UNFOLDING PLANE CURVES WITH CUSPS AND NODES

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ABSTRACT. Given an irreducible surface germ \((X,0) \subset (\mathbb{C}^3,0)\) with 1-dimensional singular set \(\Sigma\), we denote by \(\delta_1(X,0)\) the delta invariant of a transverse slice. We show that \(\delta_1(X,0) \geq m_0(\Sigma,0)\), with equality if and only if \((X,0)\) admits a corank 1 parameterisation \(f : (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)\) whose only singularities outside the origin are transverse double points and semicubic cuspidal edges. Then, we use the local Euler obstruction \(\text{Eu}(X,0)\) in order to characterize those surfaces which have finite codimension with respect to \(\mathcal{A}\)-equivalence or as a frontal type singularity.

1. Introduction

Any irreducible complex plane curve singularity \((Y,0)\) can be parameterised, that is, it can be seen as the image of a finite and generically 1-1 map germ \(\gamma : (\mathbb{C},0) \to (\mathbb{C}^2,0)\). Then, we can look at it either as a finitely determined map germ with respect to the \(\mathcal{A}\)-equivalence or also as a frontal type singularity (using Zakalyukin’s terminology [16]) of finite codimension in some sense. This phenomenon becomes explicit when we consider a suitable deformation \(Y_t\), parameterised by a stable map \(\gamma_t\). In the first case, \(Y_t\) is a Morsification of \(Y\), since the degenerated singularity splits into a finite number of nodes, that is, transverse double points \(A_1\). In the second case, besides the nodes, we also allow the birth of simple cusps \(A_2\), which are stable singularities in this context. As an example, we see in fig. 1 the two deformations of the \(E_6\) singularity, parameterised by \(\gamma(v) = (v^3, v^4)\).

The total space of the deformation \((X,0)\) is an irreducible surface in \((\mathbb{C}^3,0)\) with 1-dimensional singular locus \(\Sigma\) which has special properties. It can be parameterised as the image of a map germ \(f : (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)\) given by \(f(u,v) = (u, \gamma_u(v))\). If \(\gamma_u\) is a Morsification, then \(f\) is \(\mathcal{A}\)-finite, that is, it has finite codimension with respect to the \(\mathcal{A}\)-equivalence. Otherwise, if \(\gamma_u\)

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is a deformation as a frontal, then \( f \) is itself a frontal type surface of finite codimension as a frontal (see Section 3). We show in fig. 2 the two surfaces constructed with the two deformations of \( E_6 \). On the left hand side, we have the \( P_3(c) \) singularity of D. Mond [12] and on the right hand side we have the swallowtail.

Another interesting property of \((X,0)\) is the equality \( \delta_1(X,0) = m_0(\Sigma,0) \), where \( \delta_1(X,0) \) is the transverse delta invariant (i.e., the delta invariant of a generic plane section) and \( m_0(\Sigma,0) \) is the multiplicity of its singular locus. Since this is the minimal possible value for \( \delta_1(X,0) \), we say that \((X,0)\) is a \( \delta_1 \)-minimal surface. In fact, we show in theorem 2.1 that for any irreducible surface \((X,0)\) with non isolated singularity, we have \( \delta_1(X,0) \geq m_0(\Sigma,0) \), with equality if and only if \((X,0)\) admits a corank 1 parameterisation \( f : (\mathbb{C}^2,0) \rightarrow (\mathbb{C}^3,0) \) and such that the only singularities outside the origin are transverse double points or semicubic cuspidal edges.

In the last part of the paper, we use the local Euler obstruction \( \text{Eu}(X,0) \) in order to characterize those surfaces among the \( \delta_1 \)-minimal ones which are stable unfoldings of plane curves or frontals. We show that if \((X,0)\) is \( \delta_1 \)-minimal, then

\[
1 \leq \text{Eu}(X,0) \leq m_0(X,0).
\]

Moreover, we deduce (see corollary 4.3):

1. \((X,0)\) is the image of a corank 1 \( \mathcal{A} \)-finite map germ if and only if it is \( \delta_1 \)-minimal and \( \text{Eu}(X,0) = 1 \).

2. \((X,0)\) is the image of a corank 1 frontal of finite codimension if and only if it is \( \delta_1 \)-minimal and \( \text{Eu}(X,0) = m_0(X,0) \).

Note that Jorge-Pérez and Saia proved in [8] that if \((X,0)\) is the image of a corank 1 \( \mathcal{A} \)-finite map germ, then \( \text{Eu}(X,0) = 1 \). The results presented here are also related to those of [11], where we consider the classification and the invariants of corank 1 \( \mathcal{A} \)-finite map germs \( f : (\mathbb{C}^2,0) \rightarrow (\mathbb{C}^3,0) \) by looking at the transverse slice.

