f_2) \circ \phi = (g \circ \phi)(f_1 \circ \phi) + (f_2 \circ \phi) \in \phi^*(I)$, since $\phi^*(I)$ is an ideal in O_1. □

The proof of this for general rings is Corollary 1.3.1 of [22].

The first property is usually taken as the definition, and shows that integral dependence is an algebraic idea. This permits the extension of the concept to ideals in any ring. For the development of the idea of the integral closure of an ideal or module from the algebraic point of view see [22].

The second property is used to control equisingularity conditions. It already appeared in the discussion of Verdier's condition W in the hypersurface case earlier, and we will revisit it shortly.

The third property is convenient for computations, and often for proofs as the proof of the previous proposition shows. It is also helpful in understanding conditions involving limits. In the analytic setting, definitions that use sequences of points, such as the Whitney conditions, can be checked with curves, often leading to an interpretation of the condition in terms of the integral closure of an ideal or module. We will see an example of this in the study of limiting tangent hyperplanes in the next section.

Given a curve $\phi(s)$, and a germ f, if $f \circ \phi$ is defined, it is equal to $cs^r \mod m_1^{r+1}$ for $c \neq 0$ for some r. We call r the order of f on ϕ and write $f_\phi = r$, and J_ϕ for the order of an ideal J on ϕ. Then $f \in I^\dagger$ if and only if $f_\phi > I_\phi$, for all curves ϕ.

Because the exceptional divisor of the blow-up of the Jacobian ideal tracks limiting infinitesimal information, the fourth property is perhaps the most important. Since NB is normal, each component of the exceptional divisor is generically a smooth submanifold of a manifold, so the ideal vanishing on the component is locally principal. This means we can talk about the order of vanishing on each component. The order of the divisor $\bar{D}$ is just the order of vanishing along the component of the pullback of I to NB. Concretely, pick a local generator u of the ideal of the component, and write the elements of I in terms of u. The smallest power of u that appears is the order of I along C.

The fourth property also shows how a closure operation on rings gives a closure operation on ideals– start with a ring and an ideal, enlarge the ring by a closure operation, look at the ideal generated in the new ring, then intersect with the original ring to define the closure operation on the ideal.

The next problem is another way to see this principle for the operation of integral closure, and gives some insight into the form of the first property.

Problem 7.3.8. *Let $A = O_X[IT]$, I an ideal of O_X, and T an indeterminate. Let $B = O_X[T]$. (So A and B are rings.)Then $h \in \overline{I}$ in O_X if and only if hT is integrally dependent on A in B. (You can read the solution in [26].)*

When we have a generic property, ie. one that holds on a non-empty Zariski-open set of X, we would like to know when the property holds on all of X. The fourth property provides a key step in answering this question.

Proposition 7.3.9. *Let X, x be a germ of an analytic set, I an ideal of $O_{X,x}$ $f \in \overline{I}$ except perhaps at x and suppose $E \subset B_I(X)$ has no component that projects to x. Then $f \in \overline{I}$.*

Proof. Because $f \in \overline{I}$ except perhaps at x, it follows that the order of the pullback of f to any component of $\bar{D}$ is greater than or equal to the order of the pullback of I, except for components that project to x. But there is no component of $\bar{D}$ over x, because the normalization map is finite, and there is no component of E over x. So, the order condition holds for all components of $\bar{D}$. □

This proposition says that one way of proving integral closure conditions which hold generically, is to control the exceptional divisor. We will see different ways to do this in the rest of the notes.

Problem 7.3.10. *We can improve the last proposition. Suppose $X^d, 0$ is the germ of an analytic set, $V^k, 0$ a subvariety of X, I an ideal which vanishes along V. Suppose $f \in \bar{I}$ off V^k. Find a bound on the dimension of the fiber of E over 0, which will ensure that $f \in \bar{I}$ on X.*

Problem 7.3.11. *This is a corollary of the last problem. Suppose $\mathcal{X}^{d+k} \subset \mathbb{C}^n \times \mathbb{C}^k$ is a k-parameter family of analytic sets over $Y^k = 0 \times \mathbb{C}^k$. Suppose $f \in \bar{I}$ off V^{k-1}, $V^{k-1} \subset Y$. Find a bound on the dimension of the fiber of E over 0, which will ensure that $f \in \bar{I}$ on $\mathcal{X}$. If $I(0)$ vanishes only at the origin in $X(0)$, what is the expected dimension of the exceptional divisor of $B_I(0)(X(0))$?*

The meaning of the last two problems is that a generic integral closure condition in the set-up of the last problem, will extend over the whole space, provided that the dimension of $E(0)$ is the same as the dimension of the exceptional divisor of $B_{I(0)}(X(0))$.

Reading For detailed proofs of the equivalences between these properties see [26] p 18-27. You can download this paper from Teissier's list of publications–it is #15. Try this after reading the proofs of the equivalences contained here.

In the next example, we practice using the first property.

Example 7.3.12. *Let $A = O_2$, $I = (x^n, y^n)$. Suppose $f = x^i y^j$, $i + j \geq n$. Consider the monic polynomial $h(T) = T^n - (x^n)^i (y^n)^j$. Since $(x^n)^i (y^n)^j$ is in $(I^i)(I^j) \subset I^{i+j} \subset I^n$, and $h(f) = 0$, then $f \in \overline{I}$.*

Now we do a computation using the third property.

Example 7.3.13. *Let $A = O_2$, $I = (x^a, y^b)$. Given $m = x^i y^j$ define the weight of m to be $bi + aj$, given $f(x, y)$, define the weight of f to be the minimum weight of all monomials appearing in a power expansion of f. We will show that $\overline{I}$ consists of all f such that weight of $f \geq ab$.*

First, we'll show weight of $m \geq ab$ implies $m \in \overline{I}$. It suffices to check this for curves $\phi(t) = (t^r, t^s)$ as higher order terms don't affect the order of I or the monomial m on the curve. Since $\overline{I}$ is an ideal, this will show that $f \in \overline{I}$.

We have I_ϕ =min$\{ra, sb\}$; assume $ra \leq sb$.

It is convenient to think of the monomial $x^i y^j$ as the point (i, j) in the xy-plane. Consider the parallel lines $rx + sy = c$. Then if m is any monomial on this line, $m_\phi = c$, and $m_\phi > c$ if m lies above this line. If the weight of $m \geq ab$ then m lies above or on the line connecting $(a, 0)$ and $(0, b)$, so it will lie above or on any line passing through $(a, 0)$, which lies below or on $(0, b)$. This implies that $m_\phi \geq ra$ and shows $m \in \overline{I}$.

Suppose the power expansion of f contains a monomial m which lies below the line connecting $(a, 0)$ and $(0, b)$. Then the convex hull of the monomials appearing in f has a vertex m' which lies below the line connecting $(a, 0)$ and $(0, b)$. We can find a line passing through this vertex which lies below $(a, 0)$ and $(0, b)$. Then for the curve ψ defined by this line,

$$f_\psi = m'_\psi < I_\psi$$

which shows that $f \notin \overline{I}$.

This kind of reasoning is very useful in studying properties of ideals which are well connected to their Newton polygons. In this example, the Newton polygon of I is all the points of $\mathbb{R}^2$ above or on the line connecting $(a, 0)$ and $(0, b)$ in the first quadrant. For more examples and details see [43], which is #46 on Teissier's publication list or [40].

Next, we use property 2 to characterize Verdier's W in the hypersurface case.
Set-up: We use the basic set-up with $\mathcal{X}^{k+n}$ a family of hypersurfaces in $Y^k \times \mathbb{C}^{n+1}$.

Proposition 7.3.14. *Condition W holds for $(\mathcal{X}_0, Y)$ at $(0, 0)$ if and only if $\frac{\partial F}{\partial y_l} \in \overline{m_Y J_z(F)}$ for $1 \leq l \leq k$.*

Proof. By the last proposition of the first section we know that W holds if and only if

$$\|\frac{\partial F}{\partial y_l}(y, z)\| \leq C \sup_{i,j} \|z_i \frac{\partial F}{\partial z_j}(y, z)\|$$

But, by property 2 this is equivalent to $\frac{\partial F}{\partial y_l} \in \overline{m_Y J_z(F)}$ for $1 \leq l \leq k$. □

If we have a curve ϕ on $\mathcal{X}^{k+n}$, $\phi(0) = 0$, and the image of ϕ in $\mathcal{X}_0^{k+n}$ except at 0, and $J(F)_\phi = r$ then we can calculate the limiting tangent hyperplane to $\mathcal{X}^{k+n}$ along ϕ as

$$\underset{s\to 0}{limit}(1/s^r)(DF(\phi(s)))$$

If $\frac{\partial F}{\partial y_l} \in \overline{J_z(F)}$ for $1 \leq l \leq k$, then the limiting plane is never vertical, but it does not necessarily contain Y.

