THE CONE STRUCTURE THEOREM FOR MAP-GERMS WITH NON ISOLATED ZEROS

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Abstract. We consider the topological classification of finitely determined map germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with $f^{-1}(0) \neq \{0\}$. Associated with f we have a link diagram, which is well defined up to topological equivalence. We prove that f is topologically equivalent to the generalized cone of its link diagram.

1. Introduction

The classification problem of singular points of smooth map germs is one of the most important problems in Singularity Theory. The classical classification is done via $\mathscr{A}$-equivalence, where the changes of coordinates are given by diffeomorphisms in the source and the target. However, this is a difficult problem and it presents a lot of rigidity. Then it seems natural to investigate the classification of mappings up to weaker equivalence relations. Here we consider topological equivalence (or C^0-$\mathscr{A}$-equivalence), where the change of coordinates are homeomorphisms instead of diffeomorphisms.

It is known the topological structure of a map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ can be determined by the topological type of its associated link. The link of f is obtained by intersecting the image of f with a small enough sphere centered at the origin. If f has an isolated singularity, it follows from Milnor [13], that the topological type of f is determined by the link, which is a smooth fibre bundle from a smooth $(n-1)$-manifold with boundary N into S^{p-1}. However, isolated singularity is a very strong condition.

Later, Fukuda [4] considered a generic condition which includes the case that f is finitely determined (Fukuda's cone structure theorem). When $f^{-1}(0) = \{0\}$, the link is a topologically stable map from S^{n-1} to S^{p-1} and f is topologically equivalent to the cone of its link. The case $f^{-1}(0) \neq \{0\}$ was considered by Fukuda in [5], where it is showed that the link is a topologically stable map from a smooth $(n-1)$-manifold with boundary N into S^{p-1}. It is claimed (without proving it) that the topological type of f can be determined by its link. Recently, in [3], it was proved a version of Fukuda's theorem with respect to the topological contact equivalence (or C^0-$\mathcal{K}$-equivalence), introducing the notions of link diagram and generalized cone.

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In this paper we show that if $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is a finitely determined map germ with $f^{-1}(0) \neq \{0\}$, then f has a link diagram, which is well defined up to topological equivalence, and that f is topologically equivalent to the generalized cone of its link diagram (Theorem 4.4). This kind of result is very useful to obtain the topological classification of finitely determined map germs (see for instance [1, 2]).

2. Stability and finite determinacy

We assume that the reader is familiar with all the basic definitions and properties of singularities of mappings, such as $\mathscr{A}$-equivalence, stability, finite determinacy, etc. We refer to Wall's survey paper [15] for details.

Let $S \subset \mathbb{R}^n$ be a finite subset and $y \in \mathbb{R}^p$. Two smooth (i.e. C^∞) multi-germs $f, g : (\mathbb{R}^n, S) \to (\mathbb{R}^p, y)$ are said to be *$\mathscr{A}$-equivalent* if there exist diffeomorphism germs $\phi : (\mathbb{R}^n, S) \to (\mathbb{R}^n, S)$ and $\psi : (\mathbb{R}^p, y) \to (\mathbb{R}^p, y)$ such that $f = \psi \circ g \circ \phi^{-1}$. If ϕ, ψ are homeomorphisms instead of diffeomorphisms, then f, g are said to be *topologically equivalent* (or *C^0-$\mathscr{A}$-equivalent*).

Now we recall the concept of finite determinacy. For simplicity, we restrict ourselves to mono-germs. Given a smooth map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$, we denote by $j^k f(0)$ the k-jet of f (that is, its Taylor expansion of order k at the origin). We say that f is *k-determined* if for any g with the same k-jet, we have that g is $\mathscr{A}$-equivalent to f. We say that f is *finitely determined* (abbreviated FD) if it is k-determined for some k.

From the above definition, we deduce that if f is k-determined then f is $\mathscr{A}$-equivalent to its k-jet $j^k f(0)$. Thus, when working with FD map germs we can assume without loss of generality that f is the germ of a polynomial mapping. In particular, we can consider its complexification $\hat{f} : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ which is also FD (as a complex analytic map germ). The Mather-Gaffney criterion of finite determinacy (see [15]) says that this happens if and only if there exist small enough neighbourhoods such that the representative of f defined in these neighbourhoods has the origin as the only unstable singularity. However, in the real case, this is only a necessary condition.