## 2. \( \delta_1 \)-minimal surfaces

Let \((X,0) \subset (\mathbb{C}^3,0)\) be a singular surface. Given \( 0 \in H \subset \mathbb{C}^3 \) a generic plane we consider the plane curve \( Y = X \cap H \) and we call it a transverse slice of \( X \). The delta invariant of \( Y \) at \( 0 \) is an invariant of \((X,0)\) which is independent of the choice of \( H \). We denote \( \delta_1(X,0) := \delta(Y,0) \) and call it the transverse delta invariant.
Given an analytic set germ $(V,0) \subset (\mathbb{C}^n,0)$ we denote by $m_0(V,0)$ its multiplicity. We recall that this can be computed by means of a generic linear projection $\ell : \mathbb{C}^n \to \mathbb{C}^d$, where $d = \dim(V,0)$. Then $m_0(V,0) = \# V \cap H_t$ where $H_t = \ell^{-1}(t)$ and $t \in \mathbb{C}^d$ is a generic value.

**Theorem 2.1.** Let $(X,0) \subset (\mathbb{C}^3,0)$ be an irreducible surface with singular locus $\Sigma$ of dimension 1, then $$\delta_t(X,0) \geq m_0(\Sigma,0).$$ Moreover, the equality holds if and only if $(X,0)$ admits a corank 1 parameterisation $f : (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ such that the only singularities outside the origin are transverse double points and semicubic cuspidal edges.

**Proof.** We consider a linear projection $\ell : \mathbb{C}^3 \to \mathbb{C}$ such that $H = \ell^{-1}(0)$ is a generic plane and $Y = X \cap H$ is a transverse slice of $X$. Moreover, for each $t \in \mathbb{C}$ we can take $H_t = \ell^{-1}(t)$ in such a way that $Y_t = X \cap H_t$ defines a flat deformation of $(Y,0)$.

Since $(X,0)$ is irreducible, it has a normalization $n : (\tilde{X},0) \to (X,0)$, where $(\tilde{X},0)$ is a normal surface and $n$ is finite and generically 1-1. By taking the composition $\tilde{p} = p \circ n : (\tilde{X},0) \to (\mathbb{C},0)$ we have also a flat deformation of $Y = n^{-1}(Y)$.

We use now a result of Lejeune-Lê-Teissier [6] (see also [3, 4.1.14]): for any $t \neq 0$ small enough,

\begin{equation}
\delta(Y,0) = \delta(\tilde{Y},0) + \sum_{p \in S(Y_t)} \delta(Y_t,p),
\end{equation}

where $S(Y_t)$ denotes the singular set of $Y_t$. Obviously, $S(Y_t) = Y_t \cap \Sigma = H_t \cap \Sigma$ and for each $p \in S(Y_t)$, $\delta(Y_t,p) \geq 1$. Therefore,

$$\delta(Y,0) \geq \# H_t \cap \Sigma = m_0(\Sigma,0).$$

We have the equality in the case that $(X,0)$ admits a corank 1 parameterisation $f : (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ and the only singularities of $(X,0)$ outside the origin are transverse double points and semicubic cuspidal edges. In fact, after taking a linear coordinate change in $\mathbb{C}^3$ and after reparameterisation, we can assume that $f$ is given in the form

$$f(u,v) = (u,p(u,v),q(u,v)),$$

for some function germs $p,q$ and such that generic plane is $x = 0$ (here we denote by $(x,y,z)$ the coordinates in $\mathbb{C}^3$). Then $\tilde{Y}$ is the curve $u = 0$ which is smooth and thus $\delta(\tilde{Y},0) = 0$.

On the other hand, for each $t \neq 0$, the deformation $Y_t$ is given by $x = t$. The only singularities of $Y_t$ are cusps and nodes, both having delta invariant equal to 1. By (1), $\delta(Y,0) = m_0(\Sigma,0)$.

We see now the converse. If $\delta(Y,0) = m_0(\Sigma,0)$, we deduce from (1) that $\delta(\tilde{Y},0) = 0$ and $\delta(Y_t,p) = 1$ for each $t \neq 0$ and $p \in S(Y_t)$. In other words, $\tilde{Y}$ is smooth at 0 and the only singularities of $Y_t$ are cusps and nodes when $t \neq 0$.

Since $\delta(\tilde{Y},0) = 0$, we have from (1) that $Y_t$ is a delta constant family of curves in the sense of Teissier. By [3, 7.1.3], $Y_t$ admits a normalization in family. But the unicity of the normalization implies that $\tilde{X}$ is smooth.
at 0 and we can assume $\tilde{X} = \mathbb{C}^2$. Thus, $(X, 0)$ is the image of $f = i \circ n : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, where $i$ denotes the inclusion map.