Problem 7.3.15. *Show that if $\frac{\partial F}{\partial y_l}$ for $1 \leq l \leq k$ is strictly dependent on $J_z(F)$ then every limit of tangent planes along every curve ϕ not in $V(J_z(F))$ contains Y.*

We will prove a few of the implications showing the equivalence of the basic properties.

Proposition 7.3.16. *Property 1 implies property 3*

Proof. Let f satisfy the relation $f^k + a_1 f^{k-1} + \cdots + a_{k-1} f + a_k = 0$ in $\mathcal{O}_{X,0}$, and let $\phi : \mathbb{C}, 0 \to X, 0$. Choose $g \in I$ such that $g_\phi = I_\phi$. We may assume the image of ϕ does not lie in $V(I)$. Then

$$\frac{(f \circ \phi)^k}{(g \circ \phi)^k} + \frac{a_1 \circ \phi}{(g \circ \phi)} \frac{(f \circ \phi)^{k-1}}{(g \circ \phi)^{k-1}} + \cdots + \frac{a_{k-1} \circ \phi}{(g \circ \phi)^{k-1}} \frac{(f \circ \phi)}{(g \circ \phi)} + \frac{a_k \circ \phi}{(g \circ \phi)^k} = 0$$

and $\frac{a_i \circ \phi}{(g \circ \phi)^i}$ is smooth for all i. Since $\mathcal{O}_1$ is normal, it follows that $\frac{(f \circ \phi)}{(g \circ \phi)}$ is smooth, hence $f \circ \phi \in \phi^*(I)$. □

Proposition 7.3.17. *Property 3 implies property 4*

Proof. We will only prove this for the case where $V(I) = 0$.

Consider the components $\{C_i\}$ of $\bar{D}$. Since NB is normal and the C_i have codimension 1, we can pick out points c_i on each C_i and curves $\tilde{\phi}_i$, such that $\tilde{\phi}_i(0) = c_i$, and $\tilde{\phi}_i$ is transverse to C_i. We can choose c_i so that $\pi^*_{NB}(I)$ vanishes only on C_i in a neighborhood of c_i, and the same is true for $f \circ \pi_{NB}$. If u_i defines C_i at c_i, then we have $f \circ \pi_{NB} = h_i {u_i}^{f_i}$, h_i a unit. The exponent f_i is the order of vanishing of f along C_i. Since $\tilde{\phi}_i$ is transverse to C_i at c_i, $u_i \circ \phi_i(t) = t$, so $f \circ \pi_{NB} \circ \phi_i(t) = h_i'(t)t^{f_i}$, h' a unit.

We can also find local generators of $\pi^*_{NB}(I)$ of form ${u_i}^{I_i}$ where I_i is the order of I along C_i. Now $\pi_{NB} \circ \tilde{\phi}_i$ is a map from $\mathbb{C}, 0 \to X, 0$, since $\pi_{NB}(C_i) = 0$, and hence $\pi_{NB}(c_i) = 0$. (This is the reason for restricting to this case.) Hence, if property 3 holds, $f_i \geq I_i$ for all i. If we work at any point of $\bar{D}$ since $\pi^*_{NB}(I)$ is principal, we can find $g \circ \pi_{NB}$ a local generator then $f \circ \pi_{NB}/g \circ \pi_{NB}$ is a meromorphic function which is well defined off a set of codimension 2. Since NB is normal, the function is analytic, so $f \circ \pi_{NB} \in \pi^*_{NB}(I)$. □

Proposition 7.3.18. *Property 4 implies property 2*

Proof. Choose a compact neighborhood U of 0, and consider its inverse image in NB. The inverse image must be compact as well. So, since $f \circ \pi_{NB} \in \pi^*_{NB}(I)$, we can cover $\pi^{-1}_{NB}(U)$ with a finite number of sets and choose elements of I such that

$$\|f \circ \pi_{NB}(p')\| \leq C \max\{\|g_1 \circ \pi_{NB}(p')\|, \ldots, \|g_m \circ \pi_{NB}(p')\|\}$$

holds on $\pi^{-1}_{NB}(U)$. Then it is clear that

$$\|f(\pi_{NB}(p'))\| \leq C \max\{\|g_1(\pi_{NB}(p'))\|, \ldots, \|g_m(\pi_{NB}(p'))\|\}.$$

Since π_{NB} surjects on U, this finishes the proof. □

There is a nice corollary of the method of proof used in the previous proposition and of property 2 which we now describe. Given a subset S of an analytic set X, $f\colon X, S \to Y, y$ where $S = f^{-1}(y)$ denotes the germ of an analytic map along S. Given an ideal I in $O_{Y,y}$, $f^*(I)$ denotes the ideal sheaf along S obtained by pulling back I by f.

Proposition 7.3.19. *Suppose $f\colon X, S \to Y, y$ where $S = f^{-1}(y)$, f proper and surjective. Suppose I an ideal of $O_{Y,y}$, $h \in O_{Y,y}$. Then $h \in \overline{I}$ if and only if $h \circ f \in \overline{f^*(I)}$ along S.*

Proof. Since f is proper, S is compact, and as in the last proof we can cover S with a collection of neighborhoods such that on the union the germ of a function along S is in $\overline{f^*(I)}$ if an only if it satisfies an analytic inequality of the type described by property 2. Since f is surjective, the inequalities push down/pullback to Y, y. □

Problem 7.3.20. *Use the finite map $f(x, y) = (x^b, y^a)$ to give another proof that $\overline{(x^a, y^b)}$ consists of all g such that weight of $g \geq ab$.*

Problem 7.3.21. *Show that if $f \in O_n$, then $f \in \overline{m_n J(f)}$ in O_n. (Hint: let ϕ be a curve on $\mathbb{C}^n$ and compare f_ϕ with the order of $(f \circ \phi)'$ using the chain rule to expand $(f \circ \phi)'$. (For the solution cf [41].)*

We have Prop 2.10 to describe W for hypersurfaces, but what about sets of higher codimension? We will see that the theory of integral closure of modules provides the tools we need to describe the higher codimension case.

The Theory of Integral Closure for Modules: Motivation

Verdier's condition W is based on the distance between the tangent space TX_x to X at smooth points x and the tangent space T to Y. Recall this distance is defined as

$$\operatorname{dist}(T, TX_x) = \sup_{\substack{u \in TX_x^{\perp} - \{0\} \\ v \in T - \{0\}}} \frac{\|(u, v)\|}{\|u\|\,\|v\|}.$$

If $u \in TX_x^{\perp} - \{0\}$, then the set of points perpendicular to u consists of a hyperplane which contains TX_x. These hyperplanes are called *tangent hyperplanes*; denote a tangent hyperplane to X, x by H_x, and the collection of all tangent hyperplanes to X, x by $C(X)_x$. Then we can rephrase the distance formula as

$$\operatorname{dist}(T, TX_x) = \sup_{H_x \in C(X)_x} \operatorname{dist}(T, H_x)$$

If $X = F^{-1}(0)$ where $F\colon \mathbb{C}^n \to \mathbb{C}^p$, then at a smooth point p of X, the projectivisation of the rowspace of the matrix of partial derivatives of F is $C(X)_p$. Since the tangent hyperplanes are what we need to control the distance between the tangent space of X, p and $TY, 0$, this suggests we should look at the module generated by the partial derivatives of F denoted $JM(X)$, just as we looked at $J(F)$ in the hypersurface case.

Basic Results from the Theory of Integral Closure for Modules

Notation: $M \subset N \subset F^p$, F^p a free $O_{X,x}$ module of rank p, M, N submodules of F. If M is generated by g generators $\{m_i\}$, then let $[M]$ be the matrix of generators whose columns are the $\{m_i\}$.

We will develop properties for modules similar to those for ideals; however a convenient entry way into the theory is:

Definition 7.3.22. *If $h \in F^p$ then h is integrally dependent on M, if for all curves ϕ, $h \circ \phi \in \phi^*(M)$. The integral closure of M denoted $\overline{M}$ consists of all h integrally dependent on M.*

A good very basic reference on properties of integral closure of modules is [8, p. 301-307]. The development of these ideas in the setting of modules over commutative rings can be found in [22] starting with chapter 16.

Problem 7.3.23. *$\overline{M}$ is a module, $\overline{\overline{M}} = \overline{M}$*

Example 7.3.24. *Let $[M] = \begin{bmatrix} x & y & 0 \\ 0 & x & y \end{bmatrix}$, then $\overline{M} = m_2 O_2^2$.*

It is clear that $\overline{M} \subset m_2 O_2^2$; we will show that $\begin{pmatrix} y \\ 0 \end{pmatrix} \in \overline{M}$.

Given a curve ϕ we can assume $y_\phi < x_\phi$ otherwise $\begin{pmatrix} y \circ \phi \\ 0 \end{pmatrix} \in \begin{pmatrix} x \circ \phi \\ 0 \end{pmatrix} O_1$.