For any mapping f, we denote by $\Sigma(f)$ its critical set, that is, the set of points in where f is not submersive and the image $\Delta(f) = f(\Sigma(f))$ is called the discriminant.

Theorem 2.1. *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a FD map germ. There exists a representative $f : U \to V$, where U and V are open neighborhoods of the origin in $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively, such that $f^{-1}(0) \cap \Sigma(f) = \{0\}$ and the restriction $f : U \smallsetminus f^{-1}(0) \to V \smallsetminus \{0\}$ is locally stable (i.e., any multi-germ $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, y)$ is stable, where $S \subset U \smallsetminus f^{-1}(0)$ is finite).*

Assume f is FD and $f : U \to V$ is a representative as in Theorem 2.1. Since $f^{-1}(0) \cap \Sigma(f) = \{0\}$, we can shrink the neighborhoods U, V if necessary and assume that the restriction $f : \Sigma(f) \to V$ is finite (i.e., finite-to-one and proper).

Moreover, the 0-stable singularities (i.e., singularities of $\mathscr{K}_e$-codimension p) are isolated points in $V \smallsetminus \{0\}$. Hence, if we assume that f is polynomial, then the set of 0-stable singularities of each type is semialgebraic and by the Curve Selection Lemma (see [13]), they cannot accumulate at the origin. Thus, by shrinking again the neighborhoods U, V if necessary, we can assume that f has no 0-stable singularities in $V \setminus \{0\}$.

Definition 2.2. A smooth map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ has *isolated instability* (abbreviated II) if there exists a representative $f : U \to V$, where U and V are open neighborhoods of the origin in $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively, such that

(1) $f^{-1}(0) \cap \Sigma(f) = \{0\}$,
(2) the restriction $f : \Sigma(f) \to V$ is finite,
(3) the restriction $f : U \setminus f^{-1}(0) \to V \setminus \{0\}$ is locally stable with no 0-stable singularities.

In such case we say that $f : U \to V$ is a *good representative* of f. Moreover, if f is a polynomial mapping, we also add the condition that U, V are semialgebraic sets.

If a germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is FD, then it has finite singularity type, which is equivalent to the fact that it admits a stable unfolding F (see [6]). Given a stable type represented by the $\mathscr{A}$-class of a stable multi-germ $g : (\mathbb{R}^n, S) \to (\mathbb{R}^p, y)$, we say that F presents the stable type if for any representative $F : U \to V$ there exists $(u, y') \in V$ such that the multi-germ of f_u at y' is $\mathscr{A}$-equivalent to g at y.

Definition 2.3. We say that a smooth map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ has *discrete stable type* (abbreviated DST) if there exists a stable unfolding F of f which only presents a finite number of stable types.

Some cases in which a FD map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ has DST are:

(1) when the pair (n, p) is a pair of nice dimensions or are in the boundary of the nice dimensions in Mather's sense [12];
(2) when f has corank 1.

If $f : U \to V$ is locally stable, then for each $y \in \Delta(f)$ we denote by $\mathrm{Iso}(f, y)$ the *iso-singular locus* of f at y. By definition, this is the set of points $y' \in \Delta(f)$ such that the multi-germ $f : (\mathbb{R}^n, S') \to (\mathbb{R}^p, y')$ is $\mathscr{A}$-equivalent to the multi-germ $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, y)$, where $S' = f^{-1}(y') \cap \Sigma(f)$ and $S = f^{-1}(y) \cap \Sigma(f)$. It is well known that $\mathrm{Iso}(f, y)$ is a submanifold of V.

Definition 2.4. Let $f : U \to V$ be a good representative of a germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with II and DST. We construct a stratification $(\mathcal{A}, \mathcal{B})$ of f defined as follows:

- The strata B of $\mathcal{B}$ are either $B = \{0\}$, $B = V \setminus \Delta(f)$ or the iso-singular loci $B = \mathrm{Iso}(f, y)$, with $y \in \Delta(f) \setminus \{0\}$.
- The strata A of $\mathcal{A}$ are either strata of the form $A = f^{-1}(B) \cap \Sigma(f)$ or strata of the form $A = f^{-1}(B) \setminus \Sigma(f)$, for some $B \in \mathcal{B}$. In particular, we always have the strata $A = \{0\}$ and $A = f^{-1}(0) \setminus \{0\}$ (if $f^{-1}(0) \neq \{0\}$).