Because of $\tilde{Y}$ is smooth at 0, $f$ must have corank 1. Moreover, the only singularities of $f$ outside the origin will be semicubic cuspidal edges and transverse double points (having as transverse slice cusps and nodes, respectively).

**Definition 2.2.** We say that a surface $(X, 0) \subset (\mathbb{C}^3, 0)$ is $\delta_1$-**minimal** if it is irreducible with 1-dimensional singular locus $\Sigma$ and $\delta_1(X, 0) = m_0(\Sigma, 0)$.

It follows from the proof of theorem 2.1 that the following statements are equivalent:

1. $(X, 0)$ is $\delta_1$-minimal.
2. $(X, 0)$ admits a corank 1 parameterisation $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ such that the only singularities outside the origin are semicubic cuspidal edges and transverse double points.
3. $(X, 0)$ is the image of an unfolding of a plane curve with only cusps and nodes.

**Example 2.3.** Let $(X, 0)$ be the surface parameterised by the double fold map germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ given by $f(u, v) = (u^2, v^2, u^5 + v^5 + 2u^3v^3)$ (see [10]). Then $(X, 0)$ is irreducible, its singular set $\Sigma$ has dimension 1 and all the singularities outside the origin are semicubic cuspidal edges and transverse double points (see fig. 3). But since $f$ has corank 2, we expect to get $\delta_1(X, 0) > m_0(\Sigma, 0)$.

In fact, according to [10], $\Sigma$ is the curve in $(\mathbb{C}^3, 0)$ given by the zeros of the $3 \times 3$ minors of the following matrix:

$$
\begin{pmatrix}
-x & x^2 & y^2 & 2xy \\
x^3 & -z & 2x^2y & y^2 \\
y^3 & 2xy^2 & -z & x^2 \\
2x^2y^2 & y^3 & x^3 & -z
\end{pmatrix}
$$

With the aid of the computer algebra system SINGULAR [4], we compute $m_0(\Sigma, 0) = 13$. On the other hand, $(X, 0)$ is given by the determinant of the
above matrix:

\[\begin{align*}
&x^{10} - 8x^5y^3 + 16x^6y^6 - 2x^5y^5 - 2x^5z^2 - 16x^4y^4z \\
&- 8x^3y^2 - 8x^3y^3z^2 + y^{10} - 2y^5z^2 + z^4 = 0.
\end{align*}\]

In order to compute the transverse slice, we just substitute \( z = ax + by \) for some generic coefficients \( a, b \in \mathbb{C} \). Again with the aid of SINGULAR we get \( \delta_1(X,0) = 14 \).

We can associate two invariants to each \( \delta_1 \)-minimal surface \((X,0)\). Let \( \ell : \mathbb{C}^3 \to \mathbb{C} \) be a generic linear projection and put \( H_t = \ell^{-1}(t) \) and \( Y_t = X \cap H_t \). Since \((X,0)\) is \( \delta_1 \)-minimal, the only singularities of \( Y_t \) for \( t \neq 0 \) small enough are cusps and nodes.

**Definition 2.4.** We define the **numbers of transverse cusps** and **transverse nodes** of \((X,0)\), respectively as:

- \( \kappa \) = number of cusps \((A_2)\) of \( Y_t \),
- \( \nu \) = number of nodes \((A_1)\) of \( Y_t \).

It is obvious that the numbers \( \kappa, \nu \) are well defined and do not depend on the choice of the linear projection \( \ell \) nor the parameter \( t \). Moreover, we also deduce from the proof of theorem 2.1 that

\[ \kappa + \nu = \delta_1(X,0). \]

If \((X,0)\) admits a corank 1 parameterisation \( f : (\mathbb{C}^2,0) \to (\mathbb{C}^3,0) \), then after taking a linear coordinate change in \( \mathbb{C}^3 \) and after reparameterisation, we can assume that \( f \) is given in the form

\[ f(u,v) = (u, \gamma_u(v)), \]

where \( \gamma_u(v) \) is the parameterisation of the plane curve \( Y_u = X \cap \{ x = u \} \).

**Proposition 2.5.** Let \((X,0)\) be a \( \delta_1 \)-minimal surface, parameterised by \( f(u,v) = (u, \gamma_u(v)) \), where \( x = 0 \) is a generic plane. The following statements are equivalent:

1. \( \kappa = 0 \),
2. \( f \) is \( \mathcal{A} \)-finite,
3. for each \( t \neq 0 \), \( \gamma_t \) is \( \mathcal{A} \)-stable.