Then

$$\begin{pmatrix} y \\ 0 \end{pmatrix} \circ \phi = \begin{pmatrix} y \\ x \end{pmatrix} \circ \phi - x/y \circ \phi \begin{pmatrix} 0 \\ y \end{pmatrix} \circ \phi$$

where $x/y \circ \phi \in O_1$.

Connection with the theory of integral closure of ideals I

Notation: Given an element $h \in F$ and a submodule M, then (h, M) denotes the submodule generated by h and the elements of M. Given a submodule N of F, $J_k(N)$ denotes the ideal generated by the set of k by k minors of a matrix whose columns are a set of generators of N. If M is an O_X module then the rank of M is k on a component V of X if $J_k(M) \neq (0)$ on V and k is the largest value for which this is true. We also denote this ideal of largest non-vanishing minors by $J(M)$

Theorem 7.3.25. *(Jacobian principle) Suppose the rank of (h, M) is k on each component of (X, x). Then $h \in \overline{M}$ if and only if $J_k(h, M) \subset \overline{J_k(M)}$*

Proof. The complete proof appears in [8, p. 304]. The easy part is to show that $h \in \overline{M}$ implies $J_k(h, M) \subset \overline{J_k(M)}$.

We have

$$\phi^*(J_k(h, M)) = J_k(\phi^*(h, M)) = J_k(\phi^*(M) = \phi^*(J_k(M))$$

which implies the result.

The problem in the other direction is checking for curves which lie in the set of points where the rank is less than maximal, so that all the elements of $J_k(h, M)$ vanish, but h doesn't vanish. We approach this problem in two steps.

Assume first that the image of our curve ϕ does not lie entirely in $V(J_k(h, M))$.

Then, by hypothesis $\phi^*(J_k(h, M)) = \phi^*(J_k(M)) \neq 0$. So, there is a non-zero minor of the matrix of generators $[M]$, of M, $J(I, K)$ such that $J(I, K) \circ \phi$ is generator of $\phi^*(J_k(M))$. Here I is an index of the rows and K an index of the columns which comprise the $k \times k$ submatrix whose determinant is $J(I, K)$.

Consider $M_{I,K}$ the submodule of F^k defined using as matrix of generators the square submatrix of $[M]$ whose determinant is $J(I, K)$, and let h_I be the element obtained from h by using the entries indexed by I.

Applying Cramer's rule, we have that $h_I \circ \phi \in \phi^*(M_{I,K})$, where $h_I \circ \phi(t) = ([M_{I,K}] \circ \phi(t))\xi(t)$ for some column vector $\xi(t)$, given by composing the output of Cramer's rule with $\phi(t)$. Let $[M_K]$ be the submatrix of $[M]$ using the columns indexed by K. Consider $h_I \circ \phi(t) - ([M_K] \circ \phi(t))\xi(t)$. If this is zero, we have checked the condition for ϕ. If it is not zero, then $\phi^*(h, M)$ has rank greater than k which is a contradiction.

Now suppose the image of ϕ does lie entirely in $V(J_k(h, M))$, so $\phi^*(J_k(h, M)) = 0$.

Here the argument breaks into two parts again. We first assume X is smooth so that we can vary the curve freely, then we use the resolution of singularities to reduce to the smooth case.

Suppose $\phi^*(M) \neq \phi^*(h, M)$. Now, by the Artin-Rees theorem we know that there exists $\nu_0 > 0$, $\nu_0 \in \mathbb{Z}$ such that

$$m_1^l O_1^p \cap \phi^*(h, M) = m_1^{l-\nu_0}(m_1^{\nu_0} O_1^p \cap \phi^*(h, M)).$$

This implies, that in fact,

$$\phi^*(M) \neq \phi^*(h, M) \mod m_1^l O_1^p$$

for any $l > \nu_0$. If not, then $h \circ \phi = g \mod \phi^*(M)$, with $g \in m_1^l O_1^p$, and so

$$g \in m_1^l O_1^p \cap \phi^*(h, M),$$

hence

$$g, h \circ \phi \in \phi^*(M) + m_1(m_1^{\nu_0} O_1^p \cap \phi^*(h, M)).$$

Since $\phi^*(M) + m_1\phi^*(h, M) = \phi^*(h, M)$, Nakayma's lemma would imply the result.

Now choose $l > \nu_0$; since X is smooth, we can find a curve ϕ_1, by changing terms of the power series expansion ϕ of order $\geq l$, such that the image of ϕ_1 does not lie in $V(J_k(h, M))$.

This implies that

$$\phi_1^*(M) = \phi^*(M) \mod m_1^l O_1^p$$

$$\phi_1^*(h, M) = \phi^*(h, M) \mod m_1^l O_1^p$$

$$\phi_1^*(M) = \phi_1^*(h, M)$$

This gives a contradiction in this case.

If X is not smooth, then we can make a resolution, $\tilde{X}, \pi$, of singularities of X, lift ϕ to $\tilde{\phi}$ on $\tilde{X}$. Then $\phi^*(M) \neq \phi^*(h, M)$ if and only if $\tilde{\phi}^*\pi^*(M) \neq \tilde{\phi}^*\pi^*(h, M)$, then we can again vary $\tilde{\phi}^*$ as before.

□

If $h \in \overline{M}$, this last proposition allows us to to do more than show $h \in M$ along curves.

Proposition 7.3.26. *Suppose $h \in \overline{M}$, then there exists an open cover $\{U_{I,K}\}$ of the complement of $V(J(M))$, such that on each $U_{I,K}$, $h = [M]\xi_{I,K}$, where the entries of $\xi_{I,K}$ are locally bounded on $U_{I,K}$.*

Proof. The open cover $\{U_{I,K}\}$ is constructed by constructing an open cover $\{V_{I,K}\}$ of the fiber over the origin in $NB_{J(M)}(X)$ such that on each $V_{I,K}$, the pullback of $J(I, K)$ is a local generator of the pullback of $J(M)$. Then Cramer's rule applies, and the pullbacks of the $\xi_{I,K}$ are smooth, hence locally bounded on the images of the $V_{I,K}$ which are the $U_{I,K}$. □

As another application we can develop the analogue of property 2 for ideals.

Proposition 7.3.27. *([8], Prop 1.11) Suppose $h \in O^p_{X,x}$, M a submodule of $O^p_{X,x}$ of generic rank k on each component of X. Then $h \in \overline{M}$ if and only if for each choice of generators $\{s_i\}$ of M, there exists a constant $C > 0$ and a neighborhood U of x such that for all $\psi \in \Gamma(Hom(\mathbb{C}^p, \mathbb{C}))$,*

$$\|\psi(z) \cdot h(z)\| \leq C \sup_i \|\psi(z) \cdot s_i(z)\|$$

for all $z \in U$.

For each choice of ψ, the $\{\psi \cdot s_i(z)\}$ give a linear combination of the rows of $[M]$ at each point, while $\psi(z) \cdot h(z)$ is the analogous combination of the entries of h. So the inequality of the theorem relates the size of row vectors of $[M(x)]$ to corresponding combinations of the entries of h. The constant C and the neighborhood U depend on h and M but not on ψ.

Proof. We will use the Jacobian principle to show that the inequality implies the integral closure inclusion, by using special ψ_i.

Let S_I be a $k \times (k-1)$ submatrix of $[M]$, going through all such submatrices as I varies, let h_I be a k-tuple gotten by dropping the same entries from h as rows from $[M]$ in forming S_I. Let $\psi_I(z)(h(z)) = \det[h_I(z), S_I(z)]$. Note that $\psi_I(z)s_i(z) = \det[s_i(z), S_I(z)]$, a generator of $J_k(M)$.

The inequality which we are assuming then shows that $J_k(h, M) \subset \overline{J_k(M)}$, which gives the result by the Jacobian principle.

A weaker version of the other direction is easy; if $h \in \overline{M}$, then for any curve ϕ, $(\psi(z) \cdot h(z)) \circ \phi \in \phi^*(\{\psi(z) \cdot s_i(z)\})$, hence $(\psi(z) \cdot h(z)) \in \overline{(\{\psi(z) \cdot s_i(z)\})}$. Then the result follows by property 2 for ideals. However, here the constant does depend on ψ.