We call $(\mathcal{A}, \mathcal{B})$ the *stratification by stable types*. The fact that f has DST guarantees that the stratification is finite. If moreover, we add the hypothesis that f is polynomial, then the strata are semialgebraic sets.

Lemma 2.5. *Let $f : U \to V$ be a good representative of a germ with II and DST. Then the stratification by stable types is a Thom stratification of f.*

Proof. We have to show that $\mathcal{A}, \mathcal{B}$ satisfy the Whitney conditions and the Thom A_f condition. This is well known for $f : U \setminus f^{-1}(0) \to V \setminus \{0\}$ because of the stability of f. Moreover, f is a submersion in a neighborhood of each point of $f^{-1}(0) \setminus \{0\}$, so that we also have the Whitney conditions and the Thom A_f condition outside the origin. Finally, both the Whitney conditions and the Thom A_f condition are trivially satisfied with respect to the stratum of the origin $\{0\}$. □

In the case that f has no DST, we still have a Thom stratification of $f : U \to V$ which is called the *canonical Thom stratification* of f. Such stratification exists provided that f is good representative of a polynomial map germ with II (see [6]).

3. The link of a FD map germ

Denote by $J^r(n,p)$ the r-jet space from $(\mathbb{R}^n, 0)$ to $(\mathbb{R}^p, 0)$. For positive integers r and s with $s \geq r$, let $\pi_r^s : J^s(n,p) \to J^r(n,p)$ be the canonical projection defined by $\pi_r^s(j^s f(0)) = j^r f(0)$. For a positive number $\epsilon > 0$ we set:

$$D_\epsilon^n = \{x \in \mathbb{R}^n \mid \|x\|^2 \leq \epsilon\},$$
$$B_\epsilon^n = \{x \in \mathbb{R}^n \mid \|x\|^2 < \epsilon\},$$
$$S_\epsilon^{n-1} = \{x \in \mathbb{R}^n \mid \|x\|^2 = \epsilon\}.$$

T. Fukuda has proved the following cone structure theorem in his papers [4, 5]:

Theorem 3.1. *For any semialgebraic subset W of $J^r(n,p)$, there exists an integer s ($s \geq r$) depending only n, p and r, and there exists a closed semialgebraic subset Σ_W of $(\pi_r^s)^{-1}(W)$ having codimension ≥ 1 such that for any C^∞ mapping $f : \mathbb{R}^n \to \mathbb{R}^p$ with $j^s f(0)$ belonging to $(\pi_r^s)^{-1}(W) \setminus \Sigma_W$ we have the following properties:*

(A) *Case $f^{-1}(0) = \{0\}$. There exists $\epsilon_0 > 0$ such that for any number ϵ with $0 < \epsilon \leq \epsilon_0$, we have:*

(A-i) *$f^{-1}(S_\epsilon^{p-1})$ is diffeomorphic to the standard unit sphere S^{n-1}.*

(A-ii) *The restricted mapping $f|f^{-1}(S_\epsilon^{p-1}) : f^{-1}(S_\epsilon^{p-1}) \to S_\epsilon^{p-1}$ is topologically stable (stable if (n,p) is a nice pair) and its topological class is independent of ϵ.*

(A-iii) *The restricted mapping $f|f^{-1}(D_\epsilon^{p-1}) : f^{-1}(D_\epsilon^{p-1}) \to D_\epsilon^p$ is topologically equivalent to the cone of $f|f^{-1}(S_\epsilon^{p-1})$.*

(B) *Case $f^{-1}(0) \neq \{0\}$. There exist $\epsilon_0 > 0$ and a strictly increasing smooth function $\delta : [0, \epsilon_0] \to [0, \infty)$ with $\delta(0) = 0$ such that for any ϵ, δ with $0 < \epsilon \leq \epsilon_0$ and $0 < \delta < \delta(\epsilon)$, we have:*

(B-i) *$f^{-1}(0) \cap S_\epsilon^{n-1}$ is an $(n-p-1)$-dimensional manifold and it is diffeomorphic to $f^{-1}(0) \cap S_{\epsilon_0}^{n-1}$.*

(B-ii) *$D_\epsilon^n \cap f^{-1}(S_\delta^{p-1})$ is a smooth manifold with boundary and it is diffeomorphic to $D_{\epsilon_0}^n \cap f^{-1}(S_{\delta_0}^{p-1})$.*

(B-iii) *the restriction* $f|D^n_\epsilon \cap f^{-1}(S^{p-1}_\delta) : D^n_\epsilon \cap f^{-1}(S^{p-1}_\delta) \to S^{p-1}_\delta$ *is a topologically stable map (stable if* (n,p) *is a nice pair) and its topological class is independent of* ϵ *and* δ.