**Proof.** The equivalence between (1) and (3) follows from the fact that the only \( \mathcal{A} \)-stable singularities of plane curves are nodes. The equivalence between (1) and (2) is a consequence of the Mather-Gaffney determinacy criterion: the map germ \( f : (\mathbb{C}^2,0) \to (\mathbb{C}^3,0) \) is \( \mathcal{A} \)-finite if and only if there is a proper representative \( f : U \to V \) such that \( f^{-1}(0) = \{ 0 \} \) and the restriction to \( U \setminus \{ 0 \} \) is \( \mathcal{A} \)-stable. But since the cross-caps and the transverse triple points are isolated, by shrinking \( U \) if necessary, this is equivalent to that \( f \) has only transverse double points on \( U \setminus \{ 0 \} \).

**Example 2.6.** Let \((X,0)\) be an irreducible surface with 1-dimensional singular set whose transverse slice has type \( E_6 \). We parameterise the curve by \( \gamma(v) = (v^3, v^4) \) and take the mini-versal deformation:

\[ \Gamma(v; a, b, c) = (v^3 + av, v^4 + bv^2 + cv). \]
Then, after a linear coordinate change, \((X, 0)\) admits a parameterisation of the form
\[
f(u, v) = (u, v^3 + a(u)v, v^4 + b(u)v^2 + c(u)v),
\]
for some \(a, b, c \in \mathbb{C}\{u\}\), with \(a(0) = b(0) = c(0) = 0\).

The discriminant of the deformation \(\Delta\) is the set of points \((a, b, c) \in \mathbb{C}^3\) such that the curve \(\gamma_{a,b,c}(v) = (v^3 + av, v^4 + bv^2 + cv)\) is not \(\mathcal{A}\)-stable. According to [11], \(\Delta\) has equation
\[
P_1 = 16a^3 - 48a^2b + 36ab^2 + 27c^2,
\]
\[
P_2 = 32a^3 - 48a^2b + 24ab^2 - 4b^3 + 27c^2,
\]
\[
P_3 = a - b.
\]
The three factors \(P_1, P_2, P_3\) correspond to the strata of singular points, self-tangencies and triple points, respectively.

If we also denote \(P_i = P_i(a(u), b(u), c(u))\), we have three types of \(\delta_1\)-minimal surfaces:

1. \((X, 0)\) is \(\delta_1\)-minimal with \(\kappa = 0\) and \(\nu = 3\) if and only if \(P_1P_2P_3 \neq 0\).
2. \((X, 0)\) is \(\delta_1\)-minimal with \(\kappa = 1\) and \(\nu = 2\) if and only if \(P_1 = 0\), but \((c, 2a - 3b) \neq (0, 0)\) and \(P_2P_3 \neq 0\).
3. \((X, 0)\) is \(\delta_1\)-minimal with \(\kappa = 2\) and \(\nu = 1\) if and only if \((c, 2a - 3b) = (0, 0)\), but \(P_2P_3 \neq 0\).

### 3. Frontals

In this section, we consider frontal type singularities. This concept was introduced by Zakalyukin and Kurbatskiï in [16] and it is the generalization of a front. Roughly speaking, a frontal is the projection of a Legendrian submanifold with singularities. We refer also to Ishikawa’s paper [5] for basic definitions and notations about Legendre singularities.

Let \(PT^*\mathbb{C}^{n+1}\) be the projectivized cotangent bundle of \(\mathbb{C}^{n+1}\) with the canonical contact structure defined by the contact form \(\alpha\) and denote the projection by \(\pi : PT^*\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}\). By definition, a holomorphic map germ \(L : (\mathbb{C}^n, 0) \to PT^*\mathbb{C}^{n+1}\) is said to be integral if \(L^*\alpha \equiv 0\). This is means that \(L = (f, [\nu])\) where \(f : (\mathbb{C}^n, 0) \to \mathbb{C}^{n+1}\) is a holomorphic map germ and \(\nu : (\mathbb{C}^n, 0) \to T^*\mathbb{C}^{n+1}\) is a holomorphic non-zero 1-form along \(f\) such that \(\nu(df \circ \xi) = 0\), for any \(\xi \in V_n\), the space of all germs of vector fields in \((\mathbb{C}^n, 0)\). If \(\nu\) is given in coordinates by \(\nu = \sum_{j=1}^{n+1} \nu_j dx_j\), this is also equivalent to
\[
\sum_{j=1}^{n+1} \nu_j \frac{\partial f_j}{\partial u_i} = 0, \quad \forall i = 1, \ldots, n.
\]

**Definition 3.1.** We say that a map germ \(f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)\) is a frontal map germ if there is an integral map germ \(L : (\mathbb{C}^n, 0) \to PT^*\mathbb{C}^{n+1}\) such that \(\pi \circ L = f\). If in addition \(L\) is an embedding, then we say that \(f\) is a frontal.