Instead we argue like this. Let $\{s_i\}$ be a set of generators of M. Applying property 2 to the finite set of elements $\{g_i\}$ that make up the numerators of the entries of the $\xi_{I,K}$ in the last proposition, we have that there exists U and C such that if g_i is such a numerator, then

$$\|g_i(z)\| \le C \sup \|J_{I,K}(z)\|.$$

We have that $J_{I,K}(z)h(z) = \sum g_i s_i$ for appropriate g_i. Then working first at $z \notin V(J(M))$

$$\|\psi(z) \cdot h(z)\| = \| \sum (g_i/J(I,K))(z)\psi(z) \cdot s_i(z)\| \le CN \sup_i \|\psi(z) \cdot s_i(z)\|$$

where N is the number of terms in the sum. Since the inequality is between continuous functions and holds on an open dense subset of U it holds on U. $\square$

Corollary 7.3.28. *Suppose $h \in O^p_{X,x}$, M a submodule of $O^p_{X,x}$ of generic rank k on each component of X. Then $h \in \overline{M}$ if and only if for each choice of generators $\{s_i\}$ of M, there exists a constant $C > 0$ and a neighborhood U of x such that for all $T \in \mathbb{C}^p$,*

$$\|T \cdot h(z)\| \le C \sup_i \|T \cdot s_i(z)\|$$

for all $z \in U$.

Proof. In one direction, take ψ to be constant; in the other we can replace T by ψ, using the fact that the constant C is independent of the choice of T. $\square$

There is a useful variant of the last Proposition.

Proposition 7.3.29. *([16]) For a section $h \in O^p_X$ to be integrally dependent on M at 0, it is necessary that, for all maps $\phi\colon (\mathbb{C}, 0) \to (X, 0)$ and $\psi\colon (\mathbb{C}, 0) \to (Hom(\mathbb{C}^p, \mathbb{C}), \lambda)$ with $\lambda \neq 0$, the function $\psi(h \circ \phi)$ on $\mathbb{C}$ belong to the ideal $\psi(M \circ \phi)$.*

Conversely, it is sufficient that this condition obtain for every ϕ whose image meets any given dense Zariski open subset of X.

We will use these ideas to extend our criterion for condition W to equidimensional sets of any codimension, but first we develop the analogue of property 4 for modules.

Blowing up modules and Connection with Ideals II

We now develop the analogue of property 4 for modules. We will want a construction that works for pairs of submodules, not just a single submodule.

Given a submodule M of a free $\mathcal{O}_{X^d}$ module F of rank p, we can associate a subalgebra $\mathcal{R}(M)$ of the symmetric $\mathcal{O}_{X^d}$ algebra on p generators. This is known as the Rees algebra of M. If $(m_1, \ldots, m_p)$ is an element of M then $\sum m_i T_i$ is the corresponding element of $\mathcal{R}(M)$. Then $\operatorname{Projan}(\mathcal{R}(M))$, the projective analytic spectrum of $\mathcal{R}(M)$ is the closure of the projectivised row spaces of M at points where the rank of a matrix of generators of M is maximal. Denote the projection to X^d by c, or by c_M where there is ambiguity.

Example 7.3.30. *If M is the Jacobian module of X, then* $\operatorname{Projan}(\mathcal{R}(M))$ *is $C(X)$, the projectivised conormal space of X.*

If M is a submodule of N or h is a section of N, then h and M generate ideals on $\operatorname{Projan}\mathcal{R}(N)$; denote them by $\rho(h)$ and $\mathcal{M}$. If we can express h in terms of a set of generators $\{n_i\}$ of N as $\sum g_i n_i$, then in the chart in which $T_1 \neq 0$, we can express a generator of $\rho(h)$ by $\sum g_i T_i / T_1$.

Example 7.3.31. *If M is the Jacobian module of X and $N = F^p$ then $V(\mathcal{M})$ consists of pairs (x, L) where $x \in X$ and $L \in \mathbb{P}Hom(\mathbb{C}^p, \mathbb{C})$, and $L \circ DF(x) = 0$. If H is the hyperplane which is the kernel of L, then the image of $DF(x)$ lies in H.*

Using 7.3.29 it is easy to show that h is integrally dependent on M at the origin, if and only the ideal sheaf induced from h is integrally dependent as an ideal sheaf on $\mathcal{M}$ along $0 \times \mathbb{P}^{p-1}$. In other words, if and only if $\rho(h)$ is integrally dependent on $\mathcal{M}$. The combination $\psi(t), \phi(t)$ amounts to giving path on $X \times \mathbb{P}^{p-1}$. This is the second connection between integral closure of ideals and modules.

Looking at a pair (M, N) allows us to “strip out" one copy of N from M, as the following example shows.

Example 7.3.32. *Let $M = I = (x^2, xy, z) = J(z^2 - x^2 y)$ and $N = J = (x, z)$. M is the Jacobian ideal of the Whitney umbrella, and N defines the singular locus of the umbrella. So, working on $\mathbb{C}^3$, we have* $\operatorname{Projan}\mathcal{R}(N) = B_J(\mathbb{C}^3)$*, which has ring $R = \mathcal{O}_3[T_1, T_2]/(zT_1 - xT_2)$, and where the map from $\mathcal{R}(N)$ to R is given by $x \to T_1, z \to T_2$. Writing the generators of I in terms of the generators of J as $x^2 = x \cdot x, xy = y \cdot x, z = z$ the map from $\mathcal{R}(I)$ to R has image (xT_1, yT_1, T_2) and this induces the ideal sheaf $\mathcal{I}$ on* $\operatorname{Projan}\mathcal{R}(N)$*. We see that this is supported only at the point* $(0, [1, 0])$.

The next proposition and the ideas behind it, is very useful in the study of determinantal singularities. It is also a good example of stripping a copy of a module N from M.

Proposition 7.3.33. *Suppose $M \subset N \subset \mathcal{O}^p_{X,0}$ are $\mathcal{O}^p_X$ modules with matrix of generators* $[M]$, $[N]$*, and* $[F]$ *is a matrix such that* $[M] = [N][F]$*. Let $\mathcal{F}$ be the ideal sheaf induced on* $\operatorname{Projan}(\mathcal{R}(N))$ *by the module F with matrix of generators* $[F]$*. Then $\overline{M} = \overline{N}$ if and only if $V(\mathcal{F})$ is empty.*

Proof. We are going to apply 7.3.29, so we must show that for all maps $\phi\colon (\mathbb{C}, 0) \to (X, 0)$ and $\psi\colon (\mathbb{C}, 0) \to (Hom(\mathbb{C}^p, \mathbb{C}), \lambda)$, that the order in t of $\psi(t)[M] \circ \phi(t)$ and $\psi(t)[N] \circ \phi(t)$ are the same. We have

$$\psi(t)[M] \circ \phi(t) = \psi(t)[N][F] \circ \phi(t).$$

Suppose the order of $\psi(t)[N] \circ \phi(t)$ in t is k. Then we can lift ϕ, ψ to a curve on $\operatorname{Projan}(\mathcal{R}(N))$ as follows. Define $\Phi\colon \mathbb{C}, 0 \to X \times \mathbb{P}^{g(N)-1}$, by $\Phi(t) = (\phi(t), [(1/t^k)(\psi(t)[N] \circ \phi(t)])$. We have $\Phi(0) = (0, \lim_{t \to 0}(1/t^k)(\psi(t)[N] \circ \phi(t))$, and the image of Φ for $t \neq 0$ clearly lies in $\operatorname{Projan}(\mathcal{R}(N))$.

Given an element $f \in \mathcal{F}$, the value of f along Φ is $(\phi(t), [(1/t^k)(\psi(t)[N]\tilde{f} \circ \phi(t)])$, where $\tilde{f}$ is the element of F which induces f. Then $V(\mathcal{F})$ is empty if and only if the order of $\mathcal{F}$ along all Φ is zero. Since $[M] = [N][F]$ this is equivalent to the order of M and N being the same on (ψ, ϕ).

□

Notice that if $M \subset N$ and $\mathcal{F}$ are as above then the inclusion of M in N always induces a map from $\text{Projan}(\mathcal{R}(N)) \setminus V(\mathcal{F})$ to $\text{Projan}(\mathcal{R}(M))$. The map is given by taking (x, p) to $(x, \mathcal{F}(p))$, where $\mathcal{F}(p)$ is evaluation of the set of generators of $\mathcal{F}$ which come from the columns of $[F]$. The next corollary includes this setting in our discussion of reduction.

Corollary 7.3.34. *Suppose M and N as above, then the following are equivalent:*

1. *M is reduction of N.*
2. *$V(\mathcal{F})$ is empty.*
3. *The induced map is a finite map from* $\text{Projan}(\mathcal{R}(N))$ *to* $\text{Projan}(\mathcal{R}(M))$.

Proof. 1) and 2) are equivalent by the previous proposition. The material in section 2 of [24] shows that the induced map is finite if and only if $V(\mathcal{F})$ is empty. □

Here is a typical way that 3) is used.