Assuming that f is a FD map germ, we can assume that f is a polynomial mapping and has II. Then, we can apply Theorem 3.1 to obtain a good representative of f satisfying (A) or (B), depending on weather $f^{-1}(0) = \{0\}$ or $f^{-1}(0) \neq \{0\}$. Note that when $n \leq p$ we always have $f^{-1}(0) = \{0\}$ by the finite determinacy condition, but when $n > p$ we may have the two possibilities. The condition that (n,p) is a nice pair in (A-ii) or (B-iii) is not necessary if the map germ f has DST. In fact, the proof of the theorem is based on the stratification by stable types when it is defined or the canonical Thom stratification otherwise.

Definition 3.2. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a FD map germ. We take $f : U \to V$ a good representative and ϵ, δ as in Theorem 3.1. The link of f is defined as the map:

$$f|f^{-1}(S^{p-1}_\epsilon) : f^{-1}(S^{p-1}_\epsilon) \to S^{p-1}_\epsilon,$$

when $f^{-1}(0) = \{0\}$, or the map:

$$f|D^n_\epsilon \cap f^{-1}(S^{p-1}_\delta) : D^n_\epsilon \cap f^{-1}(S^{p-1}_\delta) \to S^{p-1}_\delta,$$

when $f^{-1}(0) \neq \{0\}$.

It follows from the definition and from Theorem 3.1 that:

(1) The link is well defined up to topological equivalence.
(2) The link is a topologically stable map (stable if f has DST).
(3) When $f^{-1}(0) = \{0\}$, f is topologically equivalent to the cone of the link.

Corollary 3.3. *Two FD map germs* $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ *with* $f^{-1}(0) = \{0\} = g^{-1}(0)$ *are topologically equivalent if their associated links are topologically equivalent.*

When $f^{-1}(0) \neq \{0\}$, Theorem 3.1 does not give that f is topologically equivalent to the cone of its link, as in the case $f^{-1}(0) = \{0\}$. In fact, the cone of $D^n_\epsilon \cap f^{-1}(S^{p-1}_\delta)$ is not homeomorphic to the closed disk D^n, hence the restriction $f|D^n_\epsilon \cap f^{-1}(S^{p-1}_\delta)$ cannot be topologically equivalent to the cone of the link. To obtain a similar result, it is necessary to use the notion of generalized cone. This was done first in [3] in the context of topological $\mathscr{K}$-equivalence.

4. The cone structure theorem for $f^{-1}(0) \neq \{0\}$

In [3], the second and third named authors introduced a generalized notion of cone and they also proved a version of the Fukuda's theorem for topological $\mathscr{K}$-equivalence.

Definition 4.1. A *link diagram* is a diagram as follows

$$V \xleftarrow{\ r\ } N \xrightarrow{\ \gamma\ } S^{p-1},$$

where N is a manifold with boundary, γ is a continuous mapping, V is a contractible space and r is a continuous and surjective mapping such that the attaching space $(N \times I) \cup_r V$ is homeomorphic to the closed disk D^n (here we set $I = [0,1]$ and we identify $N \equiv N \times \{0\} \subset N \times I$).

Definition 4.2. Given a link diagram

$$V \xleftarrow{r} N \xrightarrow{\gamma} S^{p-1},$$

we define the *generalized cone* as the induced mapping

$$C(\gamma, r) : (N \times I) \cup_r V \to c(S^{p-1})$$

defined as $[x,t] \mapsto [\gamma(x), t]$ if $(x,t) \in N \times I$ and $[y] \mapsto 0$ if $y \in V$, where $c(S^{p-1})$ is the usual cone.

Note that in the particular case that $V = \{0\}$, the generalized cone coincides with the usual notion of cone.