When \(L\) is an integral embedding, then its image in \(PT^*\mathbb{C}^{n+1}\) is called a Legendrian submanifold. If it is not an embedding, then it is usual to call the image a Legendrian submanifold with singularities. A hypersurface singularity \((X, 0)\) in \((\mathbb{C}^{n+1}, 0)\) is called a frontal (resp. front) if there is
a frontal (resp. front) map germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ whose image is $(X, 0)$.

**Remark 3.2.** If the map germ $f$ is itself an embedding, then it is always a frontal and the class $[\nu]$ is determined univocally by the components of the cross product:

$$\frac{\partial f}{\partial u_1} \wedge \cdots \wedge \frac{\partial f}{\partial u_n}.$$ 

If $f$ is not an embedding, but it is generically immersive (for instance, when it is finite and generically 1-1), then the class $[\nu]$ is also univocally determined, if it exists.

**Example 3.3.** Let us see some examples:

1. Any irreducible plane curve singularity is always a frontal. Assume $(Y, 0)$ is parameterised in $(\mathbb{C}^2, 0)$ by $\gamma(v) = (p(v), q(v))$, where

$$p(v) = a_n v^n + a_{n+1} v^{n+1} + \ldots,$$

$$q(v) = b_m v^m + b_{m+1} v^{m+1} + \ldots$$

with $a_n, b_m \neq 0$ and $n \leq m$. Then we take the 1-form:

$$\nu = \frac{1}{v^{n-1}} (-q'(v)dx + p'(v)dy).$$

Note that $(Y, 0)$ is a front if and only if $m = n + 1$.

2. The double fold surface $(X, 0)$ of Example 2.3 is a corank 2 frontal surface in $(\mathbb{C}^3, 0)$. In fact, since

$$\frac{\partial f}{\partial u} \wedge \frac{\partial f}{\partial v} = uv(-2u(5u^2 + 6v^3), -2v(6u^3 + 5v^2), 4),$$

we may take

$$\nu = -2u(5u^2 + 6v^3)dx - 2v(6u^3 + 5v^2)dy + 4dz.$$ 

3. Not every parameterised surface $(X, 0) \subset (\mathbb{C}^3, 0)$ is a frontal. For instance, given the cross-cap $f(u, v) = (u, p(u, v), q(u, v))$ we have

$$\frac{\partial f}{\partial u} \wedge \frac{\partial f}{\partial v} = (-2v^2, -u, 2v).$$

There is no a non-zero holomorphic 1-form $\nu$ such that

$$\nu \left( \frac{\partial f}{\partial u} \right) = \nu \left( \frac{\partial f}{\partial v} \right) = 0.$$ 

In general, we have the following criterion for corank 1 hypersurfaces.

**Proposition 3.4.** Consider a hypersurface $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ parameterised by a corank 1 map germ $f(u, v) = (u, p(u, v), q(u, v))$, with $u \in \mathbb{C}^{n-1}$, $v \in \mathbb{C}$. Then $(X, 0)$ is a frontal if and only if either $\frac{\partial p}{\partial u}$ divides $\frac{\partial q}{\partial u}$ or $\frac{\partial q}{\partial u}$ divides $\frac{\partial p}{\partial v}$.

**Proof.** We have that

$$\frac{\partial f}{\partial u_1} \wedge \cdots \wedge \frac{\partial f}{\partial u_{n-1}} \wedge \frac{\partial f}{\partial v} = \left( \Delta_1, \ldots, \Delta_{n-1} - \frac{\partial q}{\partial v} \frac{\partial p}{\partial v} \right)$$

where $\Delta_i = \frac{\partial q}{\partial u_i} \frac{\partial p}{\partial v} - \frac{\partial q}{\partial u_i} \frac{\partial p}{\partial v}$.
Assume, for instance, that $\frac{\partial q}{\partial v} = \lambda \frac{\partial p}{\partial v}$ for some function $\lambda$. Then, $\Delta_i = \mu_i \frac{\partial p}{\partial v}$, with $\mu_i = \lambda \frac{\partial p}{\partial u_i} - \frac{\partial q}{\partial u_i}$ and thus, we can take

$$\nu = \mu_1 dx_1 + \cdots + \mu_{n-1} dx_{n-1} - \lambda dx_n + dx_{n+1}.$$  

Conversely, suppose that there is non-zero 1-form $\nu$ such that $L = (f, [\nu])$ is integral. Then, there is a function $\alpha$, such that

$$\Delta_i = \alpha \nu_i, \quad i = 1, \ldots, n - 1,$$

and hence,

$$\alpha \nu_i = -\alpha \left( \nu_n \frac{\partial p}{\partial u_i} + \nu_{n+1} \frac{\partial q}{\partial u_i} \right), \quad i = 1, \ldots, n - 1.$$  

If $\alpha = 0$, we have $\frac{\partial p}{\partial v} = \frac{\partial q}{\partial v} = 0$ and the result is obvious. Otherwise, if $\alpha \neq 0$, we have that

$$\nu_i = -\nu_n \frac{\partial p}{\partial u_i} - \nu_{n+1} \frac{\partial q}{\partial u_i}, \quad i = 1, \ldots, n - 1.$$  

Since $\nu(0) \neq 0$, then necessarily either $\nu_n(0) = 0$ or $\nu_{n+1}(0) \neq 0$ so that either $\frac{\partial p}{\partial v} \bigg|_{\partial q/\partial v}$ or $\frac{\partial q}{\partial v} \bigg|_{\partial q/\partial v}$.