Proposition 7.3.35. *Suppose $N \subset F$, F a free $O_{X,x}$ module, and suppose the fiber of* $\text{Projan}\,\mathcal{R}(N)$ *over x has dimension k. Then N has a reduction M, where M is generated by $k+1$ elements.*

Proof. Let g be the number of generators of N, so we view $\text{Projan}\,\mathcal{R}(N)$ as a subset of $X \times \mathbb{P}^{g-1}$. For a generic choice of plane P in $\mathbb{P}^{g-1}$ of codimension $k+1$, the intersection of P and the fiber of $\text{Projan}\,\mathcal{R}(N)$ over x is empty. We can choose coordinates on $\mathbb{P}^{g-1}$ so that the plane given by $T_1 = \cdots = T_{k+1} = 0$ is such a plane, T_i coordinates on $\mathbb{P}^{g-1}$. Choosing coordinates on $\mathbb{P}^{g-1}$ is equivalent to choosing generators on N. Let M be the submodule of N generated by the first $k+1$ generators of N after the new choice of generators. Then the projection onto the first $k+1$ coordinates of $\mathbb{P}^{g-1}$, when restricted to $\text{Projan}\,\mathcal{R}(N)$ gives a finite map to $\text{Projan}\,\mathcal{R}(M)$. Hence M is a reduction of N by 3). □

Corollary 7.3.36. *Suppose $N \subset F$, F a free $O_{X,x}$ module, X^d equidimensional, N has generic rank e on each component of X, x, then N has a reduction with $d+e-1$ generators.*

Proof. Since the generic rank of N is e, the generic fiber dimension of $\text{Projan}\,\mathcal{R}(N)$ is $e-1$, so the dimension of $\text{Projan}\,\mathcal{R}(N)$ is $d+e-1$. Then $d+e-2$ is the largest the dimension of the fiber of $\text{Projan}\,\mathcal{R}(N)$ over x can be, so N has a reduction with $(d+e-2)+1$ generators. □

Having defined the ideal sheaf $\mathcal{M}$, we blow up by it. The advantages of this we will see in the notes on determinantal singularities, as it gives a constructive/geometric way to calculate the multiplicity of a pair of modules. But for now, this gives the context for which property 4 in the ideal case holds. As an example of how the blow up comes up, if we are in the basic set-up, and $M = m_Y JM(\mathcal{X})$ then the blow up by $\mathcal{M}$ is the blowup of the conormal of $\mathcal{X}$ by the ideal defining the stratum Y.

To state our result some more notation is needed. Given M a submodule of $N \subset F^p$, $h \in N$, let $NB_{\mathcal{M}}(\text{Projan}\,\mathcal{R}(N))$, $\pi_{\mathcal{M}}$ be the normalized blow-up of $\text{Projan}\,\mathcal{R}(N)$ by $\mathcal{M}$ with projection $\pi_{\mathcal{M}}$ to $\text{Projan}\,\mathcal{R}(N)$.

Proposition 7.3.37. *(Analogue of Property 4 for ideals) In the above set-up $h \in \overline{M}$ if and only if $\pi^*_{\mathcal{M}}(\rho(h)) \in \pi^*_{\mathcal{M}}(\mathcal{M})$.*

Proof. We give the proof for the case where N is free for simplicity. We apply proposition 7.3.29, so $h \in \overline{M}$ if and only if for all $\phi\colon (\mathbb{C}, 0) \to (X, 0)$ and $\psi\colon (\mathbb{C}, 0) \to (Hom(\mathbb{C}^p, \mathbb{C}), \lambda)$, we have the function $\psi(h \circ \phi)$ on $\mathbb{C}$ belongs to the ideal $\psi(M \circ \phi)$. Giving the pair (ϕ, ψ) is equivalent to giving a path on $X \times \mathbb{P}^{p-1}$, the order of $\rho(h)$ on the path is the order of $\psi(h \circ \phi)$. So 7.3.29 is equivalent to $h \in \overline{M}$ if and only if the ideal sheaf induced by $\rho(h)$ is in the integral closer of the ideal sheaf $\mathcal{M}$. In turn, by property 4 for ideals, this implies the result.

□

As an application we can extend our criterion for condition W to equidimensional sets of any codimension.
Set-up: We use the basic set-up with $\mathcal{X}^{k+n}$ an equidimensional family of equidimensional sets, $\mathcal{X}^{k+n} \subset Y^k \times \mathbb{C}^N$, $JM(X) \subset \mathcal{O}^p$.

Proposition 7.3.38. *Condition W holds for $(\mathcal{X}_0, Y)$ at $(0, 0)$ if and only if $\frac{\partial F}{\partial y_l} \in \overline{m_Y JM(F)}$ for $1 \le l \le k$.*

Proof. We re-work the form of Verdier's condition W to fit our current framework. If we work at a smooth point x of X, then a conormal vector u of X at x can always be written as $S \cdot DF(x)$, where $S \in \mathbb{C}^p$; S is not unique unless $DF(x)$ has rank p. Conversely, any such S gives a conormal vector. It is clear also that W holds if the distance inequality holds for the standard basis for the tangent space T of Y. Then

$$\operatorname{dist}(T, TX_x) = \sup_{\substack{u \in TX_x^{\perp} - \{0\} \\ v \in T - \{0\}}} \frac{\|(u, v)\|}{\|u\|\,\|v\|}.$$

becomes

$$\operatorname{dist}(T, TX_x) = \sup_{\substack{S \in \mathbb{C}^p - \{0\} \\ 1 \le i \le k,\, S \cdot DF(x) \neq 0}} \frac{\|S \cdot \frac{\partial f}{\partial y_i}\|}{\|S \cdot DF(x)\|}$$

because $\|u\| = \|S \cdot DF(x)\|$, and $\|v\| = 1$.

So Verdier's condition W becomes:

$$\sup_{\substack{S \in \mathbb{C}^p \\ 1 \le i \le k}} \|S \cdot \frac{\partial f}{\partial y_i}\| \le C\|z\|\,\|S \cdot DF(x)\|\,.$$

Since the functions are analytic and the inequality holds on a Z-open set of X, we can assume it holds on a neighborhood of the origin.

Now consider the integral closure condition, $\frac{\partial F}{\partial y_l} \in \overline{m_Y JM(F)}$ for $1 \le l \le k$. Using Corollary 2.5, we have $\frac{\partial F}{\partial y_l} \in \overline{m_Y JM(F)}$ for $1 \le l \le k$ if and only if

$$\sup_{\substack{S \in \mathbb{C}^p \\ 1 \le i \le k}} \|S \cdot \frac{\partial f}{\partial y_i}\| \le C \sup_{1 \le i \le n} \|z_i S \cdot DF(x)\| \,.$$

But this is easily seen to be equivalent to the previous inequality. □

This last result shows that Verdier's condition W is exactly the geometric meaning of the ideal sheaf induced by the $\frac{\partial f}{\partial y_i}$ being in the integral closure of the ideal sheaf induced by $m_Y JM(X)$ on $X \times \mathbb{P}^{p-1}$.

In the next section we will see how to describe and control equisingularity conditions using multiplicity of ideals and modules.

First though, we will look at an interesting variant of W-equisingularity.

We say that a deformation $F\colon Y^k \times \mathbb{C}^n \to \mathbb{C}^p$ with smooth parameter space Y^k is WV equisingular if $\frac{\partial F}{\partial y_l} \in \overline{m_Y JM_z(F) + F^*(m_Y)O^p_{n+k}}$ for $1 \le l \le k$ in O^p_{n+k}. We say each f_y is WV equivalent. This is a rephrasing of the definition of WV equivalence in [8]. In the literature, two map-germs are V-equivalent if they define isomorphic set germs. The zero sets of WV equivalent map-germs can be placed in families which are W-equisingular. Working with map-germs instead of sets often has big advantages, which we illustrate by a discussion of k-WV-determinancy.

We say that $f \in O^p_n$ is l-WV-determined, if every family of sets defined by $F(y,z) = f(z) + g(y,z)$ is WV-equisingular, where $g \in m_Y^{l+1} O^p_{n+k}$.

Theorem 7.3.39. *Suppose $\overline{T\mathcal{K}(f)} \supset m^l_n O^p_n$ in O^p_n, $n \ge p$. Then f is l-WV-determined.*

Proof. The hypothesis implies that $T\mathcal{K}(f) = O^p_n$ off the origin. This follows because $\overline{T\mathcal{K}(f)} = O^p_n$ off the origin, and this can only happen if f is a submersion at points of $X = f^{-1}(0)$ off the origin. The inequality $p \le n$ then implies X is an ICIS.