Definition 4.3. Two link diagrams

$$V_0 \xleftarrow{r_0} N_0 \xrightarrow{\gamma_0} S^{p-1}, \quad V_1 \xleftarrow{r_1} N_1 \xrightarrow{\gamma_1} S^{p-1}$$

are topologically equivalent if there exist homeomorphisms $\alpha : V_0 \to V_1$, $\phi : N_0 \to N_1$ and $\psi : S^{p-1} \to S^{p-1}$ such that $r_1 = \alpha \circ r_0 \circ \phi^{-1}$ and $\gamma_1 = \psi \circ \gamma_0 \circ \phi^{-1}$.

We present now the structure cone theorem for map germs with non isolated zeros. Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a smooth map germ, in order to simplify te notation, we put $f_{\epsilon,\delta} := f|N_{\epsilon,\delta} : N_{\epsilon,\delta} \to S^{p-1}_\delta$, where $N_{\epsilon,\delta} = D^n_\epsilon \cap f^{-1}(S^{p-1}_\delta)$ and $V_\epsilon = f^{-1}(0) \cap D^n_\epsilon$ (see fig. 1).

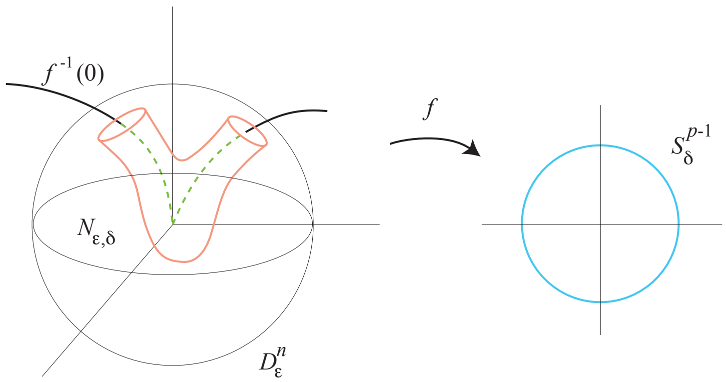

FIGURE 1

Theorem 4.4. *Let $f : U \to V$ be a good representative of a polynomial map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with II and $f^{-1}(0) \neq \{0\}$. For each $0 < \delta \ll \epsilon \ll 1$ small enough, there exists a continuous and surjective mapping $r_{\epsilon,\delta} : N_{\epsilon,\delta} \to V_\epsilon$, such that:*

(1) *The link diagram*

$$V_\epsilon \xleftarrow{r_{\epsilon,\delta}} N_{\epsilon,\delta} \xrightarrow{f_{\epsilon,\delta}} S^{p-1}_\delta$$

is independent of ϵ, δ up to topological equivalence.

(2) *The restriction* $f|D^n_\epsilon \cap f^{-1}(D^p_\delta) : D^n_\epsilon \cap f^{-1}(D^p_\delta) \to D^p_\delta$ *is topologically equivalent to the generalized cone:*

$$C(f_{\epsilon,\delta}, r_{\epsilon,\delta}) : (N_{\epsilon,\delta} \times I) \cup_{r_{\epsilon,\delta}} V_\epsilon \to c(S^{p-1}_\delta),$$

where $I = [0, \delta]$.

Proof. Let $(\mathcal{A}, \mathcal{B})$ be either the stratification by stable types if f has DST or the canonical Thom stratification otherwise (see Lemma 2.5). We choose $0 < \delta_0 \ll \epsilon_0 \ll 1$ small enough such that the following conditions hold:

(1) $f^{-1}(0) \pitchfork S^{n-1}_\epsilon$, for all ϵ with $0 < \epsilon \leq \epsilon_0$;
(2) $\mathcal{A} \pitchfork S^{n-1}_{\epsilon_0}$ on $S^{n-1}_{\epsilon_0} \cap f^{-1}(D^p_{\delta_0})$.
(3) $\mathcal{B} \pitchfork S^{p-1}_\delta$, for all δ with $0 < \delta \leq \delta_0$;

Let $B^n_{\epsilon_0}, B^p_{\delta_0}$ denote the interiors of $D^n_{\epsilon_0}, D^p_{\delta_0}$ respectively. We consider the restriction $f : D^n_{\epsilon_0} \cap f^{-1}(B^p_{\delta_0}) \to B^p_{\delta_0}$ and observe the following facts:

- f is proper, since f is the restriction of the mapping $f : D^n_{\epsilon_0} \cap f^{-1}(D^p_{\delta_0}) \to D^p_{\delta_0}$, which is obviously proper.
- $D^n_{\epsilon_0} \cap f^{-1}(B^p_{\delta_0})$ is a manifold with boundary given by $S^{n-1}_{\epsilon_0} \cap f^{-1}(B^p_{\delta_0})$.
- The restriction of f to the boundary $f : S^{n-1}_{\epsilon_0} \cap f^{-1}(B^p_{\delta_0}) \to B^p_{\delta_0}$ is a submersion.
- The restriction of the stratification $(\mathcal{A}, \mathcal{B})$ to $(D^n_{\epsilon_0} \cap f^{-1}(B^p_{\delta_0}), B^p_{\delta_0})$ provides a Thom stratification of f, taking into account that we must consider, on one hand, the strata of $\mathcal{A}$ in the interior $B^n_{\epsilon_0} \cap f^{-1}(B^p_{\delta_0})$ and on the other hand, the strata of $\mathcal{A}$ in the boundary $S^{n-1}_{\epsilon_0} \cap f^{-1}(B^p_{\delta_0})$.

On the other hand, if in the interval $[0, \delta_0)$ we consider the stratification $\mathcal{C} = \{(0, \delta_0), \{0\}\}$, then the pair $(\mathcal{B}, \mathcal{C})$ is a Thom stratification of the function $\rho : B^p_{\delta_0} \to [0, \delta_0)$, given by $\rho(y) = \|y\|^2$, which is also proper.

Let T be the stratified vector field on $[0, \delta_0)$ given by $T = \frac{d}{dt}$ on $(0, \delta_0)$ and $T_0 = 0$. Then, by [6, Theorem 3.2] there exists a stratified vector field Y on $B^p_{\delta_0}$ which is a lifting of T through ρ and there exists a stratified vector field X on $D^n_{\epsilon_0} \cap f^{-1}(B^p_{\delta_0})$ which is a lifting of Y through f. Moreover, since T is globally integrable, then Y, X are also globally integrable, by [6, Lemma 4.8].

Let $0 < \delta_1 < \delta_0$. We define the mappings Φ, Ψ:

$$\begin{array}{ccc} D^n_{\epsilon_0} \cap f^{-1}(D^p_{\delta_1}) \setminus V_{\epsilon_0} & \xrightarrow{\ f\ } & D^p_{\delta_1} \setminus \{0\} \\ \Phi \downarrow & & \Psi \downarrow \\ N_{\epsilon_0,\delta_1} \times (0, \delta_1] & \xrightarrow{f_{\epsilon_0,\delta} \times Id} & S^{p-1}_{\delta_1} \times (0, \delta_1] \end{array}$$

given by $\Phi(x) = (\phi(x), \|f(x)\|^2)$ and $\Psi(y) = (\psi(y), \|y\|^2)$, where:

- $\phi(x)$ is the point of N_{ϵ_0,δ_1} where the integral curve of X passing through x meets N_{ϵ_0,δ_1},
- $\psi(y)$ is the point of $S^{p-1}_{\delta_1}$ where the integral curve of Y passing through y meets $S^{p-1}_{\delta_1}$.

Note that Φ, Ψ are homeomorphisms. In fact, $\Phi^{-1}(x,t)$ is the point where the integral curve of X passing through x meets $N_{\epsilon_0,t}$ and $\Psi^{-1}(y,t)$ is the point where the integral curve of Y passing through y meets S_t^{p-1}. It is obvious that Φ, Ψ as well as their inverse mappings are continuous, since they are defined from the flows of X, Y respectively.

Another point is that the above diagram is commutative. We have that $\Psi(f(x)) = (\psi(f(x)), \|f(x)\|^2)$ and $f(\Phi(x)) = (f(\phi(x)), \|f(x)\|^2)$. But since that X is a lifting of Y through f, we have $df \circ X = Y \circ f$ and this implies that f maps integral curves of X into integral curves of Y, from which we deduce $\psi(f(x)) = f(\phi(x))$.