□  

Example 3.5. We apply this criterion to see some examples of frontal surfaces:

1. The swallowtail is $(X, 0)$ is a frontal surface (see the right hand side of fig. 2). In fact, it is parameterised by $f(u, v) = (u, v^3 + uv, v^4 + \frac{2}{3} uv^2)$ and we have $\frac{\partial p}{\partial v} = 3v^2 + u$ and $\frac{\partial q}{\partial v} = \frac{2}{3} v(3v^2 + u)$.

2. The folded Whitney umbrella is the surface $(X, 0)$ in $(\mathbb{C}^3, 0)$ parameterised by $f(u, v) = (u, v^2, uv^3 + v^5)$ (see fig. 4). This is also a frontal since $\frac{\partial p}{\partial v} = 2v$ and $\frac{\partial q}{\partial v} = v(3uv + 5v^3)$.

Now we define the codimension of a frontal as the codimension of the Legendrian singularity whose projection is the frontal, with respect to Legendre equivalence. Let us denote $W = PT^*\mathbb{C}^{n+1}$ for simplicity and let $L : (\mathbb{C}^n, 0) \to (W, w_0)$ be the integral map germ given by $L = (f, [\nu])$. We recall the following notations from [5]:

1. $VL_L$ is the space of all integral infinitesimal deformations of $L$, that is, germs of vector fields along $L$ which preserve the contact structure.

2. $VL_{W,w_0}$ is the space of all germs of Legendre vector fields in $(W, w_0)$.

Definition 3.6. We define the $F_e$-codimension of $f$ as

$$F_e - \text{codim}(f) = \dim_{\mathbb{C}} \{ dL \circ \xi + \tilde{\eta} \circ L : \xi \in V_n, \tilde{\eta} \in VL_{W,w_0} \}.$$  

If the $F_e$-codimension is finite, we say that $f$ is $F_e$-finite and if the $F_e$-codimension is zero, then we say that $f$ is $F_e$-stable.

According to [5], the space $VL_L$ can be interpreted as the space of all infinitesimal integral deformations of $L$ and the subspace

$$\{ dL \circ \xi + \tilde{\eta} \circ L : \xi \in V_n, \tilde{\eta} \in VL_{W,w_0} \}.$$
should be understood as the extended tangent space to the orbit of $\mathcal{L}$ under the action of Legendre equivalences. It follows from the definition that $f$ is $\mathcal{F}$-stable if and only if $\mathcal{L}$ is infinitesimally Legendre stable in the sense of [5]. By [5, 4.1], any corank 1 $\mathcal{F}$-stable frontal is the projection of an open Whitney umbrella.

All the above definitions are also valid if instead of germs we consider multigerms $f: (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, y)$, where $S \subset \mathbb{C}^n$ is any finite set and $y \in \mathbb{C}^{n+1}$. We use the above remark to classify the $\mathcal{F}$-stable singularities of curves and surfaces. Note that all the $\mathcal{F}$-stable singularities of frontal surfaces except folded Whitney umbrellas are generic fronts and their classification is well known (see for instance [1]).

**Proposition 3.7.** (1) The $\mathcal{F}$-stable singularities of a frontal curve are cusps and nodes.

(2) The $\mathcal{F}$-stable singularities of a frontal surface are either: semicubic cuspidal edges, swallowtails, folded Whitney umbrellas or their transverse self-intersections (see fig. 4).

**Figure 4**

The following property is an adapted version of the Mather-Gaffney finite determinacy criterion for frontals (see [15]).

**Proposition 3.8.** A frontal $f: (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ is $\mathcal{F}$-finite if and only if there is a proper and finite-to-one representative $\tilde{f}: U \to V$ such that $f^{-1}(0) = \{0\}$ and the multigerm at any point $y \in V \setminus \{0\}$ is $\mathcal{F}$-stable.

By shrinking the neighbourhoods $U, V$ if necessary, all the isolated $\mathcal{F}$-stable singularities can be avoided. Then, we have the following direct consequence of propositions 3.7 and 3.8.

**Corollary 3.9.** (1) A frontal curve is $\mathcal{F}$-finite if and only if it has isolated singularity.
A frontal surface of corank 1 is $F$-finite if and only if the only singularities outside the origin are transverse double points and semicubic cuspidal edges.