The proof then follows the standard line of ideas worked out by Mather, up to a certain point. Consider the trivial deformation G of f. The generators of $T\mathcal{K}(f)$ as an O_n module also generate $m_Y JM_z(G) + G^*(m_Y)O^p_{n+k}$ as an O_{n+k} module. Denote by π_n the projection to $\mathbb{C}^n$. Given a curve ϕ on $\mathbb{C}^{n+k}$ it is clear that the order of the generators of $m_Y JM_z(G) + G^*(m_Y)O^p_{n+k}$ along ϕ is the same as the order along $\pi_n \circ \phi$, and the same is true for the generators of m^l_Y. Thus

$$\overline{m_Y JM_z(G) + G^*(m_Y)O^p_{n+k}} \supset m^l_Y O^p_{k+n}$$

in O^p_{k+n}. Since the generators of $m_Y JM_z(F) + F^*(m_Y)O^p_{n+k}$ agree with the generators of $m_Y JM_z(G) + G^*(m_Y)O^p_{n+k}$ mod m^{l+1}_Y, it follows that:

$$\overline{m_Y JM_z(G) + G^*(m_Y)O^p_{n+k}} \supset \overline{m_Y JM_z(F) + F^*(m_Y)O^p_{n+k}}.$$

We show the opposite inclusion is true as well. Suppose ϕ is a curve on $\mathbb{C}^{n+k}$, $\phi(0)$ is the origin. We have

$$\phi^*(m_Y JM_z(F) + F^*(m_Y)O^p_{n+k}) + m_1(\phi^*(m_Y JM_z(G) + G^*(m_Y)O^p_{n+k})$$

$$= \phi^*(m_Y JM_z(G) + G^*(m_Y)O^p_{n+k}),$$

because

$$m_1(\phi^* m_Y JM_z(G) + G^*(m_Y)O^p_{n+k}) \supset m_1\phi^*(m_Y^l O^p_{n+k}) \supset \phi^*(m_Y^{l+1}O^p_{n+k}).$$

Applying Nakayama's lemma shows that

$$\phi^*(m_Y JM_z(F) + F^*(m_Y)O^p_{n+k}) = \phi^*(m_Y JM_z(G) + G^*(m_Y)O^p_{n+k}).$$

Since

$$\frac{\partial F}{\partial y_j} \circ \phi \in \phi^*(m_Y^{l+1}O_{n+k}) \subset \phi^*(m_Y JM_z(F) + F^*(m_Y)O_{n+k}),$$

F is WV-equisingular. □

Note that if we tried to argue on $G^{-1}(0)$ and $F^{-1}(0)$, we would be working on different spaces so that the integral closure operations would not be comparable. This problem already appears in the smooth case if we tried to compare $JM(f_y)$ and $JM(f_0)$ as submodules of their respective free modules. This is why we work with the maps defining the spaces on their ambient spaces instead of working directly with the sets.

Problem 7.3.40. *Show that if $n < p$ in the above theorem, then we still have $F^{-1}(0) = Y^k$.*

Problem 7.3.41. *Suppose $f\colon \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$, each component of f homogeneous of degree d, $f = 0$ an ICIS. Show that $\overline{T\mathcal{K}(f)} \supset m_n^d O_n^p$, hence f is d-WV-determined. (Hint: This is easier to do if you use the Jacobian Principle, Theorem 7.3.25.)*

Problem 7.3.42. *Suppose $f\colon \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$. Suppose the initial form of each component of f has degree d. Let f_d be the map-germ whose component functions are these initial forms. Suppose $f_d = 0$ is an ICIS. Show that f is d-WV-determined.*

More results on WV-equivalence can be found in [8].

Problem 7.3.43. *Find an example in which $h \in \overline{JM(X)}$ in O_X^p, but h is not in $\overline{T\mathcal{K}_e(f)}$ in O_n^p, f defines X.*

7.4 Multiplicities and Integral closure

The multiplicity of an ideal or module or pair of modules is one of the most important invariants we can associate to an m-primary module. It is intimately connected with integral closure. It has both a length theoretic definition and intersection theoretic definition. We give the definition in terms of length first, for ideals, and submodules of a free module. Denote the length of a module M by $l(M)$.

Theorem/Definition 7.4.1. *(Buchsbaum-Rim [5]) Suppose $M \subset F$, M, F both A-modules, F free of rank p, A a Noetherian local ring of dimension d, F/M of finite length, $\mathcal{F} = A[T_1, \ldots, T_p]$, $\mathcal{R}(M) \subset \mathcal{F}$, then*

$\lambda(n) = l(\mathcal{F}_n/\mathcal{M}_n)$ is eventually a polynomial $P(M, F)$ of degree d+p-1.

Writing the leading coefficient of $P(M, F)$ as $e(M)/(d + p - 1)!$, then we define $e(M)$ as the multiplicity of M.

It is possible to compute simple ideal examples by hand as we show:

Example 7.4.2. *Let* $M = I = (x^2, xy, y^2) \subset O_2$. *Then* $e(M) = 4$.

We have $p = 1$, $F = O_2$, and we work with $\mathcal{F} = O_2[T_1]$. (Notice that Projan $\mathcal{F} = \mathbb{C}^2$.) Now $\mathcal{M}_n = I^nT^n = m_2^{2n}T^n$, so

$$l(\mathcal{F}_n/\mathcal{M}_n) = l(O_2/m^{2n}) = (2n)(2n+1)/2 = 4n^2/2! + (l.o.t.)$$

So $e(M) = 4$.

Problem 7.4.3. *Let* $M = I = (x^2, y^2) \subset O_2$. *Show* $e(M) = 4$. *(Hint: Try to show that the terms that are missing in this problem due to the missing* xy *term, grow only linearly with n, so the leading term of the polynomial is the same.)*

It is possible to do the very simplest module examples by hand easily as well.

Problem 7.4.4. *Let* $M = m_2O_2^2$. *Show* $e(M) = 3$.

The next problem is harder–try to use the same strategy as in Problem 3.3.

Problem 7.4.5. *Let* $[M] = \begin{bmatrix} x & y & 0 \\ 0 & x & y \end{bmatrix}$. *Show* $e(M) = 3$.

Remark 7.4.6. *If* $O_{X^d,x}$ *is Cohen-Macaulay, and* M *has* $d + p - 1$ *generators where* $M \subset F^p$, *then there is a useful relation between* M *and its ideal of maximal minors. The multiplicity of* M *is the colength of* M, *and is also the colength of the ideal of maximal minors, by some theorems of Buchsbaum and Rim [5], 2.4 p.207, 4.3 and 4.5 p.223.*

A proof of this theorem in the context of analytic geometry using the Multiplicity Polar theorem is given in [11]. Using this result, it is easy to do Problem 4.5. Note however, that the colength of the ideal of maximal minors is, in general, not the multiplicity of the ideal of maximal minors.

Problem 7.4.7. *Let* $[M] = \begin{bmatrix} x & y & 0 \\ 0 & x & y \end{bmatrix}$. *Show* $e(J(M)) = 4$.

By the last paragraph if $I = (f_1, \ldots, f_d)$ is an ideal of finite colength with d generators and $O_{X^d,x}$ is Cohen-Macaulay, then $e(I)$ is just the degree of the mapgerm $F\colon X, x \to \mathbb{C}^d, 0$ defined by the generators of I. So, if $y \in \mathbb{C}^d - \Delta(F)$, then $F^{-1}(y)$ has $e(I)$ pre-images. Then $F^{-1}(ty)$ gives a deformation of the fiber of F over 0 to $e(I)$ smooth points.

An important theorem both for computational and theoretical purposes was proved by Rees in the ideal case. A proof of a generalization to modules appears in [24]

Theorem 7.4.8. *Suppose* $M \subset N$ *are* m *primary submodules of* F^p, *and* $\overline{M} = \overline{N}$. *Then* $e(M) = e(N)$. *Suppose further that* $O_{X,x}$ *is equidimensional, then* $e(M) = e(N)$ *implies* $\overline{M} = \overline{N}$

Several generalizations of this result exist: Kleiman and Thorup [[24], (6.8)(b)] proved a similar result in which F^p is replaced by an arbitrary finitely generated module whose support is equidimensional; they also proved an additivity result in Theorem (6.7b)(i) of [24] for the three pairs of modules arising from three nested modules. Generalizations also exist where the multiplicity is not defined. Gaffney and Gassler did the case of ideals [14], and Gaffney for modules [10], while Ulrich and Valadoshti have an approach using the epsilon multiplicity.

For computational purposes, this is coupled with another result–given any $M \subset F^p$, M a module over a local ring of dimension d, there exists a submodule R of M with $d + p - 1$ generators such that $\overline{M} = \overline{R}$. Such an R is called a *reduction* of M.

So if $O_{X^d,x}$ is Cohen-Macaulay, we can try to find a reduction R of M with the right number of generators $d + p - 1$, then calculate the length of F/R. (This length is also called the colength of R.) Here is a very simple example.

Problem 7.4.9. *Suppose I is any ideal in $m_2^n O_2$ which contains x^n, y^n. Then $e(I) = n^2$.*

Here is another example where it is easy to calculate $e(JM(X))$.