On the other hand, we also define a retraction

$$r : D^n_{\epsilon_0} \cap f^{-1}(D^p_{\delta_1}) \to V_{\epsilon_0}$$

where $r(x)$ is the point of V_{ϵ_0} where the integral curve of X passing through x meets V_{ϵ_0}. We have that r is continuous, surjective and moreover, $r(x) = x$, for all $x \in V_{\epsilon_0}$. We also have

$$\lim_{t\to 0} \Phi^{-1}(x,t) = r(x), \quad \lim_{t\to 0} \Psi^{-1}(y,t) = 0.$$

This allows us to extend the homeomorphisms Φ, Ψ to homeomorphisms $\overline{\Phi}, \overline{\Psi}$ which make commutative the following diagram:

$$\begin{array}{ccc} D^n_{\epsilon_0} \cap f^{-1}(D^p_{\delta_1}) & \xrightarrow{f} & D^p_{\delta_1} \\ \overline{\Phi}\big\downarrow & & \overline{\Psi}\big\downarrow \\ (N_{\epsilon_0,\delta_1} \times [0,\delta_1]) \cup_{r_{\epsilon_0,\delta_1}} V_{\epsilon_0} & \xrightarrow{C(f_{\epsilon_0,\delta_1}, r_{\epsilon_0,\delta_1})} & C(S^{p-1}_{\delta_1}). \end{array}$$

With this we finish the proof of (2). Let us see now (1). Given $0 < \epsilon < \epsilon_0$ and $0 < \delta < \delta(\epsilon)$, by Theorem 3.1, there exist homeomorphisms α, β which make commutative the following diagram

$$\begin{array}{ccc} N_{\epsilon_0,\delta_1} & \xrightarrow{f_{\epsilon_0,\delta_1}} & S^{p-1}_{\delta_1} \\ \alpha\big\downarrow & & \beta\big\downarrow \\ N_{\epsilon,\delta} & \xrightarrow{f_{\epsilon,\delta}} & S^{p-1}_{\delta}. \end{array}$$

On the other hand, again by Theorem 3.1 we know there exists a homeomorphism $\sigma : V_{\epsilon_0} \to V_\epsilon$. Then, it is enough to define $r_{\epsilon,\delta} : N_{\epsilon,\delta} \to V_\epsilon$ as $r_{\epsilon,\delta} = \sigma \circ r_{\epsilon_0,\delta_1} \circ \alpha^{-1}$, in such a way that we have a topological equivalence of link diagrams:

$$\begin{array}{ccccc} V_{\epsilon_0} & \xleftarrow{r_{\epsilon_0,\delta_1}} & N_{\epsilon_0,\delta_1} & \xrightarrow{f_{\epsilon_0,\delta_1}} & S^{p-1}_{\delta_1} \\ \sigma\big\downarrow & & \alpha\big\downarrow & & \beta\big\downarrow \\ V_\epsilon & \xleftarrow{r_{\epsilon,\delta}} & N_{\epsilon,\delta} & \xrightarrow{f_{\epsilon,\delta}} & S^{p-1}_{\delta}. \end{array}$$

□

Definition 4.5. Let $f : U \to V$ a good representative of a polynomial map germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with II and $f^{-1}(0) \neq \{0\}$. The *link diagram* of f

is the link diagram
$$V_\epsilon \xleftarrow{r_{\epsilon,\delta}} N_{\epsilon,\delta} \xrightarrow{f_{\epsilon,\delta}} S^{p-1}_\delta$$
given in Theorem 4.4 for $0 < \delta \ll \epsilon \ll 1$.

If $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is a FD map germ with $f^{-1}(0) \neq \{0\}$, then it has II and we can assume it is a polynomial mapping. Then, we define the link diagram of f by taking a good representative. It follows that the link diagram is well defined up to topological equivalence and that f is topologically equivalent to the generalized cone of its link diagram.

Lemma 4.6. *If two link diagrams are topologically equivalent, then their generalized cones are topologically equivalent.*

Proof. Suppose that the two link diagrams
$$V_0 \xleftarrow{r_0} N_0 \xrightarrow{\gamma} S^{p-1}, \quad V_1 \xleftarrow{r_1} N_1 \xrightarrow{\gamma} S^{p-1}$$
are topologically equivalent. Then, there are homeomorphisms $\alpha : V_0 \to V_1$, $\phi : N_0 \to N_1$ and $\psi : S^{p-1} \to S^{p-1}$ such that $r_1 = \alpha \circ r_0 \circ \phi^{-1}$ and $\gamma_1 = \psi \circ \gamma_0 \circ \phi^{-1}$.