Recall that if $(X, 0)$ is $\delta_1$-minimal then $0 \leq \kappa \leq m_0(X, 0) - 1$, where $\kappa$ is the number of cusps. Then, we have the following property, which is, in some sense, dual to Proposition 2.5.

**Proposition 3.10.** Let $(X, 0)$ be a $\delta_1$-minimal surface parameterised by $f(u, v) = (u, \gamma_u(v))$, where $x = 0$ is a generic plane. The following statements are equivalent:

1. $\kappa = m_0(X, 0) - 1$,
2. $f$ is a $F$-finite frontal,
3. $f$ is a frontal unfolding of $\gamma_0$ and for each $t \neq 0$, $\gamma_t$ is $F$-stable.

**Proof.** Since $(X, 0)$ is $\delta_1$-minimal, the only singularities outside the origin are transverse double points and semicubic cuspidal edges. Moreover, for each $t$, the transverse slice $Y_t$ is parameterised by $\gamma_t(v) = (p(t, v), q(t, v))$ and it has only cusps and nodes if $t \neq 0$. By 3.7 and 3.9, in order to show the equivalence between the three statements, we only need to show that $\kappa = m_0(X, 0) - 1$ if and only if $f$ is a frontal.

Given $h \in O_2$, we denote by $o_v(h)$ the order of $h$ in $v$, that is, the order of $h(0, v) \in O_1$. Assume that $o_v(p) = m$ and $o_v(q) = k$ with $m \leq k$. Then, because of the genericity assumption, we have that $m_0(X, 0) = m$.

For a fixed small enough $t \neq 0$, $\kappa$ is equal to the number of solutions of $p_v(t, v) = q_v(t, v) = 0$ in $v$. If $h = \gcd(p_v, q_v)$, then $\kappa$ is less than or equal to the number of solutions of $h(t, v) = 0$ in $v$. In particular,

$$\kappa \leq o_v(h) \leq o_v(p_v) = m - 1 = m_0(X, 0) - 1.$$

Thus, we have the following equivalences:

$$\kappa = m_0(X, 0) - 1 \iff o_v(h) = o_v(p_v) \iff p_v | q_v \iff f \text{ is a frontal}.$$

□

### 4. Local Euler obstruction

The local Euler obstruction was first introduced by McPherson [9] as an ingredient in the construction of characteristic classes of singular algebraic varieties. Here we prefer to use the approach of Lê-Teissier [7] in terms of polar multiplicities. Given an analytic set germ $(V, 0) \subset (\mathbb{C}^n, 0)$ of dimension $d$, its local Euler obstruction is computed as an alternate sum

$$\Eu(V, 0) = \sum_{i=0}^{d-1} (-1)^i m_i(V, 0),$$

where $m_i(V, 0)$ denotes the $i$th-polar multiplicity (see [7] for definitions and details). In particular, for a surface $(X, 0)$,

$$\Eu(X, 0) = m_0(X, 0) - m_1(X, 0),$$

and hence, $\Eu(X, 0) \leq m_0(X, 0)$.

In the next theorem, we compute the local Euler obstruction of a $\delta_1$-minimal surface in terms of the number of transverse cusps $\kappa$. To do this,
we first characterize the number $\nu$ of transverse nodes in terms of the number of vanishing cycles of the transverse slice $Y_t$.

**Lemma 4.1.** Let $(X, 0)$ be a $\delta_1$-minimal surface. Then, for each $t \neq 0$ small enough, the Euler characteristic of $Y_t$ is

$$\chi(Y_t) = 1 - \nu.$$  

**Proof.** Let us denote $\delta = \delta_1(X, 0) = \delta(Y, 0)$. Since $(X, 0)$ is $\delta_1$-minimal, we have seen in the proof of theorem 2.1 that $(Y, 0)$ is irreducible and hence its Milnor number is $\mu(Y, 0) = 2\delta$ (by Milnor’s formula).

On the other hand, $\chi(Y_t)$ is related to the Milnor number by the following formula [3]:

$$\mu(Y, 0) - \sum_{p \in S(Y_t)} \mu(Y_t, p) = \dim_C H^1(Y_t; \mathbb{C}) = 1 - \chi(Y_t).$$

For each $t \neq 0$ small enough, the only singularities of $Y_t$ are simple cusps, with Milnor number 2, and nodes, with Milnor number 1. Hence, we obtain

$$\mu(Y, 0) - \sum_{p \in S(Y_t)} \mu(Y_t, p) = 2\delta - (2\kappa + \nu) = \nu.$$

$\square$

**Theorem 4.2.** Let $(X, 0)$ be a $\delta_1$-minimal surface. Then,

$$\text{Eu}(X, 0) = 1 + \kappa.$$  

In particular, $1 \leq \text{Eu}(X, 0) \leq m_0(X, 0)$.