Proposition 7.4.10. *Let $X^1, 0 \subset \mathbb{C}^n, 0$ be an ICIS, defined by $f = (f_1, \ldots, f_{n-1})$, where f_i is homogeneous of degree d_i. Then*

$$e(JM(X)) = \left(\sum_{i=1}^{n-1}(d_i - 1)\right)(\prod_{i=1}^{n-1} d_i).$$

Proof. Note that X consists of a finite number of lines. If we treat the equations of X as equations on $\mathbb{P}^{n-1}$, the zeroes are a discrete set of points, and these points are the lines that make up X. The number of such points by Bezout's theorem is $(\prod_{i=1}^{n-1} d_i)$.

We can choose $n-1$ columns of the matrix of partial derivatives, such that the submatrix, $[N]$ gotten has rank $n - 1$ on X except at 0. Then $det[N]$ is homogeneous of degree $(\sum_{i=1}^{n-1}(d_i - 1))$, since each row is homogeneous of degree $d_i - 1$. The multiplicity of $e(N)$, N the submodule of O_X^{n-1} generated by the columns of $[N]$, is the colength of $det[N]$ in O_X by the theorem of Buchsbaum and Rim (7.4.6), since N is of finite colength and generated by $1 + (n-1) - 1 = n - 1$ generators. The colength of $det[N]$ in O_X, since X is Cohen-Macauley, is the degree of $det[N]$ on X. In turn since degree is additive, this is the sum of the degrees of $det[N]$ on each line. This degree is just the degree of $det[N]$ as a homogeneous polynomial. So,

$$e(N) = \left(\sum_{i=1}^{n-1}(d_i - 1)\right)(\prod_{i=1}^{n-1} d_i)$$

The same computation would work for any submodule of M defined using $n - 1$ linear combinations of generators of M, provided the generic rank of the submodule was $n - 1$ on each line. Hence, $e(N) = e(R)$, where R is a reduction of M, and so $e(N) = e(M)$. □

We can give a topological interpretation of the $e(JM(X))$, X an ICIS, using the Lê -Greuel formula.

Recall, Lê [44] and Greuel [1] proved the following formula:

$$\mu(X) + \mu(X') = \dim_{\mathbb{C}} \frac{O_{\mathbb{C}^n,0}}{I},$$

where X is the ICIS defined by $F\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$; F the map with components $f_1, \ldots, f_k$ and X' the ICIS defined by $F'\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{k+1}, 0)$; F' the map with components $f_1, \ldots, f_{k+1}$, I is the ideal generated by $f_1, \ldots, f_k$, and the $k + 1 \times k + 1$-minors $\frac{\partial(f_1,\ldots,f_{k+1})}{\partial(x_{i_1},\ldots,x_{i_{k+1}})}$.

Proposition 7.4.11. *(Module form of the Lê-Greuel formula) Let $X^d, 0$ be an ICIS, $d > 0$, H a hyperplane which is not a limit tangent hyperplane to X at the origin. Then*

$$e(JM(X), 0) = \mu(X) + \mu(X \cap H).$$

Proof. Let L be the linear form defining H. We let L be f_{k+1} in the formula. So the right hand side of the formula becomes $\mu(X) + \mu(X \cap H)$. Since H is a hyperplane which is not a limit tangent hyperplane to X at the origin, we know $JM(X)_H$ is a reduction of $JM(X)$. Further, the ideal of $k + 1 \times k + 1$ minors of a matrix of generators of $JM(X \cap H)$ is the same as the ideal of $k \times k$ minors of a matrix of generators of $JM(X)_H$. This implies that the colength of I in the formula is the colength of $k \times k$ minors of $JM(X)_H$, which by the Buchsbaum-Rim theorem is $e(JM(X)_H)$, which is $e(JM(X))$, since $JM(X)_H$ is a reduction of $JM(X)$. □

In the ICIS case we can use multiplicity to find Milnor numbers inductively. We first do the case of dimension 0.

Proposition 7.4.12. *Suppose I defines an ICIS X of dimension 0; then $\mu(X) = e(I, O_n) - 1$*

Proof. The hypothesis implies we can find n generators $f_1, \ldots, f_n$ of I; then $e(I) = deg(f_1, \ldots, f_n)$ at 0 as a map from $\mathbb{C}^n, 0 \to \mathbb{C}^n, 0$. Then the inverse image of a non-critical value has $e(I)$ points. Fixing one point, as a common point for every 0 sphere, we get a bouquet of $e(I) - 1$ 0-spheres. So the Milnor number is $e(I, O_n) - 1$. □

Now we show how the method works in an example.

Corollary 7.4.13. *Let X^1 be a homogeneous ICIS, then*

$$\mu(X) = \left(\sum_{i=1}^{n-1}(d_i - 1)\right)\left(\prod_{i=1}^{n-1} d_i\right) - \prod_{i=1}^{n-1} d_i + 1.$$

Proof. We know $e(JM(X), 0) = \mu(X) + \mu(X \cap H)$. Solving for $\mu(X)$ we get

$$\mu(X) = e(JM(X), 0) - \mu(X \cap H).$$

Since X has dimension 1, $\mu(X \cap H) = m(X) - 1$ by the previous proposition. Since X is a union of lines we know $e(JM(X), 0) = \sum_{i=1}^{n-1}(d_i - 1)) \prod_{i=1}^{n-1} d_i$, while $m(X) - 1 = (\prod_{i=1}^{n-1} d_i) - 1$, from which the result follows. □

Using the proof of 7.4.11, we can give another interpretation of $e(JM(X))$.

Proposition 7.4.14. *Let $\mathcal{X}$ be a versal deformation of an ICIS $(X, 0)$, defined by $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0, n > p$, let L be linear form on $\mathbb{C}^n$ such that $L^{-1}(0)$ is not a limit tangent hyperplane to $X, 0$, let π denote the projection to the base of $\mathcal{X}$. Then the degree of π restricted to $\Sigma(\pi, L)$ is the number of critical points of L restricted to a smooth fiber of $\mathcal{X}$, is $e(JM(X), 0)$.*

Proof. We know that $\Sigma(\pi, L)$ is Cohen-Macauley, so the degree of π restricted to $\Sigma(\pi, L)$, is the colength of the ideal $(u_1, \ldots, u_k)$ in the local ring of $\Sigma(\pi, L)$ at the origin, $(u_1, \ldots, u_k)$ coordinates on the base of the deformation. This the same as the colength of the ideal I in 7.4.11, which we know by the proof of 7.4.11 is $e(JM(X))$, and if we choose a non-critical value u of π restricted to $\mathcal{X}$ and to $\Sigma(\pi, L)$, the degree is the number of critical points of L on $(\mathcal{X})(u)$ which is smooth. □

This result shows we can realize $e(JM(X))$ as the number of critical points of L on any smoothing $\tilde{X}$ of X, provided L only has Morse singularities on $\tilde{X}$.

We want a theorem which extends 7.3.9 and the material in problems following it, to the module case. The next theorem due to Kleiman and Thorup provides the necessary generalization.

Set-up: X the germ of a reduced analytic space of pure dimension d, F a free O_X-module, $M \subset N \subset F$ two nested submodules with $M \neq N$, M and N are generically equal and free of rank e. Set $r := d + e - 1$. Set $C :=$ Projan$(\mathcal{R}(M))$ where $\mathcal{R}(M) \subset \mathcal{S}ym\mathcal{F}$ is the subalgebra induced by M in the symmetric algebra on F. Let $c \colon C \to X$ denote the structure map. Let W be the closed set in X where N is not integral over M, and set $E := c^{-1}W$.

Theorem 7.4.15. *(Kleiman-Thorup, [24],[25]) If N is not integral over M, then E has dimension $r - 1$, the maximum possible.*

A recent proof in a more general setting appears in [39].

If we have a family of analytic sets $\mathcal{X}$ as in the basic set-up, and M a module on $\mathcal{X}$, we can get a family of modules $M(y)$ by restricting M to the fibers of $\mathcal{X}$ over Y. There are some facts about the integral closure of ideals and their multiplicities we want to extend to modules.

Proposition 7.4.16. *Suppose $N \subset M$ are modules on $\mathcal{X}$ Assume that there is a dense Zariski open subset V of Y such that, for each y in V, the image in $O_{X(y)}$ of N is a reduction of $M(y)$. Then there is a smaller dense Zariski open subset U of Y over which N is a reduction of M.*

Proof. Cf.[16] lemma 1.2. □

Suppose M is a module on $\mathcal{X}$; assume M has finite co-support over Y, so

$$e(y) = \sum_{z \in X(y)} (e(M(y), (y, z))$$

is finite for all y. Then

Proposition 7.4.17. *The function $y \mapsto e(y)$ is Zariski upper semi-continuous on Y.*

Proof. Cf.[16] lemma 1.1. □

Theorem 7.4.18. *(Principle of Specialization of Integral Dependence PSID) Assume that X is equidimensional, and that $y \mapsto e(y)$ is constant on Y^k. Let h be a section of a free O_X module E whose image in $E(y)$ is integrally dependent on the image of $M(y)$ for all y in a dense Zariski open subset of Y. Then h is integrally dependent on M.*

Proof. (Cf. Theorem 1.8 [16])

The proof of the PSID proceeds by showing that the constancy of the multiplicity means that M has a reduction M_R which is generated by $\dim(X(y))+p-1$ generators, which is the minimum possible if $e(M(y))$ is well defined for all y. To do this, first we find such an M_R whose restriction $M_R(0)$ to $X(0)$ is a reduction of M restricted to $X(0)$, so $e(M_R(0)) = e(M(0))$ by Theorem 7.4.8. Then the uppersemicontinuity of the multiplicity implies $e(M_R(0)) \geq e(M_R(y))$, while $M_R(y) \subset M(y)$ implies $e(M_R(y)) \geq e(M(y))$. This gives us the inequality:

$$e(M(0)) = e(M_R(0)) \geq e(M_R(y)) \geq e(M(y)) = e(M(0)).$$

Thus, by Theorem 7.4.8, $M_R(y)$ is a reduction of $M(y)$ for all y.