Then we have an induced topological equivalence between the generalized cones $C(\gamma_0, r_0)$ and $C(\gamma_1, r_1)$:
$$\begin{array}{ccc} (N_0 \times I) \cup_{r_0} V_0 & \xrightarrow{C(\gamma_0, r_0)} & c(S^{p-1}) \\ \Big\downarrow \widetilde{\Phi} & & \Big\downarrow c(\psi) \\ (N_1 \times I) \cup_{r_1} V_1 & \xrightarrow{C(\gamma_1, r_1)} & c(S^{p-1}) \end{array}$$
where $\widetilde{\Phi}$ is the homeomorphism induced by ϕ and α in the following way: $\widetilde{\Phi}([x,t]) = [\phi(x), t]$ if $x \in N_0$ and $\widetilde{\Phi}([y]) = [\alpha(y)]$ if $y \in V_0$.

□

Corollary 4.7. *Let $f, g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be two FD map germs with non isolated zeros. If their link diagrams are topologically equivalent, then f and g are topologically equivalent.*

Example 4.8. Consider a FD function germ $f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$ with $f^{-1}(0) \neq \{0\}$. The finitely determinacy condition implies that f has isolated critical point in the origin. We fix $0 < \delta \ll \epsilon \ll 1$ as in Theorem 4.4. We can assume f is polynomial, hence $f^{-1}(0)$ is the algebraic curve given by $f(x,y) = 0$. Then, $V_\epsilon = f^{-1}(0) \cap D^2_\epsilon$ is made of a finite even number $2r$ of half-branches which intersect transversally the boundary S^1_ϵ and separate the disk D^2_ϵ into $2r$ sectors, so that the sign of f alternates on consecutive sectors (see fig. 2).

The manifold $N_{\epsilon,\delta}$ is given by the level curves $f(x,y) = \pm\delta$ in D^2_ϵ. It has $2r$ connected components, one in each sector of $D^2_\epsilon \setminus f^{-1}(0)$ and diffeomorphic to a closed interval. Moreover, the retraction map $r : N_{\epsilon,\delta} \to V_\epsilon$, when restricted to each connected component, is a diffeomorphism onto the two half-branches which bound the sector containing the connected component.

Thus, the topological class of f only depends on the number of half-branches of $f^{-1}(0)$. We deduce that two functions f, g are topologically equivalent if and only if the curves $f^{-1}(0)$ and $g^{-1}(0)$ have the same number of half-branches.

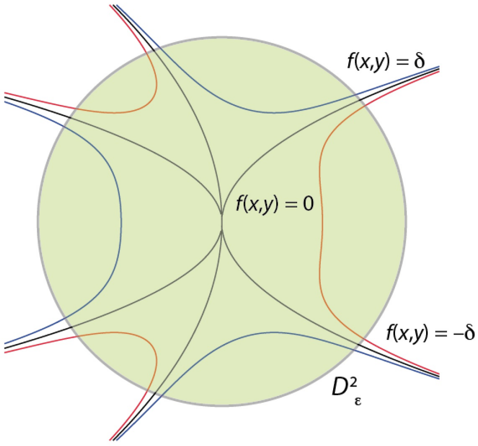

FIGURE 2

Remark 4.9. The cone structure theorem with respect to the C^0-$\mathscr{K}$-equivalence was done in [3]. The main differences are the following:

(1) if $f^{-1}(0) = \{0\}$, the link $f|f^{-1}(S^{p-1}_\epsilon) : f^{-1}(S^{p-1}_\epsilon) \to S^{p-1}_\epsilon$ is not a stable map anymore, but a smooth map whose homotopy $\mathscr{A}$-equivalence type (see [3]) is independent of ϵ and $f|f^{-1}(D^p_\epsilon)$ is topologically $\mathscr{K}$-equivalent to the cone of the link $f|f^{-1}(S^{p-1}_\epsilon)$.

(2) if $f^{-1}(0) \neq \{0\}$, the link diagram

$$V_\epsilon \xleftarrow{r_{\epsilon,\delta}} N_{\epsilon,\delta} \xrightarrow{f_{\epsilon,\delta}} S^{p-1}_\delta,$$

is independent of ϵ, δ, up to homotopy $\mathscr{A}$-equivalence and $f|D^n_\epsilon \cap f^{-1}(D^p_\delta)$ is topologically $\mathscr{K}$-equivalent to the generalized cone

$$C(f|N_{\epsilon,\delta}, r_{\epsilon,\delta}) : (N_{\epsilon,\delta} \times I) \cup_{r_{\epsilon,\delta}} V_\epsilon \to c(S^{p-1}_\delta)$$

and the map $f_{\epsilon,\delta}$ is not stable.

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