**Proof.** We use a formula of Brasselet-Lé-Seade [2] which is valid whenever $(X, 0)$ is equidimensional and has 1-dimensional singular locus $\Sigma$. We take $t \neq 0$ small enough and assume that $Y_t \cap \Sigma = \{x_1, \ldots, x_m\}$. Then,

$$\text{Eu}(X, 0) = \chi(Y_t) - m + \sum_{i=1}^{m} \text{Eu}(X, x_i).$$

Note that $Y_t \cap \Sigma$ is the singular locus of $Y_t$ and since each singular point has delta invariant 1, we have $m = \delta_1(X, 0) = \kappa + \nu$. By lemma 4.1, $\chi(Y_t) = 1 - \nu$. For each $i = 1, \ldots, m$, $\text{Eu}(X, x_i) = 2$ either if $X$ is a semicubic cuspidal edge or a transverse double point at $x_i$. Thus,

$$\text{Eu}(X, 0) = 1 - \nu - (\kappa + \nu) + 2\kappa + 2\nu = 1 + \kappa.$$  

$\square$

As a consequence, we arrive to the following result which characterizes those surfaces that are stable unfoldings of plane curves or frontals.

**Corollary 4.3.** Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be an irreducible surface with singular locus of dimension 1. Then:

1. $(X, 0)$ is the image of a corank 1 $\mathcal{A}$-finite germ if and only if it is $\delta_1$-minimal and $\text{Eu}(X, 0) = 1$.
2. $(X, 0)$ is the image of a corank 1 $\mathcal{F}$-finite front if and only if it is $\delta_1$-minimal and $\text{Eu}(X, 0) = m_0(X, 0)$.

**Proof.** It follows directly from 2.1, 2.5, 3.10 and 4.2.  

$\square$
We finish with a last result, where we consider irreducible surfaces with 1-dimensional locus in any ambient space and without any finiteness assumption. Given a space curve \((Y, 0) \subset (\mathbb{C}^N, 0)\), the first polar multiplicity was introduced by the author and Tomazella in [14]:

\[
m_1(Y, 0) := \mu(\ell|_{(Y, 0)}),
\]

where \(\ell : \mathbb{C}^N \to \mathbb{C}\) is a generic linear form and \(\mu(\ell|_{(Y, 0)})\) is the Milnor number in the sense of Mond and van Straten [13]. Then, it is showed that

\[
m_1(Y, 0) = \mu(Y, 0) + m_0(Y, 0) - 1,
\]

where \(\mu(Y, 0)\) is now the Milnor number of a space curve as defined by Buchweitz and Greuel [3].

**Proposition 4.4.** Let \((X, 0) \subset (\mathbb{C}^{N+1}, 0)\) be a equidimensional surface with 1-dimensional singular set \(\Sigma\). Then for \(t \neq 0\),

\[
m_1(X, 0) = m_1(Y, 0) - \sum_{x \in S(Y_t)} m_1(Y_t, x),
\]

where \(Y_t\) is the transverse slice of \((X, 0)\).

**Proof.** This is a consequence again of the Brasselet-Lê-Seade formula together with (2):

\[
m_1(X, 0) = m_0(X, 0) - \text{Eu}(X, 0)
\]

\[
= m_0(X, 0) - \chi(Y_t) + \sum_{x \in S(Y_t)} (\text{Eu}(X, x) - 1)
\]

\[
= m_0(Y_0, 0) - 1 + (1 - \chi(Y_t)) + \sum_{x \in S(Y_t)} (m_0(Y_t, x) - 1)
\]

\[
= m_0(Y_0, 0) - 1 + \mu(Y_0, 0) - \sum_{x \in S(Y_t)} (\mu(Y_t, x) - m_0(Y_t, x) + 1)
\]

\[
= m_1(Y_0, 0) + \sum_{x \in S(Y_t)} m_1(Y_t, x).
\]

\[\square\]

**Corollary 4.5.** With the above hypothesis, the following statements are equivalent:

1. \(m_1(X, 0) = 0\).
2. \((X, 0)\) defines a \(m_1\)-constant deformation of \((Y, 0)\).

Moreover, if \(N = 2\) and \((X, 0)\) admits a parameterisation, then any of the two above statements is also equivalent to the following one:

3. \((X, 0)\) is a frontal.

**Proof.** The equivalence between the two first statements follows directly from 4.4. According to Lê-Teissier [7], the condition \(m_1(X, 0) = 0\) is also equivalent to the fact that \((X, 0)\) has a finite number of limiting tangent planes at the origin. But in the particular case that \((X, 0)\) admits a parameterisation \(f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)\), then this condition is equivalent to that \((X, 0)\) is a frontal. \[\square\]
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References


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