Now replace M by the submodule generated by M_R and g, where g may be h or any element of M not in M_R. We know M_R is a reduction of (M, h) off a Zariski closed set of Y as this is true fiberwise.

Now, the dimension of the fiber of $\text{Projan}(\mathcal{R}(M_R))$ over our base point $x_0 \in X$ is at most $\dim(X(y)) + p - 2$, which is one less than the number of generators. Now the inverse image of W in $\text{Projan}(\mathcal{R}(M_R))$ must have dimension at most $\dim(X(y)) + p - 2 + k - 1$. Then since

$$(\dim(X(y)) + p - 2 + k - 1) \leq (\dim(X(y)) + k) + (p - 1) - 2 = ((\dim(X) + p - 1) - 2,$$

the Kleiman-Thorup theorem then shows that $\bar{M}_R = \bar{M}$, which gives the result. □

Now we come to the first of the two main results linking the equisingularity of families of ICIS with multiplicities.

Theorem 7.4.19. *Let X be a family of ICIS over Y^k as in the basic setup. Suppose $e(mJM(X(y), 0))$ is independent of y. Then $X - Y$ is smooth, and the pair $(X - Y, Y)$ satisfies W.*

Proof. Since $e(y)$ is upper semi-continuous, there can be no points on $X(y)$ except the origin in the co-support of $mJM(X(y))$; hence $JM(X(y))$ has maximal rank except at 0 so $X(y)$ is smooth except at 0. By 7.3.38 we have $\frac{\partial F}{\partial y_l} \in \overline{m_Y JM(F)}$ for $1 \leq l \leq k$ on a Z-open subset of Y. So by the PSID, we have that it holds at all points.

□

We have seen that bounding the dimension of the fiber of $C := \text{Projan}(\mathcal{R}(m_Y JM(\mathcal{X})))$ over the origin implies W. Surprisingly, in fact, by results of Teissier [42] and Lê-Teissier [45], we know that if W holds for the pair $(\mathcal{X}_0, Y)$ at the origin, then the fiber dimension of both C and $\text{Projan}(\mathcal{R}(JM(\mathcal{X})))$ over the origin in $\mathcal{X}$ is minimal, which is $n - 2$ in these cases. (Cf [42] Chap. 5, Th. 1.2, and [45] Prop. 2.2.4.2.) The proof of these results is difficult and beyond the scope of these notes.

For a beginner, it is a little hard to appreciate how significant these results are. Here is an example which uses the bound on the fiber dimension of $\text{Projan}(\mathcal{R}(JM(\mathcal{X})))$.

Theorem 7.4.20. *Suppose the pair $(\mathcal{X}_0, Y)$ satisfy W at the origin; let A_Y be the set of hyperplanes in $\mathcal{Y} \times C^n$ which contain Y. For a generic member H of A_y, $(H \cap \mathcal{X}_0, Y)$ also satisfies W.*

Proof. Let H be any hyperplane in A_Y which is not a limiting tangent hyperplane to $\mathcal{X}$ at 0. Since A_Y has dimension $n - 1$, and the fiber of $\text{Projan}(\mathcal{R}(JM(\mathcal{X})))$ over the origin, which is exactly the limiting tangent hyperplanes of $\mathcal{X}$ at the origin, has dimension at most $n - 2$.

We may assume H is defined by $z_1 = 0$. Then $JM_z(\mathcal{X})_H$ is a reduction of $JM_z(\mathcal{X})$, and $m_Y JM_z(\mathcal{X})_H$ is a reduction $m_Y JM_z(\mathcal{X})$. Restricting to $z_1 = 0$, the first statement implies that $JM_z(\mathcal{X} \cap H)$ has maximal rank at points of $H \cap \mathcal{X}_0$, hence these points are smooth. Further, the second statement implies the pair $(H \cap \mathcal{X}_0, Y)$ also satisfies W. □

Of course, repeating the construction implies that for a generic flag of planes containing Y, the induced family of sections satisfies W. If $X^d, 0 \subset \mathbb{C}^n$ is an ICIS, let $\mu_*(X, 0)$ denote the sequence of Milnor numbers $\mu_i(X) = \mu(X \cap H_i)$ where H_i is a generic plane of codimension

$i, 0 \leq i \leq d$. It is clear that $\mu_i(X)$ is a topological invariant of $X \cap H_i$ for $i < d$, while $\mu_d(X^d)$ is the multiplicity of X, less 1.

Given the product $mJM(X)$, there is an expansion formula which relates $e(mJM(X))$ our infinitesimal invariant to the μ_* invariants, which are our topological/geometric invariants.

Theorem 7.4.21. *Suppose $X^d, 0$ is an ICIS, H_i a generic plane of codimension i for X^d then*

$$e(mJM(X,0)) = \binom{n-1}{d} m(X,0) + \sum_{i=0}^{d-1} \binom{n-1}{i} e(JM(X \cap H_i, 0))$$

$$= \binom{n-1}{d} (\mu_d(X,0) + 1) + \sum_{i=0}^{d-1} \binom{n-1}{i} (\mu_i(X,0) + \mu_{i+1}(X,0))$$

Proof. [9]. Note that is suffices for H_i to be part of a flag of planes such that H_i is not a limiting tangent hyperplane to $X \cap H_{i-1}, 0$ at the origin in H_{i-1}. □

Although we refer to [9] for details, we provide some intuition in the ideal case, $mJ(X)$, $X^n \subset \mathbb{C}^{n+1}$. Suppose we have collections I and J of n elements of m and $J(X)$ which give reductions of m and $J(X)$, suppose we can form $A = (z_i f_i)$, $z_i \in I$, $f_i \in J(X)$, A a reduction of $mJ(X)$. Further suppose any of the ideals $B_k = (z_{i_1}, \ldots z_{i_k}, f_{j_1}, \ldots f_{j_{n-k}})$ have the generic value of the multiplicity for ideals of this type and $e(JM(X \cap H_k, 0))$ also has this generic value. We observe that given an ideal of the form $(f_1 f_2, g_1 \ldots, g_{d-1})$ in $\mathcal{O}_{X^d}$ of finite colength, then $e((f_1 f_2, g_1 \ldots, g_{d-1}) = e((f_1, g_1 \ldots, g_{d-1})) + e((f_2, g_1 \ldots, g_{d-1}))$. Using this observation repeatedly, and the assumptions about the multiplicity of the B_k and A, we get the formula of the theorem, as the binomial coefficients in the formula tell how many of the different B_k the expansion process yields.

Corollary 7.4.22. *Let $\mathcal{X}$ be a family of ICIS over Y^k as in the basic setup. Suppose $e(mJM(X(y), 0))$ is independent of y. Then the μ_* sequence of $X(y)$ is independent of y.*

Proof. Although this follows indirectly from 7.4.19, and the argument of the next theorem, we can give a simple, direct proof here. Since the $\mu_*(X(y))$ sequence is upper semi-continuous in y, as is $e(mJM(X(y), 0))$, all of the terms in the sum must remain constant, if the value of the sum does. □

Now we can prove our second main result.

Theorem 7.4.23. *Suppose $\mathcal{X}$ is a family of ICIS, and the pair $(\mathcal{X} - Y, Y)$ satisfies W at the origin. Then, the μ_* sequence of $X(y)$ is independent of y, as is $e(m_y JM(X(y)))$.*

Proof. Since the families of generic plane sections also satisfy W by 7.4.20, it follows that these families are topologically trivial, hence the μ_* sequence of $X(y)$ is independent of y. This implies $e(m_y JM(X(y))$ is independent of y by 7.4.21. □

Challenge Problem 7.4.24. *What is the geometric meaning of $e(T_e\mathcal{K}(f))$? This is well understood for functions; since $f \in \overline{J(f)}$, $e(T_e\mathcal{K}(f)) = e(J(f)) = \mu(f)$.*